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*A Note on Finite Groups which Act Freely on Closed Surfaces II**

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§1. Introduction

This note is a continuation of the previous note [2].

Let T_m or U_m be the orientable or non-orientable closed surface of genus m. In [2], we studied finite groups which act freely on the Klein bottle U_2 and the torus T_1, and on T_m preserving the orientation. In this note, we study what kind of finite groups can act freely on U_m, and on T_m reversing the orientation. Here we say that a finite group G acts on T_m reversing the orientation if some element of G reverses the orientation of T_m.

Let F_n be the free group generated by $x_1,\ldots, x_n$, and set $s_n=\prod_{i=1}^{n} x_i^2 \in F_n$. We say that an element w of F_n is even if w is a product of even times of generators, i.e., a form $\prod_{j=1}^{2k} x_{i_j}$, and is odd if it is not even; and also a subgroup K of F_n is even if any element of K is even, and is odd if it is not even. Also we denote by $^{\#}G$ the order of a finite group G. Then we have the following propositions.

PROPOSITION 1.1 (cf. [2, Prop. 3.2]). (i) *A finite group G acts freely on T_m reversing the orientation if and only if there exists an even normal subgroup K of F_n such that*

$$G \cong F_n/K, \quad K \ni s_n, \quad 2(1-m) = (2-n)(^{\#}G). \tag{1.2}$$

For this case, the orbit surface T_m/G is homeomorphic to U_n.

(ii) *A finite group G acts freely on U_m if and only if there exists an odd normal subgroup K of F_n such that*

$$G \cong F_n/K, \quad K \ni s_n, \quad 2-m = (2-n)(^{\#}G). \tag{1.3}$$

For this case, U_m/G is homeomorphic to U_n.

PROPOSITION 1.4 (cf. [2, Prop. 3.3]). (i) *Let G be a finite 2-group and assume that the minimum number of generators of G is n. Then G acts freely on T_m reversing the orientation, where $m=1+(n-1)(^{\#}G)$.*

(ii) *Let G be a finite group and assume that the number of generators of G is less than $n+1$. Then G acts freely on U_m, where $m=2+(2n+l-2)(^{\#}G)$*

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and $l \geqq 1$. *Moreover, if* $^{\#}G$ *is odd,* G *acts freely on* U_m, *where* $m=2+(2n-2)(^{\#}G)$.

Now, we say that a finite abelian group G, having a basis with at most n elements, is of type $e_1(n)$, $e_2(n)$ or $o(n)$, according to each case of the following:

$e_1(n)$: G includes a cyclic group Z_2 as a direct summand,

$e_2(n)$: $^{\#}G$ is even and G does not include Z_2 as a direct summand,

$o(n)$: $^{\#}G$ is odd.

Then we obtain the following results for an abelian group G by Proposition 1.1.

THEOREM 1.6 (cf. [2, Th. 1.10]). (i) *A finite abelian group* G *acts freely on* T_m *reversing the orientation if and only if there exists an integer* n *such that* $2(1-m)=(2-n)(^{\#}G)$ *and one of the following holds*:

(1) G *is of type* $e_1(n)$, (2) n *is even and* G *is of type* $e_2(n-1)$.

For this case, T_m/G *is homeomorphic to* U_n.

(ii) *A finite abelian group* G *acts freely on* U_m *if and only if there exists an integer* n *such that* $2-m=(2-n)(^{\#}G)$ *and one of the following holds*:

(3) G *is of type* $e_1(n)$ *and* $\dim(G\otimes Z_2)<n$,

(4) G *is of type* $e_2(n-1)$ *and* $\dim(G\otimes Z_2)<n-1$,

(5) n *is odd,* G *is of type* $e_2(n-1)$ *and* $\dim(G\otimes Z_2)=n-1$,

(6) G *is of type* $o(n-1)$.

For this case, U_m/G *is homeomorphic to* U_n.

Here, $\dim(G\otimes Z_2)$ *is the dimension of a vector space* $G\otimes Z_2$ *over* Z_2.

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§2. Proofs of Propositions 1.1, 1.4 and 1.5

In this note, we use the following notations:

F_n: the free group generated by $x_1,\dots, x_n$.

$\{w_1,\dots, w_k\}$: the minimal normal subgroup of F_n containing the elements $w_1,\dots, w_k$.

$H \cup K$: the minimal normal subgroup of F_n including the subgroups H and K.

As is well known, the fundamental group $\pi_1(U_n)$ of U_n is given by

$$\pi_1(U_n) = F_n/\{s_n\}, \quad s_n = \textstyle\prod_{i=1}^n x_i^2. \tag{2.1}$$

And also the Euler characteristics of T_m and U_n are given by

$$\chi(T_m) = 2(1-m), \quad \chi(U_n) = 2-n. \tag{2.2}$$

Let M be an orientable manifold and assume that a discrete group π acts properly discontinuously on M. Then we see easily that the orbit space M/π is a manifold and

(2.3) *M/π is orientable if and only if the action of π preserves the orientation of M.*

Consider the universal covering

$$u: S \longrightarrow S/\pi_1(U_n) = U_n \tag{2.4}$$

over U_n. Then S is an orientable surface, and

(2.5) (cf. [1, 5.4]) *we can choose the generators $x_1,\dots, x_n$ of $\pi_1(U_n)$ in (2.1) so that each x_i reverses the orientation of S.*

From now on, we fix such generators $x_1,\dots, x_n$.

Proof of Proposition 1.1. Let X_m denote T_m or U_m. For the case $X_m = T_m$, we only consider actions on T_m reversing the orientation.

Suppose that a finite group G acts freely on X_m. Then the orbit space X_m/G is homeomorphic to U_n for some n by (2.3), and there is a normal covering $p: X_m \to X_m/G \approx U_n$. Therefore, $p_*\pi_1(X_m)$ is a normal subgroup of $\pi_1(U_n)$ such that

$$\pi_1(U_n)/p_*\pi_1(X_m) \cong G, \tag{2.6}$$

and

$$\chi(X_m) = ({}^{\#}G)\chi(U_n). \tag{2.7}$$

By (2.7) and (2.2), we have the desired equalities

$$2(1-m) = (2-n)({}^{\#}G) \quad \text{or} \quad 2-m = (2-n)({}^{\#}G).$$

From the universal covering (2.4), we obtain the coverings

$$S \longrightarrow S/p_*\pi_1(X_m) = X \longrightarrow S/\pi_1(U_n) = U_n,$$

and X is a closed surface by (2.6). Also $\pi_1(X) \cong p_*\pi_1(X_m) \cong \pi_1(X_m)$, and so X is homeomorphic to X_m by the classification theorem of closed surfaces. Therefore, according to $X_m = T_m$ or U_m, the action of $p_*\pi_1(X_m)$ on S preserves or reverses the orientation, and hence there exists an even or odd normal subgroup K of F_n such that $K \ni s_n$ and $p_*\pi_1(X_m) = K/\{s_n\}$ by (2.1) and (2.5). Also, $G \cong F_n/K$ by (2.1) and (2.6), and the necessity is proved.

Conversely assume that there is an even or odd normal subgroup K of F_n satisfying (1.2) or (1.3), according to $X_m = T_m$ or U_m. Then $K/\{s_n\}$ is a normal subgroup of $\pi_1(U_n)$ by (2.1), and $K/\{s_n\}$ acts freely on S preserving or reversing the orientation by (2.3). Therefore, the orbit surface $X=S/(K/\{s_n\})$ is orientable or non-orientable by (2.3). Consider the normal covering

$$p: X = S/(K/\{s_n\}) \longrightarrow S/(F_n/\{s_n\}) = U_n$$

with the transformation group $G\cong F_n/K$. Then we have $\chi(X)=({}^{\#}G)\chi(U_n)=\chi(X_m)$ by (2.2) and (1.2) or (1.3). Therefore, X is homeomorphic to X_m by the classification theorem of closed surfaces, and so $G\cong F_n/K$ acts freely on X_m by (2.3), and the sufficiency is proved. *q.e.d.*

Let G be a finite 2-group and G^* be the Frattini subgroup of G, i.e., the intersection of all maximal subgroups of G. Then the following is well known.

LEMMA 2.8 (cf. [3, Th. 10.4.3, 10.3.4, 12.2.1]). *G^* includes the commutator subgroup DG of G, and G/G^* is a vector space over Z_2 and its dimension is equal to the minimum number of generators of G.*

PROOF OF PROPOSITION 1.4. (i) By the assumption, G is isomorphic to F_n/K'. Consider the projections

$$F_n \xrightarrow{\pi} F_n/DF_n \xrightarrow{p} F_n/K' \cup DF_n \cong G/DG \xrightarrow{q} G/G^*.$$

The projection qp induces an isomorphism $(F_n/DF_n)\otimes Z_2\cong G/G^*$ by the above lemma. Since DF_n is even, this shows that $\mathrm{Ker}(qp\pi)$ is even, and $K'(\subset \mathrm{Ker}(qp\pi))$ is also so. By considering the projection $\phi: F_{2n}\to F_n$, $\phi(x_{2i-1})=\phi(x_{2i}^{-1})=x_i$ $(1\leqq i\leqq n)$, we have $G\cong F_n/K'\cong F_{2n}/K$, where $K=\phi^{-1}(K')$. Since $\mathrm{Ker}\,\phi=\{x_1x_2,\dots, x_{2n-1}x_{2n}\}$ and K' is even, we see that $K\ni s_{2n}=\prod_{i=1}^{2n}x_i^2$ and K is even. Hence, the desired result follows immediately from Proposition 1.1 (i).

(ii) By the assumption, G is isomorphic to F_n/K'. By considering the projection $\psi: F_{2n+l}\to F_n$, $\psi(x_{2i-1})=\psi(x_{2i}^{-1})=x_i$ $(1\leqq i\leqq n)$, $\psi(x_{2n+j})=1$ $(1\leqq j\leqq l)$, we have $G\cong F_n/K'\cong F_{2n+l}/K$, where $K=\psi^{-1}(K')$. Since $\mathrm{Ker}\,\psi=\{x_1x_2,\dots, x_{2n-1}x_{2n}, x_{2n+1},\dots, x_{2n+l}\}$, we see that $K\ni s_{2n+l}=\prod_{i=1}^{2n+l}x_i^2$ and also K is odd if $l\geqq 1$. Suppose that ${}^{\#}G$ is odd and consider the projection

$$G\cong F_n/K' \xrightarrow{p} F_n/K' \cup \{x_1x_2^{-1},\dots, x_{n-1}x_n^{-1}\} = G'.$$

Then G' is a cyclic group of odd order, and we see that $K'\cup\{x_1x_2^{-1},\dots, x_{n-1}x_n^{-1}\}$ is an odd subgroup. Therefore K' is odd, and the above K for $l=0$ is also so. Hence, the desired results follow immediately from Proposition 1.1 (ii).

q.e.d.

§3. Proof of Theorem 1.6

In this section, we set

(3.1) $G_n = F_n/\{s_n\} \cup DF_n = Z_2 \oplus Z^{n-1}, \quad Z^{n-1} = Z \oplus \cdots \oplus Z \quad ((n-1)-\text{copies}),$

generated by $x = x_1 + \cdots + x_n$, $x_1, \ldots, x_{n-1}$, where x_i is the image of $x_i \in F_n$ by the projection $\pi\colon F_n \to G_n$. Also, we say that an element $\sum_{i=1}^n a_i x_i \in G_n$ is even if $\sum_{i=1}^n a_i$ is even, and a subgroup H of G_n is even if any element of H is even, and is odd if it is not even.

Then it is easy to see that H is an even subgroup of G_n if and only if $\pi^{-1}(H)$ is an even subgroup of F_n. Therefore, we have the following lemma.

LEMMA 3.2. *The condition of Proposition* 1.1 (i) *or* (ii) *holds for a finite abelian group G if and only if there exists an even or odd subgroup H of* G_n *such that* $G \cong G_n/H$ *and*

$$2(1-m) = (2-n)(^{\#}G) \quad or \quad 2-m = (2-n)(^{\#}G).$$

LEMMA 3.3. *A finite abelian group* $G = G_n/H$ *is of type* $e_1(n)$, $e_2(n-1)$ *or* $o(n-1)$.

PROOF. Consider the projection

$$\varphi\colon G_n = Z_2 \oplus Z^{n-1} \longrightarrow G_n/H = G,$$

and the generator x of the summand Z_2 of (3.1). For the case $\varphi(x) \neq 0$, G is of type $e_1(n)$ or $e_2(n)$ by the definition. If G is of type $e_2(n)$ in addition, then $\varphi(x) = 2\varphi(y)$ for some $y \in G_n$, and so G is of type $e_2(n-1)$. For the case $\varphi(x) = 0$, the lemma is clear. *q.e.d.*

LEMMA 3.4. *Let* $G = G_n/H$ *be a finite abelian group.*

(i) *If* $^{\#}G$ *is even and H is even, then* (1) *or* (2) *in Theorem* 1.6 *holds.*

(ii) *If H is odd, then* (3), (4), (5) *or* (6) *in Theorem* 1.6 *holds.*

PROOF. (i) Suppose that H is even. It is sufficient to show that G is not of type $e_2(n-1)$ if n is odd, by the definition of types and the above lemma. If n is odd, then the generator $x = x_1 + \cdots + x_n$ of Z_2 is not even and so $x \notin H$. Hence $\varphi(x) \in G$ is of order 2, where φ is the projection in the above proof. If G is of type $e_2(n-1)$ in addition, then there exists an element $y \in G_n$ such that $\varphi(x) = 2\varphi(y)$, i.e., $x - 2y \in H$. Since $x - 2y$ is not even, this contradicts the assumption that H is even.

(ii) By tensoring with Z_2, we have the commutative diagram

$$\begin{array}{ccc} G_n = Z_2 \oplus Z^{n-1} & \xrightarrow{\varphi} & G = G_n/H \\ \downarrow{\scriptstyle \otimes Z_2} & & \downarrow{\scriptstyle \otimes Z_2} \\ Z_2 \oplus (Z_2)^{n-1} & \xrightarrow{\bar{\varphi}} & G \otimes Z_2, \end{array} \tag{3.5}$$

where the induced homomorphism $\bar{\varphi}$ is also epimorphic. We denote by $\bar{w}$ the image of w by the vertical arrow.

Suppose that H is odd and an element y of H is not even. Then

$$(3.6) \qquad \varphi(y) = 0 \quad \text{and} \quad \bar{y} \neq 0,$$

and so $\bar{\varphi}$ in (3.5) is not isomorphic. Thus $\dim(G \otimes Z_2) < n$. Therefore by the above lemma, it is sufficient to show that

$$(3.7) \qquad n \text{ is odd if } G \text{ is of type } e_2(n-1) \text{ and } \dim(G \otimes Z_2) = n-1.$$

If $\bar{\varphi}(\bar{x}) \neq 0$ for the generator x of $Z_2 \subset G_n$, then $\varphi(x) \in G$ is of order 2 and $\varphi(x) = 2\varphi(w)$ for some element $w \in G_n$, by the assumption that G is of type $e_2(n-1)$. This is a contradiction since $0 \neq \bar{\varphi}(\bar{x}) = 2\bar{\varphi}(\bar{w}) = 0$, and we see $\bar{\varphi}(\bar{x}) = 0$. Therefore, $\operatorname{Ker}\bar{\varphi}$ is Z_2 generated by $\bar{x}$ by the assumption $\dim(G \otimes Z_2) = n-1$, and we see $\bar{x} = \bar{y}$ for an element y of (3.6). Since y is not even, $\bar{x} = \bar{y}$ implies that $x = x_1 + \cdots + x_n$ is not even, i.e., n is odd. Thus we have (3.7). *q.e.d.*

LEMMA 3.8. *The necessity of Theorem* 1.6 *is valid.*

PROOF. If G acts on T_m reversing the orientation, then we see easily that ${}^{\#}G$ is even. Therefore, the necessity of Theorem 1.6 follows immediately from Lemmas 3.2 and 3.4. *q.e.d.*

Now, we prove the sufficiency of Theorem 1.6. For any sequence

$$\tau = (t_1, \dots, t_{n-1}), \ t_i\colon \text{even} \quad (i < k_\tau), \ t_j\colon \text{odd} \ (j \geqq k_\tau),$$

and $t = 0, 1$, we consider the subgroups

$$(3.9) \qquad H_1(\tau, t) = \{tx, t_i x_i \ (i < k_\tau), (t+1)t_k(x + x_k), t_{j+1}(x_j + x_{j+1}) \ (j \geqq k_\tau)\},$$

$$H(\tau) = \{x, t_i x_i \ (i < k_\tau), t_j(x_{j-1} + x_j) \ (j \geqq k_\tau)\},$$

$$H_2(\tau, t) = \{tx, t_1 x_1, \dots, t_{n-1} x_{n-1}\}$$

of G_n. Then we have easily the following

LEMMA 3.10. *The factor group G_n/H is isomorphic to*
$Z_2 \oplus Z_{t_1} \oplus \cdots \oplus Z_{t_{n-1}}$ *if* $H = H_1(\tau, 0)$ *or* $H_2(\tau, 0)$, *or* $H = H_1(\tau, 1)$ *and* $k_\tau \leqq n-1$,
$Z_{t_1} \oplus \cdots \oplus Z_{t_{n-1}}$ *if* $H = H(\tau)$ *or* $H_2(\tau, 1)$.

LEMMA 3.11. *The sufficiency of Theorem* 1.6 *is valid.*

PROOF. By the above lemma, we see easily that any finite abelian group G of (h) $(1 \leqq h \leqq 6)$ in Theorem 1.6 is isomorphic to G_n/H, where H is given as follows:

(1) $H = H_1(\tau, 0)$ and $k_\tau = n$, or $H = H_1(\tau, 1)$, $k_\tau < n$ and n is even, or $H = H_1(\tau, 0)$

and n is odd.

(2) $H=H(\tau)$, n is even and $4|t_i$ $(i<k_\tau)$.

(3) $H=H_2(\tau, 0)$ and $k_\tau<n$.

(4) $H=H_2(\tau, 1)$, $k_\tau<n$ and $4|t_i$ $(i<k_\tau)$.

(5) $H=H_2(\tau, 1)$, $k_\tau=n$, n is odd and $4|t_i$ $(i<k_\tau)$.

(6) $H=H_2(\tau, 1)$ and $k_\tau=1$.

By the definition (3.9), it is clear that H of (h) is even for $h=1, 2$, and is odd for $3\leqq h\leqq 6$. Therefore, we obtain the sufficiency of Theorem 1.6 by Lemmas 3.2 and 3.4. *q.e.d.*

By Lemmas 3.8 and 3.11, Theorem 1.6 is proved completely.

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