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Asymptotic analysis of odd order ordinary differential equations

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1. Introduction

In this paper we consider the differential equations

$$L_n x + q(t)x = 0, \tag{1}$$

$$L_n x + q(t)f(t, x) = 0, \tag{2}$$

where $n \geq 3$ is an odd number and L_n is the differential operator of the form

$$L_n = \frac{1}{p_n(t)} \frac{d}{dt} \frac{1}{p_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{p_1(t)} \frac{d}{dt} \frac{\cdot}{p_0(t)}. \tag{3}$$

The following conditions are always assumed to hold:

(i) $p_i(t)$ $(0 \leq i \leq n)$ and $q(t)$ are continuous and positive on the interval $[a, \infty)$, and

$$\int_a^\infty p_i(t)dt = \infty \qquad \text{for} \quad 1 \leq i \leq n-1.$$

(ii) $f(t, x)$ is continuous on $[a, \infty) \times R$, $f(t, x)$ is nondecreasing in x and $xf(t, x) > 0$ for $x \neq 0$.

We introduce the notation:

$$\begin{aligned} &D^0(x; p_0)(t) = \frac{x(t)}{p_0(t)}, \\ &D^j(x; p_0,\dots, p_j)(t) = \frac{1}{p_j(t)} \frac{d}{dt} D^{j-1}(x; p_0,\dots, p_{j-1})(t), \quad 1 \leq j \leq n. \end{aligned} \tag{4}$$

Then the differential operator L_n can be rewritten as

$$L_n = D^n(\cdot\, ; p_0,\dots, p_n).$$

The domain $\mathscr{D}(L_n)$ of L_n is defined to be the set of all functions $x: [T_x, \infty) \to R$ such that $D^j(x; p_0,\dots, p_j)(t)$ $(0 \leq j \leq n)$ exist and are continuous on $[T_x, \infty)$.

A nontrivial solution of (1) (or (2)) is called oscillatory if the set of its zeros is infinite. Otherwise, it is called nonoscillatory. A nontrivial solution $x(t)$ of (1) (or (2)) is said to be strongly decreasing if it satisfies

$$(5) \qquad (-1)^j x(t) D^j(x; p_0,\dots, p_j)(t) > 0 \quad \text{for} \quad 0 \le j \le n-1$$

for all sufficiently large t. Condition (5) implies that $|D^j(x; p_0,\dots, p_j)(t)|$ $(0 \le j \le n-1)$ are decreasing and $|D^j(x; p_0,\dots, p_j)(t)| \downarrow 0$ as $t \uparrow \infty$ for $1 \le j \le n-1$. One should remark that equation (1) always has strongly decreasing solutions; see Hartman and Wintner [3].

The oscillatory behavior of even order equations of the form (1) and (2) has recently been studied by Kusano and Naito [6] and Kreith, Kusano and Naito [5]. The main purpose of this paper is to adapt their methods and techniques to establish criteria for all solutions of equations (1) and (2) with n odd to be either oscillatory or strongly decreasing. Our results generalize those of Lovelady [7] for odd order equations of the form $x^{(n)} + q(t)x = 0$.

The desired criteria for equations (1) and (2) are obtained in Sections 3 and 5, respectively. Section 4 is devoted to the study of the structure of the solution space of equation (1). Several preparatory results which are basic in these sections are summarized in Section 2.

2. Preliminaries

Let $i_k \in \{1,\dots, n-1\}$, $1 \le k \le n-1$, and $t, s \in [a, \infty)$. We define

$$(6) \qquad \begin{aligned} &I_0 = 1, \\ &I_k(t, s; p_{i_k},\dots, p_{i_1}) = \int_s^t p_{i_k}(u) I_{k-1}(u, s; p_{i_{k-1}},\dots, p_{i_1}) du. \end{aligned}$$

It is easily verified that for $1 \le k \le n-1$

$$(7) \qquad I_k(t, s; p_{i_k},\dots, p_{i_1}) = (-1)^k I_k(s, t; p_{i_1},\dots, p_{i_k}),$$

$$(8) \qquad I_k(t, s; p_{i_k},\dots, p_{i_1}) = \int_s^t p_{i_1}(u) I_{k-1}(t, u; p_{i_k},\dots, p_{i_2}) du.$$

For simplicity we put

$$(9) \qquad \begin{aligned} &J_k(t, s) = p_0(t) I_k(t, s; p_1,\dots, p_k), \qquad &J_k(t) = J_k(t, a). \\ &K_k(t, s) = p_n(t) I_k(t, s; p_{n-1},\dots, p_{n-k}), \qquad &K_k(t) = K_k(t, a). \end{aligned}$$

LEMMA 1. *If $x \in \mathscr{D}(L_n)$, then the following formula holds for $0 \le i \le k \le n-1$ and $t, s \in [T_x, \infty)$:*

$$(10) \qquad \begin{aligned} &D^i(x; p_0,\dots, p_i)(t) \\ &= \textstyle\sum_{j=i}^k (-1)^{j-i} D^j(x; p_0,\dots, p_j)(s) I_{j-i}(s, t; p_j,\dots, p_{i+1}) \\ &\quad + (-1)^{k-i+1} \int_t^s I_{k-i}(u, t; p_k,\dots, p_{i+1}) p_{k+1}(u) D^{k+1}(x; p_0,\dots, p_{k+1})(u) du. \end{aligned}$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

LEMMA 2. *If $x \in \mathscr{D}(L_n)$ satisfies $x(t)L_n x(t)<0$ on $[t_0, \infty)$, then there exist an even number l $(0 \leq l \leq n-1)$ and t_1 $(t_1 \geq t_0)$ such that for $t \geq t_1$,*

$$x(t)D^j(x;\, p_0,\dots,\, p_j)(t)>0, \qquad 0 \leq j \leq l, \tag{11}$$

$$(-1)^{j-l}x(t)D^j(x;\, p_0,\dots,\, p_j)(t) > 0, \qquad l \leq j \leq n. \tag{12}$$

This lemma generalizes a well-known lemma of Kiguradze and can be proved similarly.

Consider the n-th order differential equation

$$L_n x + F(t,\, x) = 0, \tag{13}$$

where n is either odd or even, and $F(t, x)$ is a continuous function on $[a, \infty) \times R$ such that $F(t, x)$ is nondecreasing in x and $xF(t, x)>0$ for $x \neq 0$.

LEMMA 3. *Let k, $0 \leq k \leq n-1$, be fixed. Equation* (13) *has a nonoscillatory solution $x(t)$ satisfying*

$$\lim_{t\to\infty} D^k(x;\, p_0,\dots,\, p_k)(t) = \lim_{t\to\infty}\frac{x(t)}{J_k(t)} = a_k \in R - \{0\}$$

if and only if

$$\int^{\infty} K_{n-k-1}(t)\,|F(t,\, cJ_k(t))|dt < \infty \qquad \textit{for some} \quad c \in R - \{0\}. \tag{14}$$

The proof is found in Kitamura and Kusano [4].

LEMMA 4. *If the differential inequality*

$$\{L_n x + F(t,\, x)\}\operatorname{sgn} x \leq 0$$

has a nonoscillatory solution which is not strongly decreasing, then so does the differential equation (13).

For the proof see Čanturija [1].

3. Oscillation theorems for equation (1)

As we remarked in Section 1, equation (1) always has nonoscillatory solutions which are strongly decreasing. So the strongest conclusion we can expect for oscillation of equation (1) is that all of its nonoscillatory solutions are strongly decreasing.

THEOREM 1. *Suppose that*

$$\int^{\infty} J_{i-1}(t)K_{n-i-1}(t)q(t)dt = \infty \qquad \text{for} \quad i = 2, 4,\dots, n-1. \tag{15}$$

Then every nonoscillatory solution of equation (1) *is strongly decreasing.*

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1) which is not strongly decreasing. We may suppose that $x(t)$ is eventually positive. From Lemma 2, there exist an even number l $(2 \le l \le n-1)$ and $t_1 \in [a, \infty)$ such that inequalities (11) and (12) hold for $t \ge t_1$.

Suppose $l < n-1$. From formula (10) with $i=l$, $k=n-1$, $t=t_1$, and $s \ge t_1$ it follows that

$$\begin{aligned} &D^l(x;\, p_0,\dots, p_l)(t_1) \\ &= \textstyle\sum_{j=l}^{n-1} (-1)^{j-l} D^j(x;\, p_0,\dots, p_j)(s) I_{j-l}(s, t_1;\, p_j,\dots, p_{l+1}) \\ &\quad + (-1)^{n-l} \int_{t_1}^{s} I_{n-l-1}(u, t_1;\, p_{n-1},\dots, p_{l+1}) p_n(u) D^n(x;\, p_0,\dots, p_n)(u)du. \end{aligned} \tag{16}$$

Using $D^n(x;\, p_0,\dots, p_n)(u) = -q(u)x(u)$ and (12), we have

$$\int_{t_1}^{s} p_n(u) I_{n-l-1}(u, t_1;\, p_{n-1},\dots, p_{l+1}) q(u)x(u)du < D^l(x;\, p_0,\dots, p_l)(t_1),$$

which gives in the limit as $s \to \infty$

$$\int_{t_1}^{\infty} K_{n-l-1}(t, t_1)q(t)x(t)dt < \infty. \tag{17}$$

On the other hand, by integrating $D^l(x;\, p_0,\dots, p_l)(t) > 0$ $(t \ge t_1)$ l times, we obtain

$$x(t) \ge cJ_{l-1}(t, t_1) \qquad \text{for} \quad t \ge t_2, \tag{18}$$

where c is a positive constant and $t_2 \ge t_1$ is a suitable constant. Combining (17) with (18), we get

$$\int_{t_2}^{\infty} J_{l-1}(t, t_1)K_{n-l-1}(t, t_1)q(t)dt < \infty, \tag{19}$$

which contradicts (15).

Next, suppose $l = n-1$. Multiplying both sides of equation (1) by $p_n(t)$ and integrating from t_1 to ∞, we see that

$$\int_{t_1}^{\infty} p_n(t)q(t)x(t)dt < \infty. \tag{20}$$

From (20) and (18) with $l=n-1$, we have

$$\int_{t_2}^{\infty} p_n(t)q(t)J_{n-2}(t, t_1)dt < \infty$$

or

$$\int_{t_2}^{\infty} J_{n-2}(t, t_1)K_0(t, t_1)q(t)dt < \infty, \tag{21}$$

which again contradicts (15). Therefore, every nonoscillatory solution of (1) must be strongly decreasing, and the proof is complete.

Next, we consider the case where the integrals

$$\int^{\infty} J_{i-1}(t)K_{n-i-2}(t)q(t)dt$$

are convergent for $i=2, 4,\dots, n-3$ and $n-2$. For simplicity we put

$$q_i(t) = p_{i+1}(t)\int_t^{\infty} J_{i-1}(u, t)K_{n-i-2}(u, t)q(u)du, \quad i = 2, 4,\dots, n-3, \tag{22}$$

$$q_{n-1}(t) = p_{n-2}(t)\int_t^{\infty} J_{n-3}(u, t)K_0(u, t)q(u)du. \tag{23}$$

THEOREM 2. *If all of the second order differential equations*

$$\left(\frac{z'}{p_i(t)}\right)' + q_i(t)z = 0, \quad i = 2, 4,\dots, n-1, \tag{24}$$

are oscillatory, then every nonoscillatory solution of equation (1) *is strongly decreasing.*

PROOF. We assume that $x(t)$ is a positive solution of equation (1) which is not strongly decreasing. By Lemma 2 there exist an even integer l $(2\le l\le n-1)$ and t_1 $(t_1>a)$ such that (11) and (12) hold for $t\ge t_1$.

Let $l<n-1$. Putting $i=l+1$, $k=n-1$, $s\ge t\ge t_1$ in (10), we have

$$\begin{aligned}
&D^{l+1}(x; p_0,\dots, p_{l+1})(t)\\
&\quad = \textstyle\sum_{j=l+1}^{n-1} (-1)^{j-l-1}D^j(x; p_0,\dots, p_j)(s)I_{j-l-1}(s, t; p_j,\dots, p_{l+2})\\
&\qquad + (-1)^{n-l-1}\int_t^{s} I_{n-l-2}(u, t; p_{n-1},\dots, p_{l+2})p_n(u)D^n(x; p_0,\dots, p_n)(u)du.
\end{aligned}$$

Letting $s\to\infty$ in the above, we obtain

$$-D^{l+1}(x; p_0,\dots, p_{l+1})(t) \ge \int_t^{\infty} p_n(u)I_{n-l-2}(u, t; p_{n-1},\dots, p_{l+2})q(u)x(u)du \tag{25}$$

for $t \geq t_1$. Now putting $i=0$, $k=l-2$, $t \geq s=t_1$ in (10), we have

$$D^0(x; p_0)(t)$$
$$= \sum_{j=0}^{l-2} (-1)^j D^j(x; p_0, \dots, p_j)(t_1) I_j(t_1, t; p_j, \dots, p_1)$$
$$+ (-1)^{l-1} \int_t^{t_1} I_{l-2}(u, t; p_{l-2}, \dots, p_1) p_{l-1}(u) D^{l-1}(x; p_0, \dots, p_{l-1})(u) du$$
$$= \sum_{j=0}^{l-2} D^j(x; p_0, \dots, p_j)(t_1) I_j(t, t_1; p_1, \dots, p_j)$$
$$+ \int_{t_1}^t I_{l-2}(t, u; p_1, \dots, p_{l-2}) p_{l-1}(u) D^{l-1}(x; p_0, \dots, p_{l-1})(u) du,$$

which, in view of (11), yields

$$D^0(x; p_0)(t) \geq \int_{t_1}^t I_{l-2}(t, u; p_1, \dots, p_{l-2}) p_{l-1}(u) D^{l-1}(x; p_0, \dots, p_{l-1})(u) du \tag{26}$$

for $t \geq t_1$. Combining (25) with (26), we have

$$-D^{l+1}(x; p_0, \dots, p_{l+1})(t)$$
$$\geq \int_t^\infty p_n(u) I_{n-l-2}(u, t; p_{n-1}, \dots, p_{l+2}) \cdot$$
$$\cdot q(u) p_0(u) \int_{t_1}^u I_{l-2}(u, v; p_1, \dots, p_{l-2}) p_{l-1}(v) D^{l-1}(x; p_0, \dots, p_{l-1})(v) dv du$$
$$\geq \int_t^\infty p_n(u) I_{n-l-2}(u, t; p_{n-1}, \dots, p_{l+2}) \cdot$$
$$\cdot q(u) p_0(u) \int_t^u I_{l-2}(u, v; p_1, \dots, p_{l-2}) p_{l-1}(v) D^{l-1}(x; p_0, \dots, p_{l-1})(v) dv du$$

for $t \geq t_1$. Since $D^{l-1}(x; p_0, \dots, p_{l-1})$ is increasing, it follows from the above that

$$-D^{l+1}(x; p_0, \dots, p_{l+1})(t)$$
$$\geq D^{l-1}(x; p_0, \dots, p_{l-1})(t) \int_t^\infty p_n(u) I_{n-l-2}(u, t; p_{n-1}, \dots, p_{l+2}) \cdot$$
$$\cdot q(u) p_0(u) \int_t^u I_{l-2}(u, v; p_1, \dots, p_{l-2}) p_{l-1}(v) dv du$$
$$= D^{l-1}(x; p_0, \dots, p_{l-1})(t) \int_t^\infty p_n(u) I_{n-l-2}(u, t; p_{n-1}, \dots, p_{l+2}) \cdot$$
$$\cdot q(u) p_0(u) I_{l-1}(u, t; p_1, \dots, p_{l-1}) du.$$

Let $y(t)$ be given by

$$y(t) \equiv D^{l-1}(x;\, p_0,\dots,\, p_{l-1})(t).$$

Then $y(t)>0$ on $[t_1, \infty)$ and $y(t)$ satisfies

$$(27)\qquad -D^{l+1}(x;\, p_0,\dots,\, p_{l+1})(t) \geq y(t)\int_t^\infty J_{l-1}(u,\, t)K_{n-l-2}(u,\, t)q(u)du$$

for $t \geq t_1$. Noting that

$$\left(\frac{y'(t)}{p_l(t)}\right)' = p_{l+1}(t)D^{l+1}(x;\, p_0,\dots,\, p_{l+1})(t),$$

we see from (27) that

$$\left(\frac{y'(t)}{p_l(t)}\right)' + q_l(t)y(t) \leq 0, \quad t \geq t_1.$$

Lemma 4 now implies that the equation

$$\left(\frac{z'}{p_l(t)}\right)' + q_l(t)z = 0$$

has an eventually positive solution. But this contradicts our assumption.

Let $l=n-1$. An integration of (1) yields

$$(28)\qquad D^{n-1}(x;\, p_0,\dots,\, p_{n-1})(t) \geq \int_t^\infty p_n(u)q(u)x(u)du \qquad \text{for} \quad t \geq t_1.$$

Setting $i=0$, $k=n-3$, $t \geq s=t_1$ in (10), we have

$$\begin{aligned} &D^0(x;\, p_0)(t) \\ &\quad = \textstyle\sum_{j=0}^{n-3}(-1)^j D^j(x;\, p_0,\dots,\, p_j)(t_1)I_j(t_1,\, t;\, p_j,\dots,\, p_1) \\ &\qquad + (-1)^{n-2}\int_t^{t_1} I_{n-3}(u,\, t;\, p_{n-3},\dots,\, p_1)p_{n-2}(u)D^{n-2}(x;\, p_0,\dots,\, p_{n-2})(u)du \\ &\quad = \textstyle\sum_{j=0}^{n-3} D^j(x;\, p_0,\dots,\, p_j)(t_1)I_j(t,\, t_1;\, p_1,\dots,\, p_j) \\ &\qquad + \int_{t_1}^t I_{n-3}(t,\, u;\, p_1,\dots,\, p_{n-3})p_{n-2}(u)D^{n-2}(x;\, p_0,\dots,\, p_{n-2})(u)du. \end{aligned}$$

From this we easily see that

$$(29)\quad D^0(x;\, p_0)(t) \geq \int_{t_1}^t I_{n-3}(t,\, u;\, p_1,\dots,\, p_{n-3})p_{n-2}(u)D^{n-2}(x;\, p_0,\dots,\, p_{n-2})(u)du$$

for $t \geq t_1$. From (28) and (29) it follows that for $t \geq t_1$

$$D^{n-1}(x;\ p_0,\dots,\ p_{n-1})(t)$$

$$\geq \int_t^\infty p_n(u)q(u)p_0(u)\int_{t_1}^u I_{n-3}(u, v;\ p_1,\dots, p_{n-3})p_{n-2}(v)D^{n-2}(x;\ p_0,\dots, p_{n-2})(v)dvdu$$

$$\geq \int_t^\infty p_n(u)q(u)p_0(u)\int_t^u I_{n-3}(u, v;\ p_1,\dots, p_{n-3})p_{n-2}(v)D^{n-2}(x;\ p_0,\dots, p_{n-2})(v)dvdu$$

$$= \int_t^\infty \left(\int_v^\infty p_n(u)p_0(u)I_{n-3}(u,v;p_1,\dots,p_{n-3})q(u)du\right)p_{n-2}(v)D^{n-2}(x;p_0,\dots,p_{n-2})(v)dv.$$

Therefore

$$D^{n-1}(x;p_0,\dots,\ p_{n-1})(t)$$

$$\geq \int_t^\infty \left(\int_v^\infty J_{n-3}(u,\ v)K_0(u,\ v)q(u)du\right)p_{n-2}(v)D^{n-2}(x;\ p_0,\dots,\ p_{n-2})(v)dv$$

for $t \geq t_1$. Integrating this inequality from t_1 to t, we see that $w(t) \equiv D^{n-2}(x;\ p_0,\dots,\ p_{n-2})(t) > 0$ satisfies

$$w(t) \geq w(t_1) + \int_{t_1}^t p_{n-1}(u)\int_u^\infty q_{n-1}(v)w(v)dvdu \qquad \text{for} \quad t \geq t_1. \tag{30}$$

Denoting the right side of (30) by $y(t)$, it is easy to see that

$$\left(\frac{y'(t)}{p_{n-1}(t)}\right)' + q_{n-1}(t)y(t) \leq 0, \quad t \geq t_1.$$

Again by Lemma 4 the equation

$$\left(\frac{z'}{p_{n-1}(t)}\right)' + q_{n-1}(t)z = 0$$

has an eventually positive solution, contradicting the hypothesis of the theorem.

We show that the conclusion of Theorems 1 and 2 can be strengthened if an additional condition is placed on $q(t)$.

THEOREM 3. *Suppose that all nonoscillatory solutions of equation* (1) *are strongly decreasing. Then every nonoscillatory solution* $x(t)$ *of* (1) *satisfies*

$$\lim_{t\to\infty} \frac{x(t)}{p_0(t)} = 0 \tag{31}$$

if and only if

$$\int^\infty J_0(t)K_{n-1}(t)q(t)dt = \infty. \tag{32}$$

PROOF. If (32) does not hold, then by Lemma 3 with $k=0$ equation (1) has a nonoscillatory solution $x(t)$ such that $\lim_{t\to\infty} x(t)/p_0(t)=\text{const}\neq 0$. This proves the "only if" part of the theorem.

Let $x(t)$ be a nonoscillatory solution of (1) which does not enjoy property (31). Then, there exists the limit $\lim_{t\to\infty} x(t)/p_0(t)=a\in R-\{0\}$, and from Lemma 3 with $k=0$ we have

$$\int^\infty J_0(t)K_{n-1}(t)q(t)dt < \infty.$$

This contradiction proves the "if" part of the theorem.

COROLLARY 1. *Consider the third order equation*

$$\left(\frac{1}{p_2(t)}\left(\frac{1}{p_1(t)}x'\right)'\right)' + q(t)x = 0, \tag{33}$$

where $p_1(t)$, $p_2(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$, and

$$\int^\infty p_1(t)dt = \int^\infty p_2(t)dt = \infty.$$

Suppose that either (i)

$$\int^\infty \left(\int_a^t p_1(s)ds\right)q(t)dt = \infty, \tag{34}$$

or (ii) $\int^\infty q(t)dt<\infty$ *and the equation*

$$\left(\frac{z'}{p_2(t)}\right)' + \left(p_1(t)\int_t^\infty q(s)ds\right)z = 0 \tag{35}$$

is oscillatory. Then all nonoscillatory solutions of equation (33) *are strongly decreasing. If in addition*

$$\int^\infty \left(\int_a^t p_2(s)\int_a^s p_1(\sigma)d\sigma ds\right)q(t)dt = \infty,$$

then all nonoscillatory solutions of (33) *tend to zero as $t\to\infty$.*

COROLLARY 2. *Consider the equations*

$$\left(\frac{1}{p(t)}x'\right)^{(2m)} + q(t)x = 0, \tag{36}$$

$$\left(\frac{1}{p(t)}x^{(2m)}\right)' + q(t)x = 0, \tag{37}$$

where $m\geq 1$, $p(t)$ and $q(t)$ are positive and continuous on $[a, \infty)$, and

$$\int^{\infty} p(t)dt = \infty.$$

(I) *All nonoscillatory solutions of equation* (36) *are strongly decreasing if either* (i)

$$\int^{\infty}\left(\int_a^t (s-a)^{2m-2}p(s)ds\right)q(t)dt = \infty, \tag{38}$$

or (ii)

$$\int^{\infty}\left(\int_a^t (s-a)^{2m-3}p(s)ds\right)q(t)dt < \infty \tag{39}$$

and the equation

$$z'' + \frac{1}{(2m-3)!}\left\{\int_t^{\infty}\left(\int_t^s (\sigma-t)^{2m-3}p(\sigma)d\sigma\right)q(s)ds\right\}z = 0 \tag{40}$$

is oscillatory.

If in addition

$$\int^{\infty}\left(\int_a^t (t-s)^{2m-1}p(s)ds\right)q(t)dt = \infty, \tag{41}$$

then all nonoscillatory solutions tend to zero as $t\to\infty$.

(II) *All nonoscillatory solutions of equation* (37) *are strongly decreasing if either* (i) (38) *holds and*

$$\int^{\infty} t^{2m-1}q(t)dt = \infty, \tag{42}$$

or (ii) (39) *holds,*

$$\int^{\infty} t^{2m-2}q(t)dt < \infty \tag{43}$$

and the equations (40) *and*

$$\left(\frac{z'}{p(t)}\right)' + \frac{1}{(2m-2)!}\left(\int_t^{\infty} (s-t)^{2m-2}q(s)ds\right)z = 0 \tag{44}$$

are oscillatory.

If in addition

$$\int^{\infty}\left(\int_a^t (s-a)^{2m-1}p(s)ds\right)q(t)dt = \infty, \tag{45}$$

then all nonoscillatory solutions of (37) *tend to zero as* $t\to\infty$.

EXAMPLE 1. Consider the equation

$$(46) \qquad (t^{\lambda}x^{(2m)})' + ct^{\mu}x = 0, \quad t \geq 1,$$

where $|\lambda| \leq 1$, μ and $c>0$ are constants. From Corollary 2 (II) (i) it follows that all nonoscillatory solutions of (46) tend to zero as $t \to \infty$ if $\mu \geq \max\{\lambda, 0\} - 2m$. In case $\mu < \max\{\lambda, 0\} - 2m$, (39) and (43) are satisfied and equations (40) and (44) become

$$(47) \qquad z'' + \frac{ct^{-\lambda+\mu+2m-1}}{(-\mu-1)(\lambda-\mu-2)(\lambda-\mu-3)\cdots(\lambda-\mu-2m+1)} z = 0$$

and

$$(48) \qquad (t^{\lambda}z')' + \frac{ct^{\mu+2m-1}}{(-\mu-1)(-\mu-2)\cdots(-\mu-2m+1)} z = 0,$$

respectively. By Corollary 2 (II) (ii) all nonoscillatory solutions of (46) tend to zero as $t \to \infty$ if either $\lambda - 2m - 1 < \mu < \max\{\lambda, 0\} - 2m$ or $\mu = \lambda - 2m - 1$ and

$$(49) \; c > \frac{1}{4} \max\{(-\lambda+2m)(2m-1)!, (\lambda-1)^2(-\lambda+2m)(-\lambda+2m-1)\cdots(-\lambda+2)\}.$$

Consequently if either $\mu > \lambda - 2m - 1$ or $\mu = \lambda - 2m - 1$ and (49) is satisfied, then every nonoscillatory solution of (46) tends to zero as $t \to \infty$.

We conclude this section with a theorem which gives a sufficient condition for equation (1) to have a nonoscillatory solution which is not strongly decreasing.

THEOREM 4. *Suppose there exists an odd integer l $(1 < l < n)$ such that the l-th order equation*

$$(50) \qquad D^{l}(z; p_0, p_1, \ldots, p_{l-1}, 1)(t) + K_{n-l}(t)q(t)z(t) = 0$$

has a nonoscillatory solution $z(t)$ satisfying

$$(51) \qquad z(t)D^{j}(z; p_0, p_1, \ldots, p_j)(t) > 0, \quad 0 \leq j \leq l-1,$$

for all sufficiently large t. Then equation (1) *has a nonoscillatory solution which is not strongly decreasing.*

PROOF. We may suppose that $z(t) > 0$ on $[t_0, \infty)$. Applying formula (10) to $z(t)$ with $i=0$, $k=l-2$, $t \geq s = t_0$, we obtain by use of (7) that

$$D^{0}(z; p_0)(t) - D^{0}(z; p_0)(t_0)$$
$$= \textstyle\sum_{j=1}^{l-2} (-1)^{j} D^{j}(z; p_0, \ldots, p_j)(t_0) I_j(t_0, t; p_j, \ldots, p_1)$$

$$+(-1)^{l-1}\int_t^{t_0} I_{l-2}(u, t; p_{l-2},\dots, p_1)p_{l-1}(u)D^{l-1}(z; p_0,\dots, p_{l-1})(u)du$$

$$= \textstyle\sum_{j=1}^{l-2} D^j(z; p_0,\dots, p_j)(t_0)I_j(t, t_0; p_1,\dots, p_j)$$

$$+\int_{t_0}^{t} I_{l-2}(t, u; p_1,\dots, p_{l-2})p_{l-1}(u)D^{l-1}(z; p_0,\dots, p_{l-1})(u)du.$$

In view of (51), it follows that

$$(52)\quad D^0(z; p_0)(t) \ge D^0(z; p_0)(t_0) + \int_{t_0}^{t} I_{l-2}(t, u; p_1,\dots, p_{l-2})p_{l-1}(u)D^{l-1}(z; p_0,\dots, p_{l-1})(u)du.$$

Integrating (50) from t to s ($s \ge t \ge t_0$) and letting $s \to \infty$, we obtain

$$(53)\qquad D^{l-1}(z; p_0,\dots, p_{l-1})(t) \ge \int_t^\infty K_{n-l}(u, t)q(u)z(u)du, \quad t \ge t_0.$$

Substituting (53) in (52), we obtain

$$(54)\quad z(t) \ge D^0(z; p_0)(t_0)p_0(t) + \int_{t_0}^{t} J_{l-2}(t, u)p_{l-1}(u)\int_u^\infty K_{n-l}(v, u)q(v)z(v)dvdu \quad \text{for} \quad t \ge t_0.$$

Now we define a sequence of functions $\{x_m\}_{m=0}^{\infty}$ by

$$x_0(t) = D^0(z; p_0)(t_0)p_0(t)$$

$$x_{m+1}(t) = D^0(z; p_0)(t_0)p_0(t) + \int_{t_0}^{t} J_{l-2}(t, u)p_{l-1}(u)\int_u^\infty K_{n-l}(v, u)q(v)x_m(v)dvdu, \quad m = 0, 1, 2,\dots .$$

It is easy to check that $\{x_m\}_{m=0}^{\infty}$ is well-defined as an increasing sequence and satisfies

$$D^0(z; p_0)(t_0)p_0(t) \le x_m(t) \le z(t) \qquad \text{for} \quad t \ge t_0, \quad m = 0, 1, 2,\dots .$$

Hence there exists a function $x(t)$ on $[t_0, \infty)$ such that

$$\lim_{m\to\infty} x_m(t) = x(t) \qquad \text{for} \quad t \ge t_0$$

and

$$D^0(z; p_0)(t_0)p_0(t) \le x(t) \le z(t) \qquad \text{for} \quad t \ge t_0.$$

From the Lebesgue convergence theorem it follows that

$$x(t) = D^0(z; p_0)(t_0)p_0(t) + \int_{t_0}^{t} J_{l-2}(t, u)p_{l-1}(u)\int_u^\infty K_{n-l}(v, u)q(v)x(v)dvdu$$

for $t \geq t_0$. Differentiating the above equation, we conclude that $x(t)$ is a nonoscillatory solution with the desired property. This completes the proof.

EXAMPLE 2. Let us consider equation (46). For this equation, (50) becomes

$$z^{(l)} + \frac{ct^{\mu}}{(2m-l)!}\left(\int_1^t (u-1)^{2m-l}u^{-\lambda}du\right)z = 0. \tag{55}$$

According to Foster and Grimmer [2, Theorem 1], it is seen that equation (55) has a solution satisfying (51) if and only if the second order equation

$$y'' + \frac{ct^{\mu}}{(2m-l)!}\left(\int_1^t (u-1)^{2m-l}u^{-\lambda}du\right)\frac{(t-1)^{l-2}}{(l-1)!}y = 0 \tag{56}$$

is nonoscillatory. It is easily verified that equation (56) is nonoscillatory if either $\mu < \lambda - 2m - 1$ or $\mu = \lambda - 2m - 1$ and

$$c \leq \frac{1}{4}\max\{2(2m-3)!(2m-\lambda-2),\ (2m-2)!(-\lambda+2)\}. \tag{57}$$

Hence if $\mu < \lambda - 2m - 1$ or $\mu = \lambda - 2m - 1$ and (57) is satisfied, then equation (46) has a nonoscillatory solution which is not strongly decreasing.

4. Solution space of equation (1)

Let $\mathscr{S}$ denote the set of all solutions of equation (1). It is clear that $\mathscr{S}$ is an n-dimensional linear space over the reals. We are interested in the structure of this solution space $\mathscr{S}$ in case every nonoscillatory solution of equation (1) is strongly decreasing.

THEOREM 5. *Suppose all nonoscillatory solutions of equation* (1) *are strongly decreasing. Then $\mathscr{S}$ has a basis which consists of oscillatory solutions, and $\mathscr{S}$ has an $(n-1)$-dimensional subspace whose elements are all oscillatory solutions.*

In order to prove this theorem, we need the following two lemmas.

LEMMA 5. *Let $x(t)$ be a solution of equation* (1). *If*

$$(-1)^j D^j(x;\, p_0,\dots,\, p_j)(c) > 0, \quad 0 \leq j \leq n-1,$$

for some $c \geq a$, then

$$(-1)^j D^j(x;\, p_0,\dots,\, p_j)(t) > 0 \qquad \text{for} \quad a \leq t \leq c, \quad 0 \leq j \leq n-1.$$

PROOF. Put $v(t) = x(a-t)$ for $a-c \leq t \leq 0$. Define $\bar{p}_j(t) = p_j(a-t)$ for

$a-c \le t \le 0$, $0 \le j \le n$. Then,

$$D^j(v;\ \bar{p}_0,\dots,\ \bar{p}_j)(t) = (-1)^j D^j(x;\ p_0,\dots,\ p_j)(a-t),$$

so $v(t)$ satisfies

$$D^n(v;\ \bar{p}_0,\dots,\ \bar{p}_n)(t) - q(a-t)v(t) = 0 \qquad \text{for} \quad a-c \le t \le 0$$

and

$$D^j(v;\ \bar{p}_0,\dots,\ \bar{p}_j)(a-c) > 0, \quad 0 \le j \le n-1.$$

Hence it follows that

$$D^j(v;\ \bar{p}_0,\dots,\ \bar{p}_j)(t) > 0 \qquad \text{for} \quad a-c \le t \le 0, \quad 0 \le j \le n-1,$$

which implies

$$(-1)^j D^j(x;\ p_0,\dots,\ p_j)(t) > 0 \qquad \text{for} \quad a \le t \le c, \quad 0 \le j \le n-1.$$

LEMMA 6. *Suppose that all nonoscillatory solutions of equation* (1) *are strongly decreasing. If there exists a solution* $x(t)$ *of equation* (1) *such that* $D^j(x;\ p_0,\dots,\ p_j)(t)$ *has at least one zero for some* $j \in \{0, 1,\dots, n-1\}$, *then* $x(t)$ *is oscillatory.*

PROOF. Let $x(t)$ be a positive solution of equation (1) such that (5) holds on $[c, \infty)$ for some $c > a$. Lemma 5 implies that $D^j(x;\ p_0,\dots,\ p_j)(t)$ never vanish on $[a, c]$ for $0 \le j \le n-1$. This shows our assertion.

PROOF OF THEOREM 5. For $j \in \{1,\dots, n\}$ let $z_j(t)$ be a solution of equation (1) satisfying the initial conditions

$$D^{k-1}(z_j;\ p_0,\dots,\ p_{k-1})(a) = \delta_{jk}, \quad 1 \le k \le n.$$

Clearly, $z_1,\dots, z_n$ form a basis for $\mathscr{S}$ and by Lemma 6 they are all oscillatory. On the other hand, if $x \in \operatorname{span}\{z_2,\dots, z_n\}$, then $x(a) = 0$, so that $x(t)$ is oscillatory. This implies that $\operatorname{span}\{z_2,\dots, z_n\}$ is an $(n-1)$-dimensional subspace of $\mathscr{S}$, all elements of which are oscillatory.

5. Nonlinear equations

In this section we study nonlinear equations of the form (2) which are either weakly superlinear or weakly sublinear in the sense defined below.

DEFINITION. Equation (2) is called weakly superlinear if

$$\lim_{|x| \to \infty} \frac{|f(t, x)|}{|x|} = \infty \qquad \text{uniformly for} \quad t \in [a, \infty).$$

Equation (2) is called weakly sublinear if

$$\lim_{|x|\to\infty} \frac{|f(t, x)|}{|x|} = 0 \qquad \text{uniformly for} \quad t \in [a, \infty).$$

THEOREM 6. *Suppose that equation* (2) *is weakly superlinear and* $\liminf_{t\to\infty} p_0(t) > 0$.

If, for some $M > 0$, *every nonoscillatory solution of the equation*

$$L_n x + Mq(t)x = 0 \tag{58}$$

is strongly decreasing, then every nonoscillatory solution of equation (2) *is strongly decreasing.*

If in addition

$$\int^{\infty} K_{n-1}(t)q(t)\,|f(t, cp_0(t))|dt = \infty \qquad \textit{for every} \quad c \in R - \{0\}, \tag{59}$$

then every nonoscillatory solution $x(t)$ *of equation* (2) *satisfies* $\lim_{t\to\infty} x(t)/p_0(t) = 0$.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (2) which is not strongly decreasing. We may suppose $x(t)$ is eventually positive. Then we have

$$\lim_{t\to\infty} D^0(x; p_0)(t) = \lim_{t\to\infty} \frac{x(t)}{p_0(t)} = \infty. \tag{60}$$

In fact, the integer l associated with $x(t)$ by Lemma 2 is not less than 2, and so there are a positive number N and $t_0 \geq a$ such that

$$D^1(x; p_0, p_1)(t) \geq N \qquad \text{for} \quad t \geq t_0.$$

Integrating the above inequality, we find

$$D^0(x; p_0)(t) \geq N \int_{t_0}^{t} p_1(s)ds \qquad \text{for} \quad t \geq t_0,$$

from which (60) readily follows. Now since $\liminf_{t\to\infty} p_0(t) > 0$, (60) implies that $\lim_{t\to\infty} x(t) = \infty$. By the weak superlinearity of equation (2),

$$\lim_{t\to\infty} \left(\frac{f(t', x(t))}{x(t)} \right) = \infty$$

uniformly with respect to $t' \in [a, \infty)$, so that there exists $T > t_0$ such that $f(t, x(t)) \geq Mx(t)$ for $t \geq T$. From this and (2) we have

$$L_n x(t) + Mq(t)x(t) \leq 0, \quad t \geq T.$$

We apply Lemma 4 to conclude that equation (58) has a positive solution which is not strongly decreasing. But this is a contradiction. This completes the proof of the first part of the theorem.

Suppose that (59) holds for every $c \in R-\{0\}$. If $x(t)$ is a strongly decreasing solution of equation (2), then $|D^0(x; p_0)(t)|$ is decreasing. Hence the limit $\alpha = \lim_{t\to\infty} D^0(x; p_0)(t)$ exists as a finite value. Lemma 3 with $k=0$ implies that $\alpha=0$. This proves the second part of the theorem.

COROLLARY 3. *Suppose that* $\liminf_{t\to\infty} p_0(t)>0$ *and* $f(x)$ *is a continuous and nondecreasing function on* R *which satisfies* $xf(x)>0$ *for* $x\neq 0$ *and*

$$\lim_{|x|\to\infty} \frac{|f(x)|}{|x|} = \infty. \tag{61}$$

If, for some $M>0$, *every nonoscillatory solution* $x(t)$ *of equation* (58) *satisfies* $\lim_{t\to\infty} x(t)/p_0(t)=0$, *then the same is true of every nonoscillatory solution of the equation*

$$L_n x + q(t)f(x) = 0. \tag{62}$$

PROOF. It suffices to show that (59) holds for every $c \in R-\{0\}$. Let c be any nonzero constant. Since $\liminf_{t\to\infty} p_0(t)>0$, there are $t_1>a$ and $\delta>0$ such that $|cp_0(t)|\geq\delta$ for $t\geq t_1$. Defining $\gamma=\inf_{|x|\geq\delta}(|f(x)|/|x|)$, we have $\gamma>0$ from (61). Hence

$$|f(cp_0(t))| \geq \gamma|cp_0(t)| \qquad \text{for} \quad t\geq t_1. \tag{63}$$

Now, from our hypothesis for equation (58) and Lemma 3 with $k=0$, it follows

$$M\int_{t_1}^{\infty} K_{n-1}(t)q(t)\,|cp_0(t)|dt = \infty. \tag{64}$$

(63) and (64) imply that (59) holds for any $c \in R-\{0\}$.

The following example shows that Theorem 6 becomes false if the divergence in the definition of weak superlinearity is not uniform with respect to t.

EXAMPLE 3. Consider the equation

$$(tx')'' + \frac{1}{8}t^{-2}(\log(e+t^{-1/2}))^{-1}x\log(e+t^{-1}|x|) = 0, \quad t\geq 1. \tag{65}$$

Here $n=3$, $p_0(t)=p_2(t)=p_3(t)=1$, $p_1(t)=t^{-1}$, $q(t)=\frac{1}{8}t^{-2}(\log(e+t^{-1/2}))^{-1}$ and $f(t, x)=x\log(e+t^{-1}|x|)$. It is easy to see that

$$\lim_{|x|\to\infty}\frac{|f(t, x)|}{|x|} = \lim_{|x|\to\infty}\log(e+t^{-1}|x|) = \infty,$$

but that the divergence is not uniform in $t \geq 1$. The associated linear equation is

$$(66) \qquad (tx')'' + \frac{M}{8} t^{-2}(\log(e+t^{-1/2}))^{-1}x = 0.$$

By Corollary 1 (ii) all nonoscillatory solutions of equation (66) are strongly decreasing for sufficiently large M. However equation (65) has a nonoscillatory solution $x(t) = t^{1/2}$, which is not strongly decreasing.

Our last theorem contains the result in the case that equation (2) is weakly sublinear.

THEOREM 7. *Suppose that equation* (2) *is weakly sublinear, and* $\liminf_{t\to\infty} p_0(t) > 0$. *If, for some* $m > 0$, *the equation*

$$(67) \qquad L_n x + mq(t)x = 0$$

has a nonoscillatory solution $x(t)$ *which does not satisfy* $\lim_{t\to\infty} x(t)/p_0(t) = 0$, *then so does equation* (2).

If in addition (59) *holds for every* $c \in R - \{0\}$, *then equation* (2) *has a nonoscillatory solution which is not strongly decreasing.*

PROOF. First, suppose (59) does not hold for some $c \in R - \{0\}$. By Lemma 3, equation (2) has a nonoscillatory solution $x(t)$ such that $\lim_{t\to\infty} x(t)/p_0(t) = a \neq 0$.

Next, assume that (59) is satisfied for every $c \in R - \{0\}$. By the weak sublinearity of equation (2) there is $\alpha > 0$ such that $|f(t, x)| \leq m|x|$ for $|x| \geq \alpha$, $t \geq a$. Taking $c \neq 0$ such that $|cp_0(t)| \geq \alpha$ for sufficiently large t, we obtain

$$|f(t, cp_0(t))| \leq m|cp_0(t)| \qquad \text{for sufficiently large } t.$$

Hence from (59) we see that

$$(68) \qquad \int^{\infty} K_{n-1}(t)q(t)p_0(t)dt = \infty.$$

Let $x(t)$ be a nonoscillatory solution of equation (67) which does not satisfy $\lim_{n\to\infty} x(t)/p_0(t) = 0$. Clearly $x(t)/p_0(t)$ is monotone, so that $\lim_{t\to\infty} x(t)/p_0(t)$ exists in the extended real line. By (68) Lemma 3 implies that $\lim_{t\to\infty} |x(t)/p_0(t)| = \infty$, therefore $x(t)$ is not strongly decreasing. From our assumption $\lim_{t\to\infty}|x(t)| = \infty$, so that $|f(t, x(t))| \leq m|x(t)|$ for sufficiently large t. Thus, for sufficiently large t,

$$\{L_n x(t) + q(t)f(t, x(t))\} \operatorname{sgn} x(t) \leq 0.$$

It follows from Lemma 4 that equation (2) has a nonoscillatory solution which is not strongly decreasing. The proof is complete.

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