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Module Spectra over the Moore Spectrum

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Introduction

Let q be an odd integer and $M_q(=M)$ be the Moore spectrum of type Z_q. It is a ring spectrum with multiplication $m_M: M_q \wedge M_q \to M_q$ and unit $i: S \to M_q$ [1], where S is the sphere spectrum. A finite CW-spectrum X is called an M_q-*module spectrum* if X is equipped with a left inverse $m_X: M_q \wedge X \to X$, which we call an M_q-*action* on X, of $i \wedge 1_X$. It is clear that X is an M_q-module spectrum if and only if $q1_X=0$ in $[X, X]$. When q is a prime, our M_q-module spectrum is just the Z_q-spectrum introduced by H. Toda [10].

The main purpose of this note is to investigate conditions under which an M_q-module spectrum is (non-)associative. Here an M_q-module spectrum (X, m_X) is called *associative* if $m_X(1_M \wedge m_X)=m_X(m_M \wedge 1_X)$. For an M_q-module spectrum X, the order r of 1_X is a divisor of q and the homology group $H_i(X)$ is a finite Z_r-module, and we shall obtain in §6 the following theorems on the associativity and on the non-associativity according as the case $q \not\equiv \pm 3$ mod 9 or $(r, 3)=1$ and the case $q \equiv \pm 3$ mod 9 and $3|r$.

THEOREM 6.6. *Let X be an M_q-module spectrum and in the case of $q \equiv \pm 3$* mod 9 *assume that the order of 1_X is relatively prime to* 3. *If X satisfies the following two conditions, then X admits an associative M_q-action.*

(i) *$\sharp H_i(X)$ is relatively prime to $\sharp H_{i-1}(X)$ and to $\sharp H_{i-2}(X)$.*

(ii) *For any prime p, the p-component of $H_i(X)$ is free over the p-component of Z_q.*

Here $\sharp G$ denotes the order of a finite group G. Furthermore we shall see that in the dual Postnikov system $\{X_i\}$ of X (X_i is a subspectrum of X realizing $\sum_{j \leq i} H_j(X)$ as its homology group) each X_i is also an associative M_q-module spectrum, (cf. Remark 6.7). We shall also construct, for every prime $q>3$, an example which does not satisfy the condition (i) and has a unique M_q-action, which is not associative (Example 6.8).

THEOREM 6.3. *Assume that $q \equiv \pm 3$* mod 9. *Let X be an M_q-module spectrum such that the order of 1_X is a multiple of* 3. *Then every M_q-action on X is not associative.*

In §1, we shall study elementary properties of M_q-module spectra. In §2, we shall define a derivation θ on $[X, Y]_*$ for M_q-module spectra X and Y (Definition 2.1), which is due to H. Toda [10], and consider M_q-maps. An M_q-map is a map between M_q-module spectra which is compatible with M_q-actions (Definition 2.4). In § 3, we shall quote known results on the Moore spectrum. In particular, Toda's result [9] (Theorem 3.3) on the associativity of the ring spectrum M_q plays an important role in the later sections.

In § 4, we shall explicitly construct an M_q-action on the mapping cone of an M_q-map (Theorem 4.3). Furthermore this M_q-action is admissible (Definition 4.1), i.e., compatible with the structure of the cofibering. We shall also obtain exact sequences of M_q-module spectra and M_q-maps derived as usual from a cofibering (Theorem 4.5). In § 5, we shall introduce an associator $a(m_X) \in [\Sigma^2 X, X]$, whose vanishing is equivalent to m_X being associative, and study several properties of the element $a(m_X)$. In § 6, we shall prove the above theorems using the results of § 5.

In § 7, we shall be concerned with associative M_q-module spectra X and Y such that $[X, Y]_* \xrightarrow{\theta} [X, Y]_* \xrightarrow{\theta} [X, Y]_*$ is exact. In this case, the subgroup of $[X, Y]_*$ consisting of all M_q-maps is a direct summand (Theorem 7.5). We shall also consider a modification $\bar{\theta}$ of θ (Definition 7.6) so that the discussions hold for non-associative case (Theorem 7.7). In the final section, § 8, we shall notice that the known results ([2], [4]) on the structure of the stable homotopy ring of the mod p^r Moore spectrum (p an odd prime, $p^r \neq 3$) also hold for the case $p^r = 3$ by making use of $\bar{\theta}$ instead of θ.

In this note, except for § 4, we shall work in the stable homotopy category of finite CW-spectra. In § 4 only, we shall distinguish between a map and its homotopy class.

§ 1. *M*-module spectra

We shall denote by S and $M = M_q$ the sphere spectrum $\{S^n, \varepsilon_n = 1\}$ and the Moore spectrum $\{S^n \cup_q e^{n+1}, \varepsilon_n = 1\}$ of type Z_q, respectively. Here q denotes always an *odd* integer >0 and the spectra handled in this note are suspension spectra $\{X_n, \varepsilon_n\}$ consisting of finite CW-complexes X_n and imbeddings $\varepsilon_n\colon \Sigma X_n \subset X_{n+1}$ such that $\Sigma X_n = X_{n+1}$ and $\varepsilon_n = 1$ for sufficiently large n; Σ being the suspension functor and $1 = 1_X$ being the identity map of X or its homotopy class. There is a cofiber sequence

$$(1.1) \qquad S \xrightarrow{q} S \xrightarrow{i} M_q \xrightarrow{\pi} \Sigma S.$$

For any spectra X and Y, denote by $[X, Y]$ the set of homotopy classes of maps $X \to Y$, and put $[X, Y]_k = [\Sigma^k X, Y]$ for $k \in Z$ ($[X, Y]_0 = [X, Y]$). Then the direct sum $[X, X]_* = \sum_k [X, X]_k$ forms a graded ring by the composition of maps.

PROPOSITION 1.1 ([10; Lemma 1.2]). *The following four conditions are equivalent to each other.*

(i) $i \wedge 1_X \in [X, M \wedge X]_0$ *has a left inverse* $m_X \in [M \wedge X, X]_0$.

(ii) $\pi \wedge 1_X \in [M \wedge X, X]_{-1}$ *has a right inverse* $n_X \in [X, M \wedge X]_1$.

(iii) $q1_X = 0$ *in* $[X, X]$.

(iv) $[X, X]_*$ *is an algebra over the ring* Z_q.

PROOF. From (1.1), we have the exact sequence

$$[M \wedge X, X] \xrightarrow{(i \wedge 1_X)^*} [X, X] \xrightarrow{\times q} [X, X],$$

and we see immediately that (i) is equivalent to (iii). Similarly, (ii)$\Leftrightarrow$(iii) is proved. Since 1_X is the unit of the ring $[X, X]_*$, (iv) is equivalent to (iii).

DEFINITION 1.2. A spectrum X which satisfies one of the above conditions is called an M_q- (or M-)*module spectrum*, and a left inverse m_X of $i \wedge 1_X$ is called an M_q- (or M-)*action* on X.

If X satisfies the condition (i) of above, then $M \wedge X$ is homotopy equivalent to a wedge $X \vee \Sigma X$, and hence there is a right inverse n_X of $\pi \wedge 1_X$ such that

(1.2) $$m_X n_X = 0$$

and

(1.3) $$(i \wedge 1_X)m_X + n_X(\pi \wedge 1_X) = 1_{M \wedge X}.$$

Since $n'_X = ((i \wedge 1_X)m_X + n_X(\pi \wedge 1_X))n'_X = n_X(\pi \wedge 1_X)n'_X = n_X$ for another n'_X satisfying (1.2) and (1.3), such n_X is unique for m_X, (cf. [10; Remark 1.4]). Thus we have

(1.4) *For any M-action m_X on X, there exists uniquely the right inverse n_X of $\pi \wedge 1_X$ satisfying* (1.2) *and* (1.3).

We shall write (X, m_X, n_X) (or simply (X, m_X)), when X is an M-module spectrum with the M-action m_X and the right inverse n_X of $\pi \wedge 1_X$ corresponding to m_X in the sense of (1.4).

For the wedge sum and the smash product of M-module spectra, the following are easily verified.

(1.5) *Let (X, m_X, n_X) and (Y, m_Y, n_Y) be M-module spectra. Then $X \vee Y$ is an M-module spectra equipped with $m_{X \vee Y} = m_X \vee m_Y$ and $n_{X \vee Y} = n_X \vee n_Y$ via the* *identification* $M \wedge (X \vee Y) = (M \wedge X) \vee (M \wedge Y)$.

(1.6) ([10; (2.2)]) *Let* (X, m_X, n_X) *be any M-module spectrum and* Y *be arbitrary finite CW-spectrum. Then*

$$(X \wedge Y, m_X \wedge 1_Y, n_X \wedge 1_Y),$$

$$(Y \wedge X, (1_Y \wedge m_X)(T \wedge 1_X), (T^{-1} \wedge 1_X)(1_Y \wedge n_X))$$

are M-module spectra, where $T: M \wedge Y \to Y \wedge M$ *is the switching map. In particular, via the identification* $M \wedge \Sigma^t X = \Sigma^t (M \wedge X)$, $(\Sigma^t X, \Sigma^t m_X, \Sigma^t n_X)$ *is an M-module spectrum for* $t \in Z$.

When both X and Y are M-module spectra, we can consider the two M-actions $m_X \wedge 1_Y$ and $(1_X \wedge m_Y)(T \wedge 1_Y)$ on $X \wedge Y$.

THEOREM 1.3. *Let* X *be an M-module spectrum. Then, for any M-actions* m_X *and* m'_X, *there is uniquely an element* $d(m_X, m'_X) \in [X, X]_1$ *such that*

$$m_X = m'_X + d(m_X, m'_X)(\pi \wedge 1_X). \tag{1.7}$$

The correspondence $m'_X \mapsto d(m_X, m'_X)$ *gives a bijection between the set of all M-actions on* X *and* $[X, X]_1$. *If* n_X *and* n'_X *are the right inverse of* $\pi \wedge 1_X$ *corresponding to* m_X *and* m'_X *in the sense of* (1.4), *then*

$$n_X = n'_X - (i \wedge 1_X) d(m_X, m'_X). \tag{1.7$'$}$$

PROOF. For any finite CW-spectrum Y, we have the following (split) exact sequences:

$$0 \longrightarrow [Y, X]_k \xrightarrow{(i \wedge 1_X)_*} [Y, M \wedge X]_k \xrightarrow{(\pi \wedge 1_X)_*} [Y, X]_{k-1} \longrightarrow 0,$$

$$0 \longrightarrow [X, Y]_{k+1} \xrightarrow{(\pi \wedge 1_X)^*} [M \wedge X, Y]_k \xrightarrow{(i \wedge 1_X)^*} [X, Y]_k \longrightarrow 0.$$

Then $m_X = m'_X + d(\pi \wedge 1_X)$ and $n_X = n'_X + (i \wedge 1_X) d'$ for unique d and d'. By (1.2) and easy calculations, we have $d + d' = 0$ and so (1.7) and (1.7)$'$. For any m_X and any $d \in [X, X]_1$, $m_X + d(\pi \wedge 1_X)$ is also an M-action, and hence the correspondence $m'_X \mapsto d(m_X, m'_X)$ is bijective.

REMARK. If Y is an M-module spectrum, then the exact sequences in the above proof are also split for arbitrary X. In fact, the correspondences

$$[Y, X]_{k-1} \ni f \mapsto (-1)^{k-1}(1_M \wedge f) n_Y \in [Y, M \wedge X]_k,$$

$$[X, Y]_k \ni f \mapsto m_Y (1_M \wedge f) \in [M \wedge X, Y]_k$$

give the desired splittings.

LEMMA 1.4. *Let X be an M-module spectrum such that 1_X is of order q. Then $\tilde{H}_*(X; Z_p) \neq 0$ for any prime $p|q$, and $\tilde{H}_*(X; Z_p) = 0$ for any prime $p \nmid q$.*

PROOF. The assumption asserts that the stable order of X in the sense of [8] is q. Then q divides the square of the order of $\tilde{H}_*(X; Z)$ by [8; Th. 1.5]. Hence, for $p|q$, $\tilde{H}_*(X; Z)$ has p-torsion and $\tilde{H}_*(X; Z_p) \neq 0$. If $p \nmid q$, $p1_X$ is a homotopy equivalence and induces an automorphism of $\tilde{H}_*(X; Z)$. So $\tilde{H}_*(X; Z_p) = 0$.

§2. Derivation θ and M-maps

We shall define a derivation on $[X, Y]_*$. This is due essentially to H. Toda [10], though its root goes back to P. Hoffman's D [2].

DEFINITION 2.1. Let (X, m_X) and (Y, m_Y) be M-module spectra. Then we define

$$\theta = \theta_{m_X, m_Y} \colon [X, Y]_k \longrightarrow [X, Y]_{k+1}$$

by the formula

$$\theta(f) = m_Y(1_M \wedge f) n_X,$$

where n_X is the right inverse of $\pi \wedge 1_X$ corresponding to m_X.

For the M-actions on the wedge sum and the smash product defined in (1.5–1.6), the following hold easily.

$$\theta_{m_X \vee m_V, m_Y \vee m_W}(f \vee g) = \theta_{m_X, m_Y}(f) \vee \theta_{m_V, m_W}(g) \quad \text{for} \quad f \in [X, Y]_*, \quad g \in [V, W]_*. \tag{2.1}$$

$$\theta_{m_Y \wedge 1_V, m_Y \wedge 1_W}(f \wedge g) = \theta_{m_X, m_Y}(f) \wedge g \quad \text{for} \quad f \in [X, Y]_*, \quad g \in [V, W]_*, \tag{2.2}$$

and a similar formula holds for the M-action defined from the second coordinate of the smash product.

THEOREM 2.2. *Let m_X and m'_X be M-actions on X, and m_Y and m'_Y be M-actions on Y. Then*

$$\theta_{m_X, m_Y}(f) = \theta_{m'_X, m'_Y}(f) - f d(m_X, m'_X) + (-1)^k d(m_Y, m'_Y) f$$

for any $f \in [X, Y]_k$.

PROOF. This follows immediately from (1.7) and (1.7)′.

THEOREM 2.3 ([10; Th. 2.2]). *Let X, Y, Z be M-module spectra. Then, for any $f \in [X, Y]_k$ and $g \in [Y, Z]_l$, the following formula holds:*

$$\theta(gf) = (-1)^k \theta(g) f + g\theta(f),$$

that is, the operation θ is derivative.

PROOF. This follows from (1.3) and easy calculations.

DEFINITION 2.4. Let (X, m_X), (Y, m_Y) be M-module spectra. Then an element $f \in [X, Y]_k$ is called an *M-map* (*with respect to m_X and m_Y*) if it satisfies the equality $fm_X = m_Y(1_M \wedge f)$. Denote by

$$[X, Y]_k^M$$

the subgroup of $[X, Y]_k$ consisting of all M-maps.

If q is a prime, this definition agrees with H. Toda's Z_q-map [10; p. 207]. We see immediately that the composition of two M-maps is an M-map, so

(2.3) *$[X, X]_*^M$ is a subring of $[X, X]_*$, and $[X, Y]_*^M$ is a right $[X, X]_*^M$-, left $[Y, Y]_*^M$-module.*

PROPOSITION 2.5. *Let X and Y be M-module spectra, and $f \in [X, Y]_k$. Then the following three statements are equivalent to each other.*

(i) *f is an M-map.*

(ii) $(-1)^k n_Y f = (1_M \wedge f) n_X$.

(iii) $\theta(f) = 0$.

PROOF. By (1.3), we have

$$m_Y(1_M \wedge f) = fm_X + \theta(f)(\pi \wedge 1_X),$$

$$(1_M \wedge f) n_X = (i \wedge 1_Y)\theta(f) + (-1)^k n_Y f.$$

Since $(\pi \wedge 1_X)^*$ and $(i \wedge 1_Y)_*$ are monomorphic, we obtain the proposition.

COROLLARY 2.6. *Let X and Y be M-module spectra, and $f \in [X, Y]_k$. Denote simply by θ the derivation θ_{m_X, m_Y} for fixed M-actions m_X and m_Y. Then there exist M-actions m'_X on X and m'_Y on Y such that f is an M-map with respect to m'_X and m'_Y if and only if $\theta(f)$ lies in the image of $f_* + f^*$: $[X, X]_1 \oplus [Y, Y]_1 \to [X, Y]_{k+1}$.*

PROOF. This is clear by Proposition 2.5 and Theorems 1.3 and 2.2.

§3. Moore spectrum

We denote the stable homotopy ring of sphere by

$$G_* = [S, S]_*.$$

From the known results on G_*, $*\leqq k$, the group $[M, M]_*$, $*<k$, is easily computed by using exact sequences derived from (1.1) (cf. [4; Th. 3.5] and [11]), and we obtain the following

LEMMA 3.1. *Let $M=M_q$ and p be the minimal prime dividing q. Then $[M, M]_k=0$ for $k<-1$ and for $0<k<2p-4$, $[M, M]_{-1}=Z_q$ with the generator $\delta=i\pi$, $[M, M]_0=Z_q$ with the generator 1_M, and $[M, M]_{2p-4}=Z_p$ with the generator $i\alpha_1(p)\pi$. Here $\alpha_1(p)$ is the generator of the p-component of G_{2p-3}.*

From this lemma we can easily verify the following result, which is essentially due to S. Araki and H. Toda [1].

THEOREM 3.2. *M is a ring spectrum with the unique multiplication m_M, i.e., there is uniquely the map $m_M\colon M\wedge M\to M$ such that $m_M(i\wedge 1_M)=1_M=m_M(1_M\wedge i)$.*

Also, M is an M-module spectrum with the unique M-action m_M of above. It is equipped with a right inverse n_M of $\pi\wedge 1_M$, which is unique and satisfies $(1_M\wedge\pi)n_M=-1_M$, (cf. [10; Lemma 1.3]).

Consider the element $\delta=i\pi\in[M, M]_{-1}$. Then we have

$$\delta^2 = 0, \quad \theta(\delta) = -1_M, \tag{3.1}$$

because $\delta^2=i\pi i\pi=0$ and $\theta(\delta)=m_M(1\wedge i)(1\wedge\pi)n_M=-1_M$.

The following (non-)associativity of m_M (and n_M) is proved by H. Toda ([9; p. 202], [10; §6]).

THEOREM 3.3. *In the case of $q\not\equiv\pm 3 \bmod 9$, m_M and n_M are associative, i.e.,*

$$m_M(m_M\wedge 1_M) = m_M(1_M\wedge m_M),$$

$$(n_M\wedge 1_M)n_M = -(1_M\wedge n_M)n_M.$$

In the case of $q\equiv\pm 3 \bmod 9$, these are not associative. More precisely, the following equalities hold:

$$m_M(1_M\wedge m_M) = m_M(m_M\wedge 1_M)+\varepsilon_q i\alpha_1(3)(\pi\wedge\pi\wedge\pi),$$

$$(1_M\wedge n_M)n_M = -(n_M\wedge 1_M)n_M+\varepsilon_q(i\wedge i\wedge i)\alpha_1(3)\pi,$$

where $\varepsilon_q = \pm 1$ *and* $\varepsilon_q \equiv q/3 \bmod 3$, *and we take the sign of the element* $\alpha_1(3)$ *so that* $\varepsilon_3 \equiv 1 \bmod 3$(*).

Now let r be a divisor of q, and denote by

$$\lambda: M_r \longrightarrow M_q \quad \text{and} \quad \rho: M_q \longrightarrow M_r \tag{3.2}$$

the maps which induce the canonical monomorphism $Z_r \to Z_q$ and epimorphism $Z_q \to Z_r$ of homology groups, respectively. Then these maps satisfy (cf. [4; §2])

$$\lambda i = (q/r)i, \quad \pi\lambda = \pi; \quad \rho i = i, \quad \pi\rho = (q/r)\pi. \tag{3.3}$$

$$\rho\lambda = (q/r)1_{M_r}, \quad \lambda\rho = (q/r)1_{M_q}. \tag{3.4}$$

PROPOSITION 3.4. *Let* r *be a divisor of* q, *and* (X, m_X, n_X) *be an* M_r-*module spectrum. Put*

$$m_X(q) = m_X(\rho \wedge 1_X), \quad n_X(q) = (\lambda \wedge 1_X)n_X. \tag{3.5}$$

Then X *is the* M_q-*module spectrum having the* M_q-*action* $m_X(q)$ *and the right inverse* $n_X(q)$ *of* $\pi \wedge 1_X$ *corresponding to* $m_X(q)$.

PROOF. By virtue of (3.3), $m_X(q)$ is a left inverse of $i \wedge 1_X$ and $n_X(q)$ is a right inverse of $\pi \wedge 1_X$. By (3.4), $m_X(q)n_X(q) = (q/r)m_X n_X = 0$. The equality (1.3) for $m_X(q)$ and $n_X(q)$ is obtained from the fact:

(3.6) *Let* m_X *and* n_X *be arbitrary left inverse of* $i \wedge 1_X$ *and right inverse of* $\pi \wedge 1_X$, *respectively. Then* (1.2) *and* (1.3) *are equivalent.*

PROPOSITION 3.5. *Let* r *be a divisor of* q. *Let* (X, m_X) *and* (Y, m_Y) *be* M_r-*module spectra. Then*

$$\theta_{m_X(q), m_Y(q)}(f) = (q/r)\theta_{m_X, m_Y}(f)$$

for any $f \in [X, Y]_*$.

PROOF. This is immediate from (3.4) and (3.5).

COROLLARY 3.6. *Let* r *be an integer such that* $r^2 | q$. *Let* (X, m_X) *and* (Y, m_Y) *be* M_r-*module spectra. Then any map* $f \in [X, Y]_*$ *is an* M_q-*map with respect to the* M_q-*action* (3.5).

PROOF. Since $r[X, Y]_* = 0$, this follows from Propositions 2.5 and 3.5.

(*) Toda's result [9; §4] does not make mention of the sign ε_q depending on q. By [4; Lemma 2.1 (iv)], the map ρ in (3.2) is a morphism of ring spectra, and hence $\varepsilon_q \equiv (q/3)^3 \varepsilon_3 \equiv q/3 \bmod 3$ for $q \equiv 0 \bmod 3$.

REMARK. More precise discussions show the following result: *Let r and s be integers such that $rs|q$. Let X and Y be M_r- and M_s-module spectra. Then any map $f \in [X, Y]_*$ is an M_q-map with respect to the M_q-action* (3.5).

Let G be a finite Z_q-module, i.e., a finite abelian group such that $qx=0$ for any $x \in G$. Let $M(G)$ be the Moore spectrum of type G. Then we have a decomposition $G=Z_{r_1}\oplus\cdots\oplus Z_{r_l}$ for some $r_i|q$, and hence $M(G)=M_{r_1}\vee\cdots\vee M_{r_l}$. In the same way as Lemma 3.1, we see $[M_r, M_s]_1=0$ for odd r and s, and so $[M(G), M(G)]_1=0$. Hence we have obtained the following

PROPOSITION 3.7. *Let G be as above. Then $M(G)$ has a unique M_q-action $m_1(q)\vee\cdots\vee m_l(q)$, where m_i is the multiplication (M_{r_i}-action) on M_{r_i}. In particular, for any $r|q$, M_r has a unique M_q-action $m_{M_r}(q)=m_{M_r}(\rho\wedge 1_{M_r})$.*

§4. Mapping cone

In this section only, we shall usually *distinguish between a map and its homotopy class.*

For any map $f\colon \Sigma^k X \to Y$, we shall denote by

$$\Sigma^k X \xrightarrow{f} Y \xrightarrow{i_f} C(f) \xrightarrow{\pi_f} \Sigma^{k+1} X \tag{4.1}$$

the cofiber sequence for the mapping cone $C(f)$ of f.

DEFINITION 4.1. Let (X, m_X) and (Y, m_Y) be M-module spectra and $f\colon \Sigma^k X \to Y$ be any map. Assume that $C(f)$ is also an M-module spectrum. Then an M-action m_C on $C(f)$ is called *admissible* if i_f and π_f in (4.1) are the M-maps with respect to m_C.

We shall construct an admissible M-action on $C(f)$ for any M-map f.

CONSTRUCTION 4.2. Let (X, m_X) and (Y, m_Y) be M-module spectra and $f\colon X \to Y$ be an M-map. We shall distinguish a map from its homotopy class. By the homotopy extension property for the pair $(M\wedge W, W)$, $W=X, Y$, we can take the map m_W so that $m_W(i\wedge 1_W)$ is equal to 1_W as a map. Let $F_t\colon M\wedge X \to Y$ be a homotopy from $F_0=m_Y(1_M\wedge f)$ to $F_1=fm_X$. Define a map

$$\tilde{m}_C = \tilde{m}_C(F_t)\colon M \wedge C(f) \longrightarrow C(f)$$

by $\tilde{m}_C(m\wedge y)=m_Y(m\wedge y)$ for $m\wedge y \in M\wedge Y \subset M\wedge C(f)$ and

$$\tilde{m}_C(m \wedge s \wedge x) = \begin{cases} F_{2s}(m \wedge x) & \text{if } 0 \leqq s \leqq 1/2, \\ (2s-1) \wedge m_X(m \wedge x) & \text{if } 1/2 \leqq s \leqq 1, \end{cases}$$

for $m \wedge s \wedge x \in M \wedge I \wedge X \subset M \wedge C(f)$, where $I=[0, 1]$ with the base point 1 (so $I \wedge X$ is the cone CX over X). In the notation of [5; Lemma 2.5], $\tilde{m}_C = e(F_t)$ via the identification $C(1_M \wedge f) = M \wedge C(f)$. This map $\tilde{m}_C$ satisfies $\tilde{m}_C(1_M \wedge i_f) = i_f m_Y$ and $\pi_f \tilde{m}_C \sim \Sigma m_X(1_M \wedge \pi_f)$, where $\sim$ means "is homotopic to". Put $f_t = F_t(i \wedge 1_X): X \to X$. This is a homotopy from $f_0 = f$ to $f_1 = f$, and the map $e(f_t): C(f) \to C(f)$ constructed in the same way as above is a homotopy equivalence such that $\tilde{m}_C(i \wedge 1_{C(f)}) = e(f_t)$, $e(f_t) i_f = i_f$ and $\pi_f e(f_t) \sim \pi_f$. Thus we obtain a map

$$m_C = m_C(F_t): M \wedge C(f) \longrightarrow C(f)$$

by the formula $m_C = e(f_{1-t})\tilde{m}_C$.

THEOREM 4.3. *The map m_C constructed above is an admissible M-action on $C(f)$, namely, the mapping cone of any M-map has always an admissible M-action.*

PROOF. Since $e(f_{1-t})$ is a homotopy inverse of $e(f_t)$ such that $e(f_{1-t}) i_f = i_f$ and $\pi_f e(f_{1-t}) \sim \pi_f$, we have $m_C(i \wedge 1_{C(f)}) \sim 1_{C(f)}$, $m_C(1_M \wedge i_f) = i_f m_Y$ and $\pi_f m_C \sim \Sigma m_X(1_M \wedge \pi_f)$ as desired.

For any maps $f: X \to Y$ and $g: Y \to Z$, we denote by

$$(4.2) \qquad \begin{aligned} &(1_Z, f): C(gf) \longrightarrow C(g), \\ &(g, 1_X): C(f) \longrightarrow C(gf) \end{aligned}$$

the maps defined by $(1_Z, f)|Z = 1_Z$, $(1_Z, f)|CX = Cf$ and by $(g, 1_X)|Y = g$, $(g, 1_X)| CX = 1_{CX}$. It is easy to see that $(1_Z, ff') = (1_Z, f)(1_Z, f')$ and $(g'g, 1_X) = (g', 1_X)(g, 1_X)$ for $f': W \to X$ and $g': Z \to U$.

THEOREM 4.4. *Let X, Y and Z be M-module spectra, and $f: X \to Y$ and $g: Y \to Z$ be M-maps. Then there exist admissible M-actions on $C(f)$, $C(g)$ and $C(gf)$ such that the maps $(1_Z, f)$ and $(g, 1_X)$ of (4.2) are M-maps.*

PROOF. Let $F_t: M \wedge X \to Y$ and $G_t: M \wedge Y \to Z$ be homotopies with $F_0 = m_Y(1_M \wedge f)$, $F_1 = f m_X$ and $G_0 = m_Z(1_M \wedge g)$, $G_1 = g m_Y$. Define the homotopy $H_t: M \wedge X \to Z$ from $H_0 = m_Z(1_M \wedge gf)$ to $H_1 = gf m_X$ by

$$H_t = \begin{cases} G_{2t}(1_M \wedge f) & \text{for} \quad 0 \leqq t \leqq 1/2, \\ g F_{2t-1} & \text{for} \quad 1/2 \leqq t \leqq 1. \end{cases}$$

Then we shall prove that $m_C(F_t)$, $m_C(G_t)$ and $m_C(H_t)$ are the desired M-actions on $C(f)$, $C(g)$ and $C(gf)$.

Define a homotopy $K_\theta: M \wedge C(gf) \to C(g)$ from $K_0 = (1, f)\tilde{m}_C(H_t)$ to K_1

$=\tilde{m}_C(G_t)(1_M\wedge(1,f))$ by

$$K_\theta(m \wedge z) = m_Z(m \wedge z),$$

$$K_\theta(m \wedge s \wedge x) = \begin{cases} G_{4s/(1+\theta)}(m \wedge f(x)) & \text{for } 0 \leqq s \leqq (1+\theta)/4, \\ gF_{4s-\theta-1}(m \wedge x) & \text{for } (1+\theta)/4 \leqq s \leqq 1/2, \\ (2s-1) \wedge F_{1-\theta}(m \wedge x) & \text{for } 1/2 \leqq s \leqq 1, \end{cases}$$

where $m\in M$, $z\in Z$, $s\in I$, $x\in X$. Then $k_\theta=K_\theta(i\wedge 1_{C(gf)})$ is a homotopy from $k_0=(1,f)e(h_t)$ to $k_1=e(g_t)(1,f)$, where $h_t=H_t(i\wedge 1_X)$ and $g_t=G_t(i\wedge 1_Y)$. Therefore $(1,f)m_C(H_t)\sim m_C(G_t)(1\wedge(1,f))$, and $(1,f)$ is an M-map.

Next define a homotopy L_θ: $M\wedge C(f)\to C(gf)$ by

$$L_\theta(m \wedge y) = G_\theta(m \wedge y),$$

$$L_\theta(m \wedge s \wedge x) = \begin{cases} G_{4s+\theta}(m \wedge f(x)) & \text{for } 0 \leqq s \leqq (1-\theta)/4, \\ gF_{(4s+\theta-1)/(1+\theta)}(m \wedge x) & \text{for } (1-\theta)/4 \leqq s \leqq 1/2, \\ (2s-1) \wedge m_X(m \wedge x) & \text{for } 1/2 \leqq s \leqq 1. \end{cases}$$

Then $L_0=\tilde{m}_C(H_t)(1\wedge(g,1))$, $L_1=(g,1)\tilde{m}_C(F_t)$, $L_0(i\wedge 1_{C(f)})=e(h_t)(g,1)$ and $L_1(i\wedge 1_{C(f)})=(g,1)e(f_t)$. Hence $(g,1)$ is also an M-map.

THEOREM 4.5. *Let (X, m_X) and (Y, m_Y) be M-module spectra and f: $\Sigma^k X\to Y$ be an M-map. Then, with respect to any admissible M-action on $C(f)$ in Construction 4.2, the following sequences are exact for any M-module spectrum (Z, m_Z):*

$$\cdots \longrightarrow [Z, X]^M_{j-k} \xrightarrow{f_*} [Z, Y]^M_j \xrightarrow{i_{f*}} [Z, C(f)]^M_j \xrightarrow{\pi_{f*}} [Z, X]^M_{j-k-1} \longrightarrow \cdots,$$

$$\cdots \longrightarrow [X, Z]^M_{j+k+1} \xrightarrow{\pi_f^*} [C(f), Z]^M_j \xrightarrow{i_f^*} [Y, Z]^M_j \xrightarrow{f^*} [X, Z]^M_{j+k} \longrightarrow \cdots.$$

To prove the theorem, we prepare the following

LEMMA 4.6. *Let X and Y be M-module spectra and f: $X\to Y$ be an M-map with a homotopy F_t: $M\wedge X\to Y$ from $F_0=m_Y(1_M\wedge f)$ to $F_1=fm_X$. Assume that f is homotopic to the constant map. Then there are a retraction r: $C(f)\to Y$ and an inclusion l: $\Sigma X\to C(f)$ which are M-maps with respect to the M-action $m_C(F_t)$ on $C(f)$.*

PROOF. Let f_t: $X\to Y$ be a homotopy from $f_0=f$ to $f_1=*$. Then r is defined by $r(y)=y$ for $y\in Y$, $r(t\wedge x)=f_t(x)$ for $t\wedge x\in I\wedge X=CX$. Since $I\times 0\cup I\times 1\cup 0\times I$ is a retract of $I\times I$, we can construct a double homotopy $H_{s,t}$: $M\wedge X\to Y$,

$(s, t) \in I \times I$, such that $H_{s,0}=m_Y(1_M \wedge f_s)$, $H_{s,1}=f_s m_X$ and $H_{0,t}=F_t$. Define a homotopy K_θ: $M \wedge C(f) \to Y$ by

$$K_\theta(m \wedge y) = m_Y(m \wedge y),$$

$$K_\theta(m \wedge t \wedge x) = \begin{cases} F_{2\theta t}(m \wedge x) & \text{for } 0 \leqq t \leqq 1/2, \\ H_{2t-1,\theta}(m \wedge x) & \text{for } 1/2 \leqq t \leqq 1. \end{cases}$$

Then K_0 is homotopic to $m_Y(1_M \wedge r)$ and $K_1 = r\tilde{m}_C(F_t)$. Therefore r is an M-map. The proof of l being an M-map is similar.

PROOF OF THEOREM 4.5. It suffices to show the theorem for the case $j=k=0$. We take an m_C on $C(f)$ in Construction 4.2. Let g be any element in $[Z, X]_{-1}^M = [Z, \Sigma X]^M$ such that $f_*(g)=0$. Then the composition

$$\bar{g} = (1, g)l \colon Z \longrightarrow C(fg) \longrightarrow C(f)$$

satisfies $\pi_{f*}(\bar{g})=g$ and is an M-map by Theorem 3.4 and Lemma 3.6. This shows the exactness at $[Z, X]_{-1}^M$ in the first sequence. From the definition of $m_C = m_C(F_t)$, we can easily construct a homotopy P_t from $P_0 = m_X(1 \wedge \Sigma^{-1}\pi_f)$ to $P_1 = (\Sigma^{-1}\pi_f)m_C$ such that the M-action $m_C(P_t)$ on $Y = C(\Sigma^{-1}\pi_f)$ coincides with the original one m_Y. So we can replace (Y, m_Y) by $(C(\Sigma^{-1}\pi_f), m_C(P_t))$, and hence the exactness at $[Z, C(f)]_0^M$ follows from the same discussion as above. The exactness at $[Z, Y]_0^M$ is also the same. The proof for the second sequence is similar.

The following result is an improvement of the second half of [10; Lemma 2.3].

LEMMA 4.7. *Let X be an M-module spectrum, and Y be a finite CW-spectrum. Let f: $\Sigma^k X \to Y$ be a map such that $C(f)$ is an M-module spectrum. If $[Y, X]_{-k}=0$, then Y is also an M-module spectrum and there are M-actions m_X on X and m_Y on Y such that f is an M-map with respect to m_X and m_Y. Furthermore there is a homotopy from $m_Y(1_M \wedge f)$ to fm_X such that the M-action on $C(f)$ given by Construction 4.2 using this homotopy coincides with the original one on $C(f)$.*

PROOF. From the assumption, π_f^*: $[X, X]_1 \to [C(f), X]_{-k}$ is an epimorphism. Hence $\theta(\pi_f)=0$ for suitable m_X by Corollary 2.6. (Here we notice that the M-action on $C(f)$ is fixed). Since $Y = C(\Sigma^{-1}\pi_f)$, there is an m_Y on Y such that f and i_f are M-maps. The last statement on the M-action on $C(f)$ is proved by a similar discussion to the proof of Theorem 4.5.

§5. Associator

In this section, we always consider the smash product $M \wedge X$ to be the M-*module spectrum with the M-action $m_M \wedge 1_X$ even if X is also an M-module spectrum.*

LEMMA 5.1. *For any M-module spectrum (X, m_X, n_X), there hold $\theta(i \wedge 1_X) = n_X$ and $\theta(\pi \wedge 1_X) = -m_X$.*

PROOF. This is immediately obtained from $m_M(1_M \wedge i) = 1_M$ and $(1_M \wedge \pi)n_M = -1_M$.

The following result is originally proved by H. Toda [10; Prop. 2.1] under the assumption $[X, X]_1 = 0$.

THEOREM 5.2. *For each M-action m_X on X, there exists uniquely an element $a(m_X) \in [X, X]_2$ such that*

$$
(5.1) \qquad \begin{aligned} m_X(1_M \wedge m_X) &= m_X(m_M \wedge 1_X) + a(m_X)(\pi \wedge \pi \wedge 1_X), \\ (1_M \wedge n_X)n_X &= -(n_M \wedge 1_X)n_X + (i \wedge i \wedge 1_X)a(m_X). \end{aligned}
$$

PROOF. Operating θ to (1.2) and using Theorem 2.3, we have $\theta(m_X)n_X = m_X\theta(n_X)$. So we put

$$
(5.2) \qquad a(m_X) = \theta(m_X)n_X = m_X\theta(n_X).
$$

Since 1_X is clearly the M-map, $\theta(m_X(i \wedge 1_X)) = 0$ and $\theta((\pi \wedge 1_X)n_X) = 0$. So we have

$$
(5.3) \qquad \theta(m_X) = a(m_X)(\pi \wedge 1_X), \quad \theta(n_X) = (i \wedge 1_X)a(m_X),
$$

by the above lemma and (1.3). By the definition of $\theta(m_X)$ and (1.3) for $X = M$, we have

$$
\begin{aligned}
a(m_X)(\pi \wedge \pi \wedge 1_X) &= \theta(m_X)(\pi \wedge 1_M \wedge 1_X) \\
&= m_X(1_M \wedge m_X)(n_M \wedge 1_X)(\pi \wedge 1_M \wedge 1_X) \\
&= m_X(1_M \wedge m_X) - m_X(1_M \wedge m_X)(i \wedge 1_M \wedge 1_X)(m_M \wedge 1_X) \\
&= m_X(1_M \wedge m_X) - m_X(i \wedge 1_X)m_X(m_M \wedge 1_X) \\
&= m_X(1_M \wedge m_X) - m_X(m_M \wedge 1_X).
\end{aligned}
$$

Hence the first equality of (5.1) is obtained, and the second one is similarly obtained. Since $\pi \wedge \pi \wedge 1_X$ has a right inverse $(n_M \wedge 1_X)n_X$, $a(m_X)$ is unique.

DEFINITION 5.3. An M-module spectrum (X, m_X) (or an M-action m_X on X) is called *associative* if the equality $m_X(m_M \wedge 1_X) = m_X(1_M \wedge m_X)$ holds. The element $a(m_X)$ in the above theorem is called an *associator* of m_X.

Then (5.1) and (5.3) imply the following

PROPOSITION 5.4. *The following five statements are equivalent to each other.*

(i) m_X *is associative.*

(ii) m_X *is an M-map.*

(iii) n_X *is associative, i.e.,* $(n_M \wedge 1_X)n_X = -(1_M \wedge n_X)n_X$.

(iv) n_X *is an M-map.*

(v) $a(m_X) = 0$.

For the wedge sum and the smash product of (1.5–1.6), the following are easily verified.

$$a(m_{X \vee Y}) = a(m_X) \vee a(m_Y). \tag{5.4}$$

$$a(m_X \wedge 1_Y) = a(m_X) \wedge 1_Y, \tag{5.5}$$

and a similar formula holds for $(1_Y \wedge m_X)(T \wedge 1_X)$.

The following is a restatement of Theorem 3.3.

(5.6) $a(m_M) = 0$ *if* $q \not\equiv \pm 3$ mod 9, and $a(m_M) = \pm i\alpha_1(3)\pi \neq 0$ *if* $q \equiv \pm 3$ mod 9.

For the M_q-action $m_X(q)$ of (3.5) defined from an M_r-action m_X, $r|q$, we have

$$a(m_X(q)) = (q/r)^2 a(m_X), \tag{5.7}$$

by (5.2), Proposition 3.5 and (3.4). As a corollary we see that *any* $m_X(q)$ *is associative if* $r^3|q^2$.

By Proposition 3.7 and (5.6–5.7), we have immediately

PROPOSITION 5.5. *Let G be a finite* Z_q*-module. Then, in the case of* $q \not\equiv \pm 3$ mod 9, *M(G) is always an associative* M_q*-module spectrum, but in the case of* $q \equiv \pm 3$ mod 9, *M(G) is associative if and only if G does not contain* Z_3 *as a direct summand.*

THEOREM 5.6. *Let* m_X *and* m'_X *be M-actions on X, and write simply* $\theta = \theta_{m_X, m_X}$ *and* $\theta' = \theta_{m'_X, m'_X}$. *Then*

$$\begin{aligned} a(m_X) &= a(m'_X) - \theta'(d(m_X, m'_X)) + d(m_X, m'_X)^2 \\ &= a(m'_X) - \theta(d(m_X, m'_X)) - d(m_X, m'_X)^2. \end{aligned}$$

PROOF. Put $d=d(m_X, m'_X)$. Then $m_X=m'_X+d(\pi\wedge 1_X)$ and $n_X=n'_X-(i\wedge 1_X)d$ by (1.7) and (1.7)$'$, and $\theta(m_X)=\theta'(m_X)+dm_X$ by Theorem 2.2. So $\theta(m_X)n_X=\theta'(m_X)n_X$ by (1.2). Hence $a(m_X)=\theta(m_X)n_X=\theta'(m'_X+d(\pi\wedge 1_X))(n'_X-(i\wedge 1_X)d)=\theta'(m'_X)n'_X-\theta'(d)(\pi\wedge 1_X)n'_X-dm'_Xn'_X-\theta'(m'_X)(i\wedge 1_X)d+\theta'(d)(\pi\wedge 1_X)(i\wedge 1_X)d+dm'_X(i\wedge 1_X)d=a(m'_X)-\theta'(d)+d^2$ as desired. Since $d(m'_X, m_X)=-d$, the second formula is obtained by interchanging m_X with m'_X.

COROLLARY 5.7. *Let (X, m_X) be an associative M-module spectrum, and write simply $\theta=\theta_{m_X,m_X}$. Then the set of associative M-actions on X corresponds in a one-to-one onto fashion to the subset $\{f|\theta(f)+f^2=0\}$ of $[X, X]_1$.*

The following theorem is just the result of H. Toda [10; Th. 6.1, (i)].

THEOREM 5.8. *Let X and Y be M-module spectra. Then, for any $f\in[X, Y]_k$,*

$$\theta(\theta(f)) = fa(m_X)-a(m_Y)f.$$

In particular, θ is a differential on $[X, Y]_*$: $\theta\theta=0$, if X and Y are associative.

PROOF. Together with (5.1), easy calculations lead to the theorem. The details are the same as H. Toda's [10; p. 238].

COROLLARY 5.9. *M-maps commute with associators, i.e., for any $f\in[X, Y]^M_*$, $fa(m_X)=a(m_Y)f$.*

THEOREM 5.10. *Let (X, m_X) be any M_q-module spectrum, and denote simply θ_{m_X,m_X} by θ. Then, in the case of $q\not\equiv\pm 3$ mod 9, $\theta(a(m_X))=0$, but in the case of $q\equiv\pm 3$ mod 9, $\theta(a(m_X))=\mp\alpha_1(3)\wedge 1_X$.*

PROOF. By (5.3), $\theta(m_X)\theta(n_X)=0$, so we have

$$\begin{aligned}
\theta(a(m_X)) &= \theta(\theta(m_X)n_X) && \text{(by (5.2))}\\
&= -\theta^2(m_X)n_X && \text{(by Theorem 2.3)}\\
&= -m_Xa(m_{M\wedge X})n_X+a(m_X)m_Xn_X && \text{(by Theorem 5.8)}\\
&= -m_Xa(m_{M\wedge X})n_X && \text{(by (1.2))}\,.
\end{aligned}$$

Since $a(m_{M\wedge X})=a(m_M\wedge 1_X)=a(m_M)\wedge 1_X$ by (5.5), it follows from (5.6) that $m_Xa(m_{M\wedge X})n_X=0$ for $q\not\equiv\pm 3$ mod 9 and $m_Xa(m_{M\wedge X})n_X=\pm\alpha_1(3)\wedge 1_X$ for $q\equiv\pm 3$ mod 9.

§6. (Non-)associativity

LEMMA 6.1. *Let p be an odd prime and $\alpha_1(p)$ be a generator of the p-*

component of G_{2p-3}. Then, for any finite CW-spectrum X, $\alpha_1(p)\wedge 1_X\neq 0$ if and only if $\tilde{H}_*(X; Z_p)\neq 0$.

PROOF. Denote simply $\alpha_1(p)$ by α. If $\tilde{H}_*(X; Z_p)=0$, the order of 1_X is finite and relatively prime to p by [8; Th. 1.5]. Since α is of order p, $\alpha\wedge 1_X$ is trivial.

Next assume that $\alpha\wedge 1_X=0$. Then there is a left inverse $m\colon C(\alpha)\wedge X\to X$ of $i_\alpha\wedge 1_X$, where $i_\alpha\colon S\to C(\alpha)$ is the inclusion. Let $u\in H^0(C(\alpha); Z_p)$ be the class of the bottom sphere. It is well known that $P^1u\neq 0$ and this generates $H^{2p-2}(C(\alpha); Z_p)$, where P^n denotes the reduced power operation for the prime p. Take l such that $H^{l-2p+2}(X; Z_p)=0$. Then $m^*\colon H^l(X; Z_p)\to H^l(C(\alpha)\wedge X; Z_p)$ is isomorphic and $(i_\alpha\wedge 1_X)^*$ is its inverse. So $m^*(x)=u\otimes x$ for $x\in H^l(X; Z_p)$. Then $m^*(P^nx)=u\otimes P^nx+P^1u\otimes P^{n-1}x$, while there is an n such that $P^nx=0$. Hence $x=0$ and $H^l(X; Z_p)=0$. Thus $\tilde{H}^*(X; Z_p)=0$ and $\tilde{H}_*(X; Z_p)=0$ as desired.

REMARK 6.2. By using the squaring operation Sq^n, Sq^{2n}, Sq^{4n} or Sq^{8n} instead of P^n, we also obtain the following mod 2 version of the above lemma.

Let X be a non-trivial finite CW-spectrum, and denote the generators of the 2-components of G_1, G_3 and G_7 by η, ν and σ, respectively (these are odd multiples of the Hopf classes). Then

(1) $2\cdot 1_X\neq 0$, *i.e., there is no non-trivial finite M_2-module spectrum*;

(2) $\tilde{H}_*(X; Z_2)\neq 0 \Longleftrightarrow \eta\wedge 1_X\neq 0 \Longleftrightarrow \nu\wedge 1_X\neq 0 \Longleftrightarrow \sigma\wedge 1_X\neq 0.$

REMARK. In the above lemma and remark, the finiteness of X is essential. In fact, the Brown-Peterson spectrum BP at p gives a counterexample for Lemma 6.1 and Remark 6.2 (2), and the spectrum $M_2\wedge BP$ gives a counterexample for Remark 6.2 (1).

THEOREM 6.3. *Assume that $q\equiv\pm 3$ mod 9. Let X be an M_q-module spectrum such that the order of 1_X is a multiple of 3. Then every M_q-action on X is not associative.*

PROOF. From Lemma 1.4 together with the assumption on 1_X, $\tilde{H}_*(X; Z_3)\neq 0$. Hence $\alpha_1(3)\wedge 1_X\neq 0$ by Lemma 6.1. So $\theta(a(m_X))\neq 0$ by Theorem 5.10 and $a(m_X)\neq 0$. Thus m_X is not associative by Proposition 5.4.

LEMMA 6.4. *Assume that a finite Z_q-module G satisfies the following condition.*

(6.1) *For any prime p, the p-component of G is free over the p-component of Z_q.*

Then, for any M-module spectrum X,

$$[M(G), X]_k^M \subset \theta[M(G), X]_{k-1},$$

$$[X, M(G)]_k^M \subset \theta[X, M(G)]_{k-1}.$$

PROOF. By Proposition 3.7, it suffices to show the lemma for the case $G=Z_r$ for r such that $q=rs$ and $(r, s)=1$.

Consider the element $\delta \in [M_r, M_r]_{-1}$ and put $\delta'=x\delta$ for an integer x with $xs\equiv -1 \bmod r$. Then $\theta(\delta)=-s1_{M_r}$ by (3.1) and Proposition 3.5, so $\theta(\delta')=1_{M_r}$. For any $f\in[M_r, X]_k^M$ and $g\in[X, M_r]_k^M$, $\theta(f\delta')=f$ and $\theta(\delta' g)=(-1)^k g$ by Theorem 2.3, and hence the lemma for $G=Z_r$ is proved.

In the next section, we shall generalize the above result (Proposition 7.2, Theorems 7.5 and 7.7).

LEMMA 6.5. *Let G be a finite Z_q-module satisfying the condition* (6.1), *and G' be any finite Z_q-module. Let (X, m_X) be associative and $f\colon \Sigma^k M(G)\to X$ be an M_q-map. In the case of $q\equiv \pm 3$* mod 9, *assume further that $H_*(C(f))$ has no* 3-*torsion and G' does not contain Z_3 as a direct summand. Then, for any M_q-map $g\colon \Sigma^l M(G')\to C(f)$ with respect to some admissible M_q-action on $C(f)$ of Construction* 4.2, *there exists an admissible and associative M_q-action on $C(f)$ such that g is also an M_q-map.*

PROOF. Let m_C be an admissible M-action on $C(f)$ of Construction 4.2. By Theorem 5.10, Lemma 6.1 and Proposition 2.5, the associator $a(m_C)$ is an M-map. By Corollary 5.9, $i_f^* a(m_C)=i_f a(m_X)=0$, so $a(m_C)=\pi_f^* h_0$ for some $h_0\in[M(G), C(f)]_{k+3}^M$ by Theorem 4.5. Since $[M(G), M(G)]_1=0$, we have $[M(G), M(G)]_2^M=0$ by Lemma 6.4, and hence $h_0=i_{f*}h$ for some $h\in[M(G), X]_{k+3}^M$ by Theorem 4.5. Again by Lemma 6.4, $h=\theta(h')$ for some $h'\in[M(G), X]_{k+2}$, and so $\pm\theta(i_f h'\pi_f g)=i_f h\pi_f g=a(m_C)g=ga(m_{MG'})=0$ by Theorem 2.3, Corollary 5.9 and Proposition 5.5. Hence $i_f h'\pi_f g$ is an M-map. Then Theorem 4.5 implies that $i_f h'\pi_f g=i_f h''\pi_f g$ for some $h''\in[M(G), X]_{k+2}^M$. Put $d=(-1)^{k+1}i_f(h'-h'')\pi_f\in[C(f), C(f)]_1$. Then $\theta(d)=i_f\theta(h')\pi_f=a(m_C)$ and $d^2=0$. Define another M-action m_C' on $C(f)$ by $m_C'=m_C+d(\pi\wedge 1_{C(f)})$. Then $a(m_C')=a(m_C)-\theta(d)+d^2=0$ by Theorem 5.6 and m_C' is associative. By the relations $di_f=0$, $\pi_f d=0$ and $dg=0$ together with Theorem 2.2, i_f, π_f and g are again M-maps with respect to m_C'. Thus m_C' is the desired M-action on $C(f)$.

Now we are ready to prove the following

THEOREM 6.6. *Let X be an M_q-module spectrum and in the case of $q\equiv\pm 3$* mod 9 *assume that the order of 1_X is relatively prime to* 3, *or equivalently $H_*(X)$ has no* 3-*torsion. If X satisfies the following two conditions, then X admits an associative M_q-action.*

(i) *$\# H_i(X)$ is relatively prime to $\# H_{i-1}(X)$ and to $\# H_{i-2}(X)$, where $\# G$ denotes the order of a finite group G.*

(ii) *The group* $H_i(X)$ *satisfies the condition* (6.1).

PROOF. Denote simply $H_i(X)$ by H_i and take integers r and $s>0$ such that $\tilde{H}_i=0$ for $i<r$ and for $i>r+s$. By a dual consideration of the Postnikov system, there is a filtration $\{X_k\}_{r\leqq k\leqq r+s}$ of subspectra of X together with maps f_k: $\Sigma^{k-1}M(H_k)\to X_{k-1}$ $(r+1\leqq k\leqq r+s)$ such that $X_r=\Sigma^r M(H_r)$, $X_{r+s}=X$, $X_k=C(f_k)$ and the inclusion $X_k\subset X$ induces isomorphisms $H_i(X_k)\approx H_i(X)$ for $i\leqq k$. Since $\dim X_k=k+1$, $[X_{k-3}, M(H_k)]_{-k+1}=0$. By the condition (i), $[\Sigma^{-1}M(H_{k-2})\vee M(H_{k-1}), M(H_k)]=0$. So we have

$$(*)\qquad [X_{k-1}, M(H_k)]_{-k+1}=0 \qquad \text{for} \quad r+1\leqq k\leqq r+s.$$

Let m_X be an M-action on X. By applying Lemma 4.7 to $(*)$, we can inductively construct M-actions m_k on $X_k(m_{r+s}=m_X)$ such that f_k is an M-map and m_k is the M-action on $C(f_k)$ given by Construction 4.2. We shall prove the following statements by the induction on k.

$(**)_k$ *There is an associative and admissible M-action* m'_k *on* $X_k=C(f_k)$ *such that* f_{k+1} *is also an M-map with respect to* m'_k.

Obviously, $(**)_r$ is valid by Propositions 3.7 and 5.5. Assume that $(**)_{k-1}$ is valid. Then, by the condition (ii), all the assumptions of Lemma 6.5 are satisfied for the case $G=H_k$, $(X, m_X)=(X_{k-1}, m'_{k-1})$, $f=f_k$, $G'=H_{k+1}$ and $g=f_{k+1}$. So we obtain $(**)_k$. The theorem is a restatement of $(**)_{r+s}$.

REMARK 6.7. In the above dual Postnikov system $\{X_k\}$ of X, each X_k also admits an associative M-action by $(**)_k$. But it may have no M-action if the condition (i) does not satisfied. Let p be an odd prime and consider the case $q=p$, $M=M_p$. The group $[M, M]^M_{2p-2}$ is Z_p and its generator α satisfies $\pi\alpha i=\alpha_1(p)$ ([11], [10; §§5–6] and [4; Th. 5.1]). Let f: $N=\Sigma^{2p-3}M\vee\Sigma^{2p-2}M\to M$ be the map such that $fi_1=\delta\alpha$ and $fi_2=\alpha$, i.e., $f=\delta\alpha p_1+\alpha p_2$, where i's are the inclusions and p's are the projections. Put $X=C(f)$. For this X, the condition (i) does not hold and X is the $(4p-5)$-skeleton of the Eilenberg-MacLane spectrum $K(Z_p)$. So X is an M-module spectrum, but $X_{2p-2}=M\cup C\Sigma^{2p-3}M=C(\delta\alpha)$ is not by Lemma 4.7. In this case, f is not an M-map with respect to the canonical M-action $m_N=m_M\vee m_M$ on N, while it is an M-map by a twisted one $m_N+i_2p_1(\pi\wedge 1_N)$.

In a similar manner to the above, we can construct an M-module spectrum having no associative M-action.

EXAMPLE 6.8. Let p be a prime$\geqq 5$, $M=M_p$ and put $N=M\vee\Sigma^2M$. Denote by $i_1\in[M, N]_0$, $i_2\in[M, N]_2$ the inclusions and by $p_1\in[N, M]_0$, $p_2\in[N, M]_{-2}$ the projections. Define $f\in[N, N]_{2p-2}$ by

$$f=i_1\alpha p_1-i_2\delta\alpha\delta p_1+i_2\alpha p_2\colon \Sigma^{2p-2}N\longrightarrow N,$$

where $\alpha \in [M, M]^M_{2p-2}$ is the same as in the above remark. Denote by $m_N(0)$ the canonical M-action $m_M \vee \Sigma^2 m_M$ on N. Since $[N, N]_1 = Z_p$, generated by $i_2\delta p_1$, by Lemma 3.1, we have

(1) *there are just p distinct M-actions on N, which are written as* $m_N(x) = m_N(0) + x i_2 \delta p_1(\pi \wedge 1_N)$, $x \in Z_p$.

Since i_1, i_2, p_1 and p_2 are the M-maps with respect to $m_N(0)$, we see from Theorem 2.2 that

(2) $f\colon (\Sigma^{2p-2}N, m_N(x)) \to (N, m_N(y))$ *is an M-map if and only if* $x = y = -1$.

Since $m_N(0)$ is associative by (5.4) and (5.6), and since $[N, N]_2 = Z_p$, generated by $i_2 p_1$, it follows from Theorem 5.6 that

(3) $a(m_N(x)) = x i_2 p_1$, *and hence only* $m_N(0)$ *is associative.*

Let X be the mapping cone of f. Then easy calculations show that $[X, X]_1 = 0$ and $[X, X]_2 = Z_p$ with the generator h satisfying $i_f^*(h) = i_{f*}(i_2 p_1)$. So

(4) *X has a unique M-action* m_X.

By Corollary 5.9 and (3), $a(m_X) i_f = i_f a(m_N(-1)) = -i_f i_2 p_1$. Hence

(5) $a(m_X) = -h \neq 0$, *i.e., X has no associative M-action.*

It is clear that $H_i(X) = Z_p$ for $i = 0, 2, 2p-1, 2p+1$, so

(6) *X does not satisfy the condition* (i) *in Theorem* 6.6.

Let V be the mapping cone of α. This is just the spectrum $V(1)$ [10] and has a unique associative M-action. It is easy to see that X is the mapping cone of some map $g\colon \Sigma^{-1}V \to \Sigma^2 V$. By H. Toda's result [10; Th. 3.6] on $[V, V]_*$, g is an M-map (in fact, $g = \pm \alpha' \delta_0$). Thus

(7) *there is an M-map* $X \to Y$ *such that all the M-actions on X and on Y are associative but its mapping cone has no associative one.*

§7. Hoffman's decomposition

P. Hoffman [2; Th. A] obtained the direct sum decompositions

$$\begin{aligned} [M, M]_k &= [M, M]^M_k \oplus \delta_*[M, M]^M_{k+1} \\ &= [M, M]^M_k \oplus \delta^*[M, M]^M_{k+1} \end{aligned}$$

and the split exact sequence

$$[M, M]_{k-1} \xrightarrow{\theta} [M, M]_k \xrightarrow{\theta} [M, M]_{k+1}$$

for $M=M_q$, $q\not\equiv \pm 3 \bmod 9$. We shall generalize these results.

We first consider the following condition for an M-module spectrum (X, m_X).

CONDITION 7.1. *There exists an element $\delta(m_X)\in [X, X]_{-1}$ such that*

(i) $\delta(m_X)\delta(m_X)=0$,

(ii) $\theta(\delta(m_X))=-1_X$

and

(iii) $\delta(m_X)a(m_X)=0$.

By Theorems 5.8 and 5.10, we see easily that the condition (iii) can be replaced by one of the following.

(iii)′ $a(m_X)\delta(m_X)=0$.

(iii)″ $a(m_X)=0$ *if* $q\not\equiv \pm 3 \bmod 9$, and $a(m_X)=\pm(\alpha_1(3)\wedge 1_X)\delta(m_X)=\mp\delta(m_X)(\alpha_1(3)\wedge 1_X)$ *if* $q\equiv \pm 3 \bmod 9$.

We shall give several examples of (X, m_X) satisfying Condition 7.1. First, it is clear by (3.1) and (5.6) that

(7.1) *M satisfies Condition 7.1 by putting $\delta(m_M)=\delta$.*

PROPOSITION 7.2. *Let G be a finite Z_q-module. Then the Moore spectrum $M(G)$ satisfies Condition 7.1 if and only if G satisfies (6.1).*

PROPOSITION 7.3. *Let $f\colon \Sigma^k M\to M$ be an M-map such that*

$$f\delta=(-1)^k\delta f. \tag{7.2}$$

Put $X=C(f)$. Then there exists uniquely an element $D\in[X, X]_{-1}$ such that $Di_f=i_f$, $\pi_f D=(-1)^k\delta\pi_f$, $D^2=0$ and $\theta(D)=-1_X$ for any admissible M-action on X. Furthermore there exists an admissible M-action m_X on X such that $Da(m_X)=0$. Hence (X, m_X) satisfies Condition 7.1 for $\delta(m_X)=D$.

REMARK. According to the discussion in [4; § 3] and a similar one for the case $q\equiv \pm 3 \bmod 9$ in the next section, we see that $f\in[M, M]_k^M$ *satisfies* (7.2) *if and only if $f=h\wedge 1_M+g$ for $h\in G_k$ and $g\in[M, M]_k^M$ such that $\pi g i\in G_{k-1}$ is divisible by q.* By [2; Th. A], for any $g'\in[M, M]_*^M$ of even degree, the element $g=(g')^q$ satisfies such a condition.

LEMMA 7.4. *Let r and s be divisors of q, and denote by $d=(r, s)$ and $l=\{r, s\}$ the greatest common divisor and the least common multiple of r and s, respectively. Then*

$$[M_r, M_s]_{-1} = Z_d, \quad \textit{generated by} \quad \delta_{r,s} = i\pi\colon M_r \longrightarrow \Sigma S \longrightarrow \Sigma M_s,$$

$$[M_r, M_s]_0 = Z_d, \quad \textit{generated by} \quad \varepsilon_{r,s} = \lambda\rho\colon M_r \longrightarrow M_d \longrightarrow M_s,$$

and the relation $\theta(\delta_{r,s})=(q/l)\varepsilon_{r,s}$ holds as M_q-module spectra.

PROOF. Except for the statement on θ, everything is clear, (cf. Lemma 3.1). We see easily that $\theta(\delta_{r,s})=\rho'\lambda'\colon M_r\to M_q\to M_s$, and so $\theta(\delta_{r,s})=(q/r)/(s/d)\varepsilon_{r,s}=(q/l)\varepsilon_{r,s}$.

PROOF OF PROPOSITION 7.2. We have proved (6.1)$\Rightarrow$7.1 in the proof of Lemma 6.4. Assume that $M(G)$ satisfies Condition 7.1, and put $G=Z_{r_1}\oplus\cdots\oplus Z_{r_l}$, so $M(G)=M_{r_1}\vee\cdots\vee M_{r_l}$ for $r_j|q$. Let $i_j\colon M_{r_j}\to M(G)$ be the inclusion and $p_j\colon M(G)\to M_{r_j}$ be the projection. By the above lemma, $[MG, MG]_{-1}=\sum_{j,k}A_{j,k}$ and $[MG, MG]_0=\sum_{j,k}B_{j,k}$, where $A_{j,k}$ and $B_{j,k}$ are the cyclic groups generated by $i_k\delta_{r_j,r_k}p_j$ and $i_k\varepsilon_{r_j,r_k}p_j$ of order (r_j, r_k). Then, again by Lemma 7.4, $1_{MG}=\sum_k i_k\varepsilon_{r_k,r_k}p_k\in\sum_{j,k}\theta(A_{j,k})$ implies the congruences $qx_k/r_k\equiv 1 \bmod r_k$ for some x_k. So r_k and q/r_k are relatively prime. This means that G satisfies (6.1).

PROOF OF PROPOSITION 7.3. By (7.2), there is an element $D_0\in[X, X]_{-1}$ such that $D_0 i_f=i_f\delta$ and $\pi_f D_0=(-1)^k\delta\pi_f$. From the exact sequences derived from the cofibering for $X=C(f)$ together with Lemma 3.1, we see the following results on $[X, X]_*$:

$$[X, X]_l = i_{f*}\pi_f^*[M, M]_{k+l+1} \qquad \text{for} \quad l = -3,\ -2,\ 1,$$

$$[X, X]_{-1} = \{D_0\} \oplus i_{f*}\pi_f^*[M, M]_k,$$

$$[X, X]_0 = \{1_X\} \oplus i_{f*}\pi_f^*[M, M]_{k+1},$$

and the kernel of $i_{f*}\pi_f^*\colon [M, M]_{k+l+1}\to[X, X]_l$ is 0, $\{f\delta\}$, $\{f\}$ and 0 for $l=-3$, -2, -1 and 0, respectively, where $\{g\}$ means the subgroup generated by g.

Put $D_0^2=i_f g\pi_f$, $g\in[M, M]_{k-1}$. Then $i_f\delta g\pi_f=D_0D_0^2=D_0^2D_0=(-1)^k i_f g\delta\pi_f$, and hence $\delta g=(-1)^k g\delta$. Applying θ to this, we have $\theta(g)\delta+(-1)^k\delta\theta(g)=-2g$. So we define

$$D = D_0+((-1)^k/2)i_f\theta(g)\pi_f.$$

Then $Di_f=D_0i_f=i_f\delta$, $\pi_f D=\pi_f D_0=(-1)^k\delta\pi_f$ and $D^2=D_0^2+((-1)^k/2)i_f((-1)^k\theta(g)\delta+\delta\theta(g))\pi_f=D_0^2-i_f g\pi_f=0$. We can put $\theta(D)=x1_X+i_f g'\pi_f$, $x\in Z_q$, $g'\in[M, M]_{k+1}$. Then $xi_f=\theta(D)i_f=\theta(Di_f)=-i_f$, and $x=-1$. Since $i_f(\delta g'-$

$(-1)^k g'\delta)\pi_f = -\theta(D)D + D\theta(D) = \theta(D^2) = 0$, we have $\delta g' \equiv (-1)^k g'\delta \bmod \{f\}$. Applying θ to this, we have $-2g' = \theta(g')\delta + (-1)^k \delta\theta(g')$, so $\theta(D) + 1_X = -((-1)^k/2)(i_f\theta(g')\pi_f D + D i_f \theta(g')\pi_f) = -(1/2)(\theta^2(D)D + D\theta^2(D)) = 0$, because $(-1)^{k+1} i_f \theta(g')\pi_f = \theta^2(D) = Da(m_X) - a(m_X)D$. Thus D satisfies the desired relations.

Let D' also satisfy the above relations. Then $D' = D + i_f h \pi_f$ for some $h \in [M, M]_k$. Applying θ to this, $i_f \theta(h)\pi_f = 0$, so $\theta(h) = 0$. From the relation $(D + i_f h\pi_f)^2 = 0$, we have $(-1)^k h\delta + \delta h \equiv 0 \bmod \{f\delta\}$ and so $2h \equiv 0 \bmod \{f\}$. Hence $i_f h \pi_f = 0$ and $D' = D$. Thus D is unique.

Take an admissible M-action m'_X and put $Da(m'_X) = i_f h' \pi_f$ for some $h' \in [M, M]_{k+2}$. Then $0 = D^2 a(m'_X) = i_f \delta h' \pi_f$ and $\delta h' = 0$, so $h' = (-1)^k \delta\theta(h')$. Hence $Da(m'_X) = (-1)^k D i_f \theta(h')\pi_f = -D\theta(i_f h'\pi_f)$. Put $d = -i_f h' \pi_f$ and $m_X = m'_X + d(\pi \wedge 1_X)$. Then m_X is also admissible and satisfies $Da(m_X) = 0$.

We now generalize P. Hoffman's results at the beginning of this section.

THEOREM 7.5. *Let (X, m_X) be an M-module spectrum satisfying Condition 7.1. Then, for any associative M-module spectrum (Y, m_Y), the sequences*

$$[X, Y]_{k-1} \xrightarrow{\theta} [X, Y]_k \xrightarrow{\theta} [X, Y]_{k+1},$$

$$[Y, X]_{k-1} \xrightarrow{\theta} [Y, X]_k \xrightarrow{\theta} [Y, X]_{k+1}$$

are split exact and there are the direct sum decompositions:

$$[X, Y]_k = [X, Y]_k^M \oplus \delta(m_X)^*[X, Y]_{k+1}^M,$$

$$[Y, X]_k = [Y, X]_k^M \oplus \delta(m_X)_*[Y, X]_{k+1}^M.$$

PROOF. Since Y is associative, we see by Theorem 6.3 that $q \not\equiv \pm 3 \bmod 9$ or that the order of 1_Y is relatively prime to 3. In the latter case, $\alpha_1(3) \wedge 1_Y = 0$ by Lemma 6.1. Hence $\operatorname{Im}\theta \subset \operatorname{Ker}\theta$ by Theorem 5.8 and the condition (iii)$''$. In the same way as Lemma 6.4, $\operatorname{Ker}\theta \subset \operatorname{Im}\theta$ and the above sequences are exact. The desired splittings are given by $\delta(m_X)^*$ and $\delta(m_X)_*$, and we have the direct sum decompositions.

We next consider a non-associative version of the above theorem.

DEFINITION 7.6. *Let (X, m_X) satisfy Condition 7.1, and (Y, m_Y) be arbitrary M-module spectrum. Define*

$$\bar{\theta}\colon [X, Y]_k \longrightarrow [X, Y]_{k+1},$$

$$\bar{\theta}\colon [Y, X]_k \longrightarrow [Y, X]_{k+1}$$

by the formulas

$$\bar{\theta}(f) = \theta(f) - a(m_Y) f\delta(m_X),$$

$$\bar{\theta}(f) = \theta(f) + (-1)^k \delta(m_X) f a(m_Y),$$

respectively.

Clearly $\bar{\theta}=\theta$ if Y is associative. If Y satisfies Condition 7.1, both $\bar{\theta}$ are coincident. For, $a(m_Y)f\delta(m_X)=\pm(\alpha_1(3)\wedge 1_Y)\delta(m_Y)f\delta(m_X)=\pm(-1)^k\delta(m_Y) f\delta(m_X)(\alpha_1(3)\wedge 1_X)=-(-1)^k\delta(m_Y)fa(m_X)$ if $q\equiv\pm 3 \bmod 9$.

THEOREM 7.7. *Let (X, m_X) satisfy Condition 7.1. Then, for any M-module spectrum (Y, m_Y),*

$$\operatorname{Ker}\theta \subset \operatorname{Ker}\bar{\theta} = \operatorname{Im}\bar{\theta} \subset \operatorname{Im}\theta \quad in \quad [X, Y]_* \quad and\ in \quad [Y, X]_*.$$

Hence Theorem 7.5 with θ replaced by $\bar{\theta}$ holds even if (Y, m_Y) is not associative.

PROOF. Since $\bar{\theta}(f)=-\theta(\theta(f)\delta(m_X))$ for $f\in[X, Y]_k$ and $\bar{\theta}(f)=(-1)^k\theta(\delta(m_X)\theta(f))$ for $f\in[Y, X]_k$, the theorem is easily derived from the following algebraic lemma with $\psi=\delta(m_X)_*$ or $\delta(m_X)^*$ up to sign.

LEMMA. *Let θ and ψ be endomorphisms (of degree $+1$ and -1) of a (graded) abelian group A such that $\theta\psi+\psi\theta=1_A$ and $\psi^2=0$. Put $\bar{\theta}=\theta\psi\theta$. Then $\bar{\theta}\psi+\psi\bar{\theta}=1_A$, $\bar{\theta}^2=0$ and $\operatorname{Ker}\theta\subset\operatorname{Ker}\bar{\theta}=\operatorname{Im}\bar{\theta}\subset\operatorname{Im}\theta$.*

Concerning (2.3), we see that $\operatorname{Ker}\theta$ acts on $\operatorname{Ker}\bar{\theta}$ from the both sides:

(7.3) *If $\theta(f)=0$ and $\bar{\theta}(g)=0$, then $\bar{\theta}(fg)=0$. If $\bar{\theta}(f)=0$ and $\theta(g)=0$, then $\bar{\theta}(fg)=0$.*

EXAMPLE. Let $\beta=\beta_{(1)}\in[M_3, M_3]_{11}$ be the element defined by N. Yamamoto [11] and H. Toda [10; §6]. Then $\theta(\beta)=\delta\alpha\delta\beta\delta=a(m_M)\beta\delta(m_M)\neq 0$ by [10; Th. 6.4], and so $\bar{\theta}(\beta)=0$. $[M_3, M_3]_{11}$ is generated by the elements β, $\alpha^2\delta\alpha$ and $\alpha^3\delta$, and $\bar{\theta}[M_3, M_3]_{11}=0$. So $\delta\alpha\delta\beta\delta$ lies in $\operatorname{Im}\theta$ but not in $\operatorname{Im}\bar{\theta}$. Thus we can not, in general, replace the mark $\subset$ in Theorem 7.7 by the equal mark $=$.

By [10; Th. 6.8], $\beta\beta=\delta\alpha\delta\beta\delta\beta\delta$ and its $\bar{\theta}$-image is $\alpha\delta\beta\delta\beta\delta-\delta\alpha\delta\beta\delta\beta\neq 0$. Thus $\operatorname{Ker}\bar{\theta}$ can not, in general, form a ring by the composition product.

§8. Remark on $[M_3, M_3]_*$

In [4] and [6; §8], we studied the ring structure of $[M_q, M_q]_*$ for q a power of an odd prime p, in connection with the p-component of the stable homotopy ring G_* of spheres. But only the case $q=3$ is exceptional, since M_3 is not associative and θ on $[M_3, M_3]_*$ is not a differential. For this case, similar discussions can be done by considering $\bar{\theta}$ instead of θ.

For any $\alpha \in G_k$ and $\beta \in G_{k-1} * Z_3 \subset G_{k-1}$, define $\langle\alpha\rangle \in [M_3, M_3]_k$ and $[\beta] \in [M_3, M_3]_k$ by

$$\langle\alpha\rangle = \alpha \wedge 1_M \quad \text{and} \quad [\beta] = (-1)^{k-1}\theta(i\bar{\beta}),$$

where $\bar{\beta} \in [M_3, S]_{k-1}$ is an extension of β, i.e., $\bar{\beta}i = \beta$. Then we have

$$\langle\alpha\rangle i = i\alpha, \quad \pi\langle\alpha\rangle = (-1)^k \alpha\pi, \quad \theta(\langle\alpha\rangle) = 0;$$

$$\pi[\beta]i = \beta, \quad \bar{\theta}([\beta]) = 0,$$

since $\theta([\beta]) = (-1)^{k-1} i\bar{\beta}i\alpha_1(3)\pi = i\alpha_1(3)\beta\pi = a(m_M)[\beta]\delta$, (cf. [4; Lemmas 3.1–3.2]). Denote by $[G_{k-1} * Z_3]$ and $\langle G_k\rangle$ the subgroups generated by those elements $[\beta]$ and $\langle\alpha\rangle$, and also by $\bar{K}_k$ and K_k the subgroups $[M_3, M_3]_k \cap \operatorname{Ker}\bar{\theta}$ and $[M_3, M_3]_k^M = [M_3, M_3]_k \cap \operatorname{Ker}\theta$. Then the following direct sum decompositions are obtained:

$$[M_3, M_3]_k = \bar{K}_k \oplus \delta_* \bar{K}_{k+1} = \bar{K}_k \oplus \delta^* \bar{K}_{k+1}, \tag{8.1}$$

$$\bar{K}_k = \langle G_k\rangle \oplus [G_{k-1} * Z_3] \approx G_k \otimes Z_3 \oplus G_{k-1} * Z_3, \tag{8.2}$$

$$K_k = \langle G_k\rangle \oplus [H_{k-1}] \approx G_k \otimes Z_3 \oplus H_{k-1}, \tag{8.3}$$

where $H_{k-1} = \{\beta \in G_{k-1} * Z_3 \mid \alpha_1(3)\beta$ is divisible by 3$\}$ and $[H_{k-1}]$ is the subgroup generated by $[\beta]$ for $\beta \in H_{k-1}$. The decomposition of [4; Th. 3.5] is also obtained.

For the composition, the formulas (3.7–3.8) and Proposition 3.8 of [4] also hold in $[M_3, M_3]_*$, but we must correct Proposition 3.9 of [4] as follows:

Let $\xi \in G_{k-1}$ *and* $\eta \in G_{l-1}$ *be elements of order* 3 *such that* $\zeta = \langle\xi, 3, \eta\rangle$ *has trivial indeterminacy. Then* $[\xi][\eta] = [\zeta] - (-1)^l i\alpha_1(3)\pi[\xi]\delta[\eta]\delta$.

Concerning the formula (1.11) of [4], we obtain the following result: $\eta\xi - (-1)^{kl}\xi\eta = (-1)^{kl+l+1} i\alpha_1(3)\pi\xi\delta\eta\delta = (-1)^k i\alpha_1(3)\pi\eta\delta\xi\delta$ *for* $\xi \in \bar{K}_k$ *and* $\eta \in \bar{K}_l$, *and in particular* $\eta\xi = (-1)^{kl}\xi\eta$ *further if one of* ξ *and* η *lies in* K_*.

The ring structure of $[M_3, M_3]_*$ has been determined up to degree 31 by N. Yamamoto [11] and H. Toda [10; §6]. Applying the results on G_* [3; Th. B] to the decompositions (8.1–8.3), we can continue to compute $[M_3, M_3]_*$. The following result is proved similarly to the case $p \geqq 5$ [4; Th. 0.1], and we omit the proof.

THEOREM 8.1. *The ring* $[M_3, M_3]_*$ *is multiplicatively generated up to degree* 66 *by the following six elements*

$$\delta = i\pi \in [M_3, M_3]_{-1}, \quad \alpha = [\alpha_1(3)] \in K_4,$$

$$\beta_{(1)} = [\beta_1] \in \bar{K}_{11}, \qquad \beta_{(2)} = [\beta_2] \in \bar{K}_{27},$$

$$\varepsilon = [\varepsilon_1] \in K_{39}, \qquad \bar{\varphi} = \langle \varphi \rangle \in K_{45},$$

and a Z_3-basis for $[M_3, M_3]_*$ is given in the cited range by the following elements (a, $b=0$ *or* 1 *unless otherwise stated*):

$$\delta, \ 1_M; \ \alpha^r\delta^a, \ \alpha^{r-1}\delta\alpha\delta^a \quad \textit{for} \quad 1 \leqq r \leqq 16, \ \alpha^{16}\delta\alpha\delta;$$

$$\delta^a(\beta_{(1)}\delta)^r\beta_{(1)}\delta^b \quad \textit{for} \quad 0 \leqq r \leqq 4, \qquad \delta^a\alpha\delta(\beta_{(1)}\delta)^r\beta_{(1)}\delta^b \quad \textit{for} \quad r = 0, 1;$$

$$\delta^a(\beta_{(1)}\delta)^r\beta_{(2)}\delta^b \quad \textit{for} \quad 0 \leqq r \leqq 2, \qquad \delta^a\alpha\delta(\beta_{(1)}\delta)^r\beta_{(2)}\delta^b \quad \textit{for} \quad 0 \leqq r \leqq 2;$$

$$\delta^a(\beta_{(1)}\delta)^r\beta_{(1)}\beta_{(2)}\delta^b \quad \textit{for} \quad r = 0, 1;$$

$$\delta^a(\beta_{(1)}\delta)^r\beta_{(2)}\delta\beta_{(2)}\delta^b \quad \textit{for} \quad r = 0, 1, \qquad \delta^a\alpha(\delta\beta_{(1)})^r(\delta\beta_{(2)})^2\delta^b \quad \textit{for} \quad r = 0, 1;$$

$$\delta^a\varepsilon\delta^b, \quad \delta^a\varepsilon\alpha\delta^b, \quad \bar{\varphi}\delta^a.$$

The element $\bar{\varepsilon} = [\varepsilon'] \in \bar{K}_{38}$ is decomposable while it is not for the case $p \geqq 5$, and the element corresponding to $\beta_{(p+1)}$ for $p \geqq 5$ does not exist. We can also determine the multiplicative structure in the cited range, but the result is more complicated than the case $p \geqq 5$ and we omit the detail. For example, we obtain the following relations which are different from the case $p \geqq 5$.

$$\varepsilon\alpha^2 = \pm(\beta_{(1)}\delta\beta_{(1)}\beta_{(2)}\delta - (\delta\beta_{(1)})^2\beta_{(2)}),$$

$$\bar{\varphi}\alpha = \pm(\alpha(\delta\beta_{(1)})^2\delta\beta_{(2)}\delta - \delta\alpha(\delta\beta_{(1)})^2\delta\beta_{(2)} - (\delta\beta_{(1)})^5\delta),$$

$$\bar{\varphi}\beta_{(2)} = \pm((\beta_{(1)}\delta)^2(\beta_{(2)}\delta)^2 + (\delta\beta_{(1)})^2(\delta\beta_{(2)})^2).$$

The first relation is a restatement of [7; (5.1)], and the last two are induced by $\alpha\beta_{(2)}\bar{\varphi} = (\beta_{(1)}\delta)^2\beta_{(1)}\bar{\varphi} = \pm\alpha(\delta\beta_{(1)})^2(\delta\beta_{(2)})^2$. In $[M_3, M_3]_{69}$, there appears new indecomposable element $\lambda_{(1)}$ with $\theta(\lambda_{(1)}) = 0$. This is introduced in the proof of [3; Prop. 17.5], and for the case $p \geqq 5$ there is no element corresponding to $\lambda_{(1)}$.

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