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Note on KO-Rings of Lens Spaces Mod 2^r

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§1. Introduction

Let η be the canonical complex line bundle over the standard lens space mod p^r:

$$L^n(p^r) = S^{2n+1}/Z_{p^r} \qquad (p\text{: prime, } r \geqq 1;\ n \geqq 0).$$

Then, we have the stable classes

$$(1.1) \qquad \sigma = \eta - 1 \in \tilde{K}(L^n(p^r)), \quad r\sigma = r\eta - 2 \in \widetilde{KO}(L^n(p^r)),$$

where r is the real restriction. On the orders of the powers of these elements, the following results are proved in [1, Th. 1.1]:

(1.2) *$\sigma^i \in \tilde{K}(L^n(p^r))$ is of order $p^{r+[(n-i)/(p-1)]}$ for $1 \leqq i \leqq n$, and $\sigma^{n+1} = 0$.*

(1.3) *If p is an odd prime, then $(r\sigma)^i \in \widetilde{KO}(L^n(p^r))$ is of order $p^{r+[(n-2i)/(p-1)]}$ for $1 \leqq i \leqq [n/2]$, and $(r\sigma)^{[n/2]+1} = 0$.*

The purpose of this note is to prove the following theorem, by using the partial result of M. Yasuo [5, Prop. (3.5)] which shows the theorem under the assumption $n \not\equiv 1\,(4)$:

THEOREM 1.4. *In the reduced KO-group $\widetilde{KO}(L^n(2^r))$ $(r \geqq 2)$, the order of $(r\sigma)^i$ is equal to*

$$2^{r+n-2i+1} \ \textit{if } n \equiv 0\,(2), \quad 2^{r+n-2i} \ \textit{if } n \equiv 1\,(2), \quad \textit{for } 1 \leqq i \leqq [n/2];$$

$$1 \ \textit{if } n \not\equiv 1\,(4), \quad 2 \ \textit{if } n \equiv 1\,(4), \quad \textit{for } i = [n/2]+1;$$

and 1 *for $i \geqq [n/2]+2$.*

As an application of this theorem, we have the following corollary by the method of M. F. Atiyah using the γ-operation.

COROLLARY 1.5 (cf. [3, Th. C, Prop. 7.6]). *The $(2n+1)$-manifold $L^n(2^r)$ $(r \geqq 2)$ cannot be immersed in the Euclidean space R^{2n+2L} and cannot be imbedded in $R^{2n+2L+1}$, where*

$$L=\begin{cases}\max\left\{i \mid 1 \leqq i \leqq [n/2], \binom{n+i}{i} \not\equiv 0\ (2^{r+n-2i+1})\right\} & \text{if } n \equiv 0\ (2),\\ \max\left\{i \mid 1 \leqq i \leqq [n/2], \binom{n+i}{i} \not\equiv 0\ (2^{r+n-2i})\right\} & \text{if } n \equiv 1\ (2).\end{cases}$$

§2. Some relations in $\tilde{K}(L^n(2^r))$

In this section, we study some relations in the reduced K-group

$$\tilde{K}(L^n(2^r)) \qquad (r \geqq 2).$$

The element $\sigma \in \tilde{K}(L^n(2^r))$ in (1.1) satisfies the relations

$$\sigma^{n+1} = 0, \quad (1+\sigma)^{2^r} - 1 = 0, \tag{2.1}$$

(cf., e.g., [1, Prop. 2.6]). Consider the following elements in $\tilde{K}(L^n(2^r))$:

$$\sigma(0) = \sigma, \ \sigma(s) = (1+\sigma)^{2^s} - 1 = 2\sigma(s-1) + \sigma(s-1)^2 \qquad (0 < s \leqq r). \tag{2.2}$$

LEMMA 2.3 ([2, Prop. 3.2]). *For any integers $k_0, \dots, k_{s-1} \geqq 0$ and $k_s > 0$ $(0 \leqq s \leqq r)$, we have the following in $\tilde{K}(L^n(2^r))$:*

$$2^{r-s+h}\textstyle\prod_{t=0}^{s} \sigma(t)^{k_t} = 0 \text{ if } r-s+h \geqq 0,$$

$$\textstyle\prod_{t=0}^{s} \sigma(t)^{k_t} = 0 \text{ if } r-s+h < 0,$$

where $h = h(k_0, \dots, k_s) = 1 + [(n-1-\sum_{t=0}^{s} 2^t k_t)/2^s]$.

PROOF. If $s=0$ and $h=n-k_0<0$, then the relation is obtained from $\sigma^{n+1}=0$ in (2.1).

Assume inductively that $h \geqq 0$ and the relation on $\alpha\sigma(s)^k$ $(\alpha=\prod_{t=0}^{s-1}\sigma(t)^{k_t})$ holds for $k>k_s$. Since $(1+\sigma(s))^{2^{r-s}}-1=0$ by (2.1–2), we have

$$2^{r-s+h}\alpha\sigma(s)^{k_s} + \textstyle\sum_{i=2}^{2^{r-s}}\binom{2^{r-s}}{i} 2^h\alpha\sigma(s)^{k_s-1+i} = 0.$$

If $i=2^v j \geqq 2$ and j is odd, then $h(k_0, \dots, k_s-1+i) = h-(i-1) \leqq h-v$. Thus the above equality and the inductive assumption imply $2^{r-s+h}\alpha\sigma(s)^{k_s}=0$.

Assume inductively that $s \geqq 1$, $h<0$ and the relation on $\alpha\sigma(s-1)^k$ $(\alpha=\prod_{t=0}^{s-1}\sigma(t)^{k_t})$ holds for $k>0$. Then, by using (2.2), we see

$$2^{r-s+h}\alpha\sigma(s)^{k_s} = \textstyle\sum_{i=0}^{k_s}\binom{k_s}{i} 2^{r-s+h+i}\alpha\sigma(s-1)^{2k_s-i} = 0,$$

as desired, since $h(k_0, \dots, k_{s-2}, k_{s-1}+2k_s-i) \leqq 2h+i < h+i$.

Therefore, we have the lemma by the induction. *q.e.d.*

LEMMA 2.4. *For any integers* $k_0, \dots, k_{s-1} \geqq 0$ *and* $k_s > l \geqq 0$ $(0 \leqq s < r)$, *we have the following in* $\tilde{K}(L^n(2^r))$:

$$2^{h'} \alpha \sigma(s)^{k_s} = (-1)^l 2^{h'+l} \alpha \sigma(s)^{k_s - l} \quad (\alpha = \textstyle\prod_{t=0}^{s-1} \sigma(t)^{k_t}),$$

where h' *is any non-negative integer such that*

$$h' \geqq r - s + [(n - 1 - \textstyle\sum_{t=0}^{s} 2^t k_t)/2^{s+1}].$$

PROOF. We see easily that $2^{h'+l} \alpha \sigma(s)^{k_s - l - 2} \sigma(s+1) = 0$ if $k_s - 1 > l \geqq 0$, by the above lemma. Thus we have the lemma by $\sigma(s+1) = \sigma(s)^2 + 2\sigma(s)$ in (2.2). *q.e.d.*

LEMMA 2.5. *If* $0 < s < r$, $d \geqq 0$, $k > 0$ *is even and* $n < d + 2^s k$, *then we have the following in* $\tilde{K}(L^n(2^r))$ $(r \geqq 2)$:

$$2^{r-s-2+k} \textstyle\sum_{t=0}^{s} 2^{k(2^t - 1)} \sigma^d \sigma(s - t) = 0.$$

PROOF. For any $0 < t \leqq s$, we show the equality

$$(*) \qquad 2^{r-s-1} \sigma^d (\sigma(s-t+1)^{2^{t-1}k} - \sigma(s-t)^{2^t k}) = 2^{r-s-2+2^t k} \sigma^d \sigma(s-t).$$

By (2.2), the left hand side of $(*)$ is equal to

$$\textstyle\sum_{i=1}^{2^{t-1}k} \binom{2^{t-1}k}{i} 2^{r-s-1+i} \sigma^d \sigma(u)^{2^t k - i} \qquad (u = s - t \geqq 0).$$

If $i = 2^v j$ and j is odd, then we see easily from $n < d + 2^s k$ that

$$r - u - 1 + 1 + [(n - 1 - d - 2^u(2^t k - i))/2^{u+1}]$$
$$\leqq r - s - 1 + i + t - v.$$

Thus, by the above lemma and the assumption that k is even, the above sum is equal to

$$\textstyle\sum_{i=1}^{2^{t-1}k} (-1)^{i-1} \binom{2^{t-1}k}{i} 2^{r-s-1+i+2^t k - i - 1} \sigma^d \sigma(u),$$

which is equal to the right hand side of $(*)$.

Since $n < d + 2^s k$ by the assumption, we see that $2^{r-s-1} \sigma^d \sigma(s)^k = (-1)^{k-1} 2^{r-s-2+k} \sigma^d \sigma(s)$ by the above lemma and that $\sigma^{d+2^s k} = 0$ by (2.1). Therefore, we obtain the desired equality by summing up the equalities $(*)$. *q.e.d.*

LEMMA 2.6. *If* $0 < s < r$, $d \geqq 0$, $k \geqq 3$ *is odd and* $n < d + 2^s k$, *then we have the following in* $\tilde{K}(L^n(2^r))$ $(r \geqq 2)$:

$$2^{r-s-2+k} \{\sigma^d \sigma(s) + \textstyle\sum_{t=1}^{s} 2^{(k-1)(2^t - 1) - 1} \sigma^{d+2^s} \sigma(s-t) + \sigma^{d+2^{s-1}} \sigma(s)\} = 0,$$

where $2^{r-s-2+k}\sigma^{d+2^{s-1}}\sigma(s)=0$ *if* $n+2^{s-1}\leqq d+2^s k$.

PROOF. We see easily that

$$2^{r-s-1}\sigma^d\sigma(s)^k = \textstyle\sum_{i=0}^{2^s-1}\binom{2^s}{i}2^{r-s-1}\sigma^{d+2^s-i}\sigma(s)^{k-1} \qquad \text{(by (2.2))}$$

$$= -\,2^{r-s-1}\sigma^{d+2^s}\sigma(s)^{k-1} + 2^{r-s}\sigma^{d+2^{s-1}}\sigma(s)^{k-1}$$

by the assumption $n<d+2^s k$ and Lemma 2.3. Hence we have

$$2^{r-s-2+k}\sigma^d\sigma(s) = 2^{r-s-3+k}\sigma^{d+2^s}\sigma(s) - 2^{r-s-2+k}\sigma^{d+2^{s-1}}\sigma(s),$$

by Lemma 2.4. Since $n<d+2^s+2^s(k-1)$ and $k-1>0$ is even, this implies the desired equality by the above lemma, where the last term is zero if $n+2^{s-1}\leqq d+2^s k$ by Lemma 2.3. *q.e.d.*

§3. Proof of Theorem 1.4

To study some relations in $\widetilde{KO}(L^n(2^r))$, we use the following result due to M. Yasuo [5, (A. 13)]:

(3.1) *The complexification* $c\colon \widetilde{KO}(L^n(2^r))\to\tilde{K}(L^n(2^r))$ *is monomorphic if* $n\equiv 3$ (4).

LEMMA 3.2. *For the real restriction* $r\sigma(s)\in\widetilde{KO}(L^n(2^r))$ *of* $\sigma(s)$ *in* (2.2), *we have*

$$r\sigma(s+1) = 4r\sigma(s) + (r\sigma(s))^2 \qquad (0\leqq s<r).$$

PROOF. Since $1+\sigma(s)=\eta^{2^s}$ is a complex line bundle, we see that

(3.3) $\quad cr\sigma(s) = -2 + (1+\sigma(s)) + 1/(1+\sigma(s)) = \sigma(s)^2/(1+\sigma(s)).$

Therefore, by the fact that c is multiplicative and (2.2), it holds that

$$c(4r\sigma(s) + (r\sigma(s))^2) = (2\sigma(s)+\sigma(s)^2)^2/(1+\sigma(s))^2 = cr\sigma(s+1).$$

Thus we have the desired equality for $n\equiv 3(4)$ by (3.1) and so for any n by the naturality. *q.e.d.*

LEMMA 3.4. *For any integers* $k_0,\dots,k_{s-1}\geqq 0$ *and* $k_s>0$ $(0\leqq s\leqq r)$, *we have the following in* $\widetilde{KO}(L^n(2^r))$ $(n\leqq 4m+3)$:

$$2^{r-s+k}\textstyle\prod_{t=0}^{s}(r\sigma(t))^{k_t}=0 \textit{ if } r-s+k\geqq 0, \quad \prod_{t=0}^{s}(r\sigma(t))^{k_t}=0 \textit{ if } r-s+k<0,$$

where $k = 1 + [(4m+2-\sum_{t=0}^{s}2^{t+1}k_t)/2^s]$.

PROOF. By (3.3) and Lemma 2.3, the c-image of the left hand side is zero in $\tilde{K}(L^{4m+3}(2^r))$. Thus we see the equality for $n=4m+3$ by (3.1) and so for $n \leqq 4m+3$ by the naturality. *q.e.d.*

LEMMA 3.5. *For any integers $k_0,\ldots, k_{s-1} \geqq 0$ and $k_s > l \geqq 0$ $(0 \leqq s < r)$, we have the following in $\widetilde{KO}(L^n(2^r))$ $(n \leqq 4m+3)$:*

$$2^{k'}\prod_{t=0}^{s-1}(r\sigma(t))^{k_t}\{(r\sigma(s))^{k_s} - (-1)^l 2^{2l}(r\sigma(s))^{k_s-l}\} = 0,$$

where k' is any non-negative integer such that

$$k' \geqq r - s + [(4m + 2 - \textstyle\sum_{t=0}^{s} 2^{t+1}k_t)/2^{s+1}].$$

PROOF. We see the lemma using Lemmas 3.4 and 3.2, by the same way as Lemma 2.4. *q.e.d.*

Now, we are ready to prove Theorem 1.4.

LEMMA 3.6. *The following holds in $\widetilde{KO}(L^{4m+1}(2^r))$ $(r \geqq 2,\ m > 0)$:*

$$2^{r+4m+1-2i}(r\sigma)^i = 0 \qquad \textit{for} \quad 1 \leqq i \leqq 2m.$$

PROOF. By applying the above lemma for $s=0$, $k_0=2m$, $k'=r+1$, we see that $2^{r+1}(r\sigma)^{2m}=(-1)^i 2^{r+1+2(2m-i)}(r\sigma)^i$ for $1 \leqq i \leqq 2m$. Hence, it is sufficient to show the equality for $i=1$, which is a consequence of

$$(*) \qquad 2^{r+2m-2}r\sigma(1) + 2^{r+4m}r\sigma = 0 \quad \text{in} \quad \widetilde{KO}(L^{4m+3}(2^r)),$$

$$(**) \qquad 2^{r+2m-2}r\sigma(1) + 2^{r+4m-1}r\sigma = 0 \quad \text{in} \quad \widetilde{KO}(L^{4m+1}(2^r)).$$

By Lemma 2.5 for $n=4m+3$, $s=1$, $d=0$ and $k=2m+2$, we have

$$2^{r+2m-1}\sigma(1) + 2^{r+4m+1}\sigma = 0 \quad \text{in} \quad \tilde{K}(L^{4m+3}(2^r)).$$

This and Lemmas 2.3–4 show the equality

$$2^{r+2m-2}\sigma(1)^2 + 2^{r+4m}\sigma^2(1 + \sigma) = 0 \quad \text{in} \quad \tilde{K}(L^{4m+3}(2^r)).$$

Multiplying this by $1/(1+\sigma)^2$, we obtain the c-image of $(*)$ by (3.3), and hence $(*)$ by (3.1).

By Lemma 2.6 for $n=4m+1$, $s=1$, $d=0$ and $k=2m+1$, we have

$$2^{r+2m-2}\sigma(1) + 2^{r+4m-3}\sigma^3 = 0 \quad \text{in} \quad \tilde{K}(L^{4m+1}(2^r)).$$

Thus, we see by Lemma 2.4 that

$$2^{r+2m-2}\sigma(1) + 2^{r+4m-1}\sigma = 0 \quad \text{in} \quad \tilde{K}(L^{4m+1}(2^r)),$$

whose r-image is $(**)$. *q.e.d.*

PROOF OF THEOREM 1.4. By [5, Prop. (3.5)], it is sufficient to prove the theorem for the case $n=4m+1$.

Since $c(r\sigma)^i=\sigma^{2i}/(1+\sigma)^i$ by (3.3), we see immediately that $(r\sigma)^i\in \widetilde{KO}(L^{4m+1}(2^r))$ $(r\geqq 2,\ m>0)$ is of order $2^{r+4m+1-2i}$ for $1\leqq i\leqq 2m$, by the above lemma and (1.2).

Now consider the commutative diagram

$$\begin{array}{ccccc} \widetilde{KO}(S^{8m+4}) & \xrightarrow{j^!} & \widetilde{KO}(CP^{4m+2}) & \xrightarrow{i^!} & \widetilde{KO}(CP^{4m+1}) \\ \| & & \downarrow{\scriptstyle \pi^!} & & \downarrow{\scriptstyle \pi^!} \\ \widetilde{KO}(S^{8m+4}) & \xrightarrow{j^!} & \widetilde{KO}(L_0^{4m+2}) & \xrightarrow{i^!} & \widetilde{KO}(L^{4m+1}(2^r)) \end{array}$$

for $m\geqq 0$, where $L_0^{4m+2}=L^{4m+1}(2^r)\cup e^{8m+4}$ is the $(8m+4)$-skeleton of $L^{4m+2}(2^r)$ and π is the restriction of the natural projection π: $L^{4m+2}(2^r)\rightarrow CP^{4m+2}$ onto the complex projective space CP^{4m+2}. It is proved by B. J. Sanderson [4, Th. (3.9)] that the image of $\widetilde{KO}(S^{8m+4})=Z$ by the upper $j^!$ is generated by $2y^{2m+1}$, where y is the real restriction of the stable class of the canonical complex line bundle over CP^{4m+2}. Hence $(r\sigma)^{2m+1}=i^!\pi^!y^{2m+1}\in \widetilde{KO}(L^{4m+1}(2^r))$ is of order 2, since the lower sequence in the above diagram is exact. *q.e.d.*

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