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Invariant sequences in Brown-Peterson homology and some applications

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§0. Introduction

Let BP be the Brown-Peterson ring spectrum at p, where p is a prime number. Then

$$BP_* = Z_{(p)}[v_1, v_2, \ldots], \quad \dim v_n = 2(p^n - 1),$$

where the v_n's are Hazewinkel's generators. A sequence of elements $a_0, a_1, \ldots, a_s$ of BP_* is said to be *invariant* if

$$\eta_R a_i = \eta_L a_i \bmod (a_0, a_1, \ldots, a_{i-1}) \cdot BP_*BP \qquad \text{for} \quad i = 0, 1, \ldots, s,$$

where η_R, η_L: $BP_* \to BP_*BP$ are the right and the left units of the Hopf algebroid BP_*BP over BP_*.

The purpose of this note is to prove the following

THEOREM 1.5. *Let $s_0, s_1, \ldots, s_n$ be positive integers, and let p^{e_i} be the largest power of p dividing s_i. Then the sequence $p^{s_0}, v_1^{s_1}, \ldots, v_n^{s_n}$ is invariant if and only if $s_0 - 1 \leqq e_1$ and $s_i \leqq p^{e_{i+1} - s_0 + 1}$ for $i = 1, \ldots, n-1$.*

The case $s_0 = 1$ of this theorem has been given by Baird [4; Lemma 7.6].

As an application, we obtain some γ-elements in H^3BP_* of order p^{s_0} in Corollary 2.5 (p: odd prime). Furthermore, we consider the non-realizability of some cyclic BP_*-modules in Corollary 2.7.

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§1. Invariant sequences in BP_*

Let p be a prime number, and let BP denote the Brown-Peterson ring spectrum at p. Then, it is known that

$$BP_* = Z_{(p)}[v_1, v_2, \ldots, v_n, \ldots], \quad \dim v_n = 2(p^n - 1),$$

where the v_n's are Hazewinkel's generators, and the Hopf algebroid

$$BP_*BP = BP_*[t_1, t_2, \ldots, t_n, \ldots], \quad \dim t_n = 2(p^n - 1),$$

over BP_* admits the left unit and the right unit

$$\eta_L\colon BP_* \longrightarrow BP_*BP, \quad \eta_R\colon BP_* \longrightarrow BP_*BP$$

satisfying the following equalities:

$$\eta_L v_n = v_n; \tag{1.1}$$

$$\eta_R v_n = v_n + \textstyle\sum_{i=1}^{n-1} v_{n-i} f_{n,i} + p f_{n,n} \tag{1.2}$$

where

(1.3) *$f_{n,i} \in BP_*BP$ is a polynomial in $t_1,\dots, t_n$ with coefficients in $Z_{(p)}[v_1,\dots, v_{n-1}]$ and $f_{n,i}=t_i^{p^{n-i}}+$monomials having lower degree with respect to t_i, for $i=1,\dots, n$.*

(Cf. [1], [5]; especially, we see immediately (1.2–3) by the results of Hazewinkel [2; Lemma 6.2].)

DEFINITION 1.4. An ideal I of BP_* is said to be *invariant* if $I\cdot BP_*BP = BP_*BP\cdot I$; and an element $a\in BP_*$ is said to be *invariant mod I* if $\eta_R a\equiv\eta_L a \bmod I\cdot BP_*BP$. A sequence $a_0, a_1,\dots, a_s$ of elements in BP_* is said to be *invariant* if a_i is invariant modulo the ideal $(a_0,\dots, a_{i-1})$ generated by $a_0,\dots, a_{i-1}$ for $i=0, 1,\dots, s$.

The purpose of this section is to prove the following

THEOREM 1.5. *Let p be a prime number and $s_0, s_1,\dots, s_n$ $(n\geqq 1)$ be positive integers, and let p^{e_i} be the largest power of p dividing s_i. Then, the sequence*

$$p^{s_0}, v_1^{s_1},\dots, v_n^{s_n}$$

of elements in BP_* is invariant if and only if

$$s_0 - 1 \leqq e_1 \quad \text{and} \quad s_i \leqq p^{e_{i+1}-s_0+1} \qquad \text{for} \quad i = 1,\dots, n-1. \tag{1.6}$$

PROOF. (Sufficiency) In (1.2), we put

$$F_k = v_n + \textstyle\sum_{i=1}^{n-k} v_{n-i} f_{n,i} \qquad \text{for} \quad k = 1,\dots, n,$$

and we shall prove the following (1.7) by the induction on k:

(1.7) If $s_0-1\leqq e_n$ and $s_i\leqq p^{e_n-s_0+1}$ for $i=1,\dots, k-1$, then

$$\eta_R(v_n^{s_n}) \equiv F_k^{s_n} \bmod J_k\cdot BP_*BP \qquad (J_k = (p^{s_0}, v_1^{s_1},\dots, v_{k-1}^{s_{k-1}})).$$

Since $\eta_R v_n=F_1+pf$ $(f=f_{n,n})$ by definition,

$$\eta_R(v_n^{s_n}) = \textstyle\sum_{j=0}^{s_n}\binom{s_n}{j}p^j F_1^{s_n-j}f^j.$$

If $s_0-1\leqq e_n$, then $\binom{s_n}{j}\equiv 0 \bmod p^{s_0-j}$ for $1\leqq j<s_0$ because $s_n\equiv 0 \bmod p^{e_n}$. Thus, the above equality implies (1.7) for $k=1$.

By the same way, for $k\geqq 1$, $F_k=F_{k+1}+v_kf$ $(f=f_{n,n-k})$ and hence

$$F_k^{s_n} = \sum_{j=0}^{s_n}\binom{s_n}{j}F_{k+1}^{s_n-j}v_k^jf^j \equiv F_{k+1}^{s_n} \bmod (p^{s_0}, v_k^{s_k})\cdot BP_*BP$$

if $s_k\leqq p^{e_n-s_0+1}$, because $\binom{s_n}{j}\equiv 0 \bmod p^{s_0}$ for $1\leqq j<s_k$. Thus, we see (1.7) by induction.

Now, the conclusion of (1.7) for $k=n$ means that $v_n^{s_n}$ is invariant mod J_n. Therefore, (1.6) is a sufficient condition.

(Necessity) Let $p^{s_0}, v_1^{s_1},\dots, v_n^{s_n}$ $(n\geqq 1)$ be invariant. Then

(1.8) $$s_0-1\leqq e_n \quad \text{and} \quad s_i\leqq p^{e_n} \qquad \text{for} \quad i=1,\dots,n-1.$$

In fact, consider the ideal

$$J_{n,i} = (v_0,\dots, v_{i-1}, v_i^{s_i}, v_{i+1},\dots, v_{n-1}) \qquad (v_0 = p)$$

containing J_n for $i=0, 1,\dots, n-1$. Then, $v_n^{s_n}$ is invariant mod $J_{n,i}$, and

(*) $$v_n^{s_n} \equiv \eta_R(v_n^{s_n}) \equiv (v_n + v_if)^{s_n} = \sum_{j=0}^{s_n}\binom{s_n}{j}v_n^{s_n-j}v_i^jf^j \bmod J_{n,i}\cdot BP_*BP$$

by (1.1–2), where $f=f_{n,n-i}$ satisfies (1.3). Therefore, we see that

$$\binom{s_n}{j}v_i^jf^j \equiv 0 \bmod J_{n,i}\cdot BP_*BP \qquad \text{for} \quad 1\leqq j\leqq s_n,$$

and hence

$$ps_n \equiv 0 \bmod p^{s_0}, \quad \text{if} \quad i=0;$$

$$s_i\leqq s_n \quad \text{and} \quad \binom{s_n}{j}\equiv 0 \bmod p \quad \text{for} \quad 1\leqq j\leqq s_i, \quad \text{if} \quad i\geqq 1.$$

Since p^{e_n} is the largest power of p dividing s_n, p^{e_n-k} is that dividing $\binom{s_n}{p^k}$ for $k\leqq e_n$. Thus, these imply (1.8).

Now, the first inequality in (1.6) is seen by that in (1.8) since the sequence p^{s_0}, $v_1^{s_1}$ is invariant. Assume inductively that the inequality in (1.6) holds for $i=1,\dots, n-2$ $(n\geqq 2)$. Then, the assumptions of (1.7) for $k=n-2$ hold, since $e_i\leqq e_n$ by (1.8). Therefore, $\eta_R(v_n^{s_n})\equiv F_{n-1}^{s_n} \bmod J_{n-1}\cdot BP_*BP$ by (1.7), and (*) for $i=n-1$ is also valid mod $J_n\cdot BP_*BP$. Thus, by the same way as the above proof, we see that

$$s_{n-1}\leqq s_n \quad \text{and} \quad \binom{s_n}{j}\equiv 0 \bmod p^{s_0} \qquad \text{for} \quad 1\leqq j<s_{n-1},$$

and hence $s_{n-1} \leqq p^{e_n - s_0 + 1}$. These show the necessity by induction. q.e.d.

§2. Some applications

In the first place, we consider some γ-elements in H^3BP_*.

Let p be an odd prime number. For positive integers s_1, s_2, s_3 with

$$s_1 \leqq p^{e_2}, \quad s_2 \leqq p^{e_3} \tag{2.1}$$

(p^{e_i} is the largest power of p dividing s_i), by using the invariant sequence p, $v_1^{s_1}$, $v_2^{s_2}$, $v_3^{s_3}$ in Theorem 1.5 for $s_0 = 1$, Miller-Ravenel-Wilson [4; Corollary 7.8] defined the element

$$\gamma_{s_3/s_2,s_1} = \eta(v_3^{s_3}/pv_1^{s_1}v_2^{s_2}) \in H^3BP_* \tag{2.2}$$

and proved that it is nontrivial unless $s_1 < s_2 = p^{e_3} = s_3$.

Now, let s_0, s_1, s_2 and s_3 be positive integers with

$$1 \leqq s_0 - 1 \leqq e_1 \quad \text{and} \quad s_i \leqq p^{e_{i+1} - s_0 + 1} \qquad \text{for} \quad i = 1, 2. \tag{2.3}$$

Then, the sequence p^{s_0}, $v_1^{s_1}$, $v_2^{s_2}$, $v_3^{s_3}$ is invariant by Theorem 1.5, and by the same way as the definition of the element in (2.2), this sequence determines the element

$$\gamma_{s_3/s_2,s_1,s_0} = \eta(v_3^{s_3}/p^{s_0}v_1^{s_1}v_2^{s_2}) \in H^3BP_*. \tag{2.4}$$

Since (2.3) implies (2.1) and $s_2 < p^{e_3}$, the element $\gamma_{s_3/s_2,s_1}$ in (2.2) is also defined and is nontrivial; and there holds clearly the relation

$$p^{s_0-1}\gamma_{s_3/s_2,s_1,s_0} = \gamma_{s_3/s_2,s_1} \quad \text{in} \quad H^3BP_*.$$

Thus, we have the following

COROLLARY 2.5. *Let p be an odd prime number. Then, for positive integers s_0, s_1, s_2 and s_3 with* (2.3), *the element $\gamma_{s_3/s_2,s_1,s_0} \in H^3BP_*$ of* (2.4) *is defined and is of order p^{s_0}.*

In the second place, we consider some cyclic BP_*-modules for any prime p. For the invariantness in Definition 1.4, we notice the following lemma which may be known:

LEMMA 2.6. *For positive integers $s_0, s_1, \ldots, s_n$ ($n \geqq 1$), the sequence p^{s_0}, $v_1^{s_1}, \ldots, v_n^{s_n}$ is invariant if and only if the ideal $(p^{s_0}, v_1^{s_1}, \ldots, v_n^{s_n})$ is invariant.*

PROOF. The necessity is seen immediately by definition.

Suppose that $J = (p^{s_0}, v_1^{s_1}, \ldots, v_n^{s_n})$ is invariant, and set

$$\eta_R v_k = v_k + f_k, \quad f_k \in Z_{(p)}[v_1, \ldots, v_{k-1}; t_1, \ldots, t_k]$$

by (1.2–3). Then, for $i=1,\ldots, n$,

$$(v_i + f_i)^{s_i} = \eta_R(v_i^{s_i}) \in BP_*BP \cdot J = J \cdot BP_*BP,$$

and we see that $\eta_R(v_i^{s_i}) - v_i^{s_i} = (v_i + f_i)^{s_i} - v_i^{s_i} \in (p^{s_0}, v_1^{s_1}, \ldots, v_{i-1}^{s_{i-1}}) \cdot BP_*BP$. Thus the sequence $p^{s_0}, v_1^{s_1}, \ldots, v_n^{s_n}$ is invariant, as desired. q.e.d.

COROLLARY 2.7. *Let p be a prime number and $s_0, s_1, \ldots, s_n$ $(n \geqq 1)$ be positive integers such that* (1.6) *does not hold. Then, there exists no finite CW-complex X whose BP-homology $BP_*(X)$ is isomorphic to $BP_*/(p^{s_0}, v_1^{s_1}, \ldots, v_n^{s_n})$ as BP_*-modules.*

PROOF. If there exists such a finite CW-complex X, then we see that the ideal $(p^{s_0}, v_1^{s_1}, \ldots, v_n^{s_n})$ is invariant by the same way as the proof of [7; Corollary 4], using the result of Landweber [3] that annihilator ideals of primitive elements are invariant. Thus, the corollary follows immediately from Theorem 1.5 and the above lemma. q.e.d.

EXAMPLE 2.8. For $n, m \geqq 0$ and $t, s \geqq 1$ with $tp^n > p^{m-n}$ and $t, s \not\equiv 0 \bmod p$, $BP_*/(p^{n+1}, v_1^{tp^n}, v_2^{sp^m})$ is not realizable; while the realizability of $BP_*/(p^{n+1}, v_1^{tp^n})$ was shown by S. Oka, (for any odd prime p, L. Smith [6] also showed that of $MU_*/(p^{n+1}, [CP(p-1)]^{tp^n})$).

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