HIROSHIMA MATH. J.
17 (1987), 415–431

Global existence of nonoscillatory solutions of perturbed general disconjugate equations

Takaŝi KUSANO and William F. TRENCH

(Received January 9, 1987)

1. Introduction

Let L_n be the general disconjugate operator

$$L_n = \frac{1}{p_n}\frac{d}{dt}\frac{1}{p_{n-1}}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_1}\frac{d}{dt}\frac{\cdot}{p_0} \qquad (n \geqq 2), \tag{1}$$

with $p_i > 0$ and $p_i \in C[a, \infty)$, $0 \leqq i \leqq n$. Let (1) be in canonical form [10] at ∞; i.e.,

$$\int^{\infty} p_j(t)dt = \infty, \quad 1 \leqq j \leqq n-1. \tag{2}$$

With the operator (1) we associate the quasi-derivatives $L_0 u, \ldots, L_{n-1}u$ defined by

$$L_0 u = \frac{u}{p_0}; \quad L_r u = \frac{1}{p_r}(L_{r-1}u)', \quad 1 \leqq r \leqq n-1. \tag{3}$$

We give conditions which imply that the equation

$$L_n u + f(t, L_0 u, \ldots, L_{n-1}u) = 0 \tag{4}$$

has solutions which behave as $t \to \infty$ like solutions of the unperturbed equation

$$L_n x = 0. \tag{5}$$

Several authors [e.g. 2, 3, 5, 6, 9] have studied perturbed disconjugate equations of the simpler form

$$L_n u + f(t, L_0 u) = 0. \tag{6}$$

The more general equation (4), in which the perturbing terms depend also on $L_1 u, \ldots, L_{n-1}u$ have been studied in [4], [11] and [12]. However, to the authors' knowledge, all the results previously obtained for nonlinear equations of the forms (4) or (6) are "local" near ∞, in that the desired solutions are shown to exist only for t sufficiently large. Although one of our results given below is a local theorem of this kind which extends a result of Fink and Kusano [4], our main thrust here is in the direction of global results, in which the desired solution is shown to exist on a given interval. This continues a theme —global existence of solutions of nonlinear equations with specified asymptotic properties— which has recently been developed by the authors in [7] and [8].

2. Formulation of the problem

Following Willett [13], we define the iterated integrals $I_0=1$,

$$I_j(t, s; q_j,\dots, q_1) = \int_s^t q_j(\lambda)I_{j-1}(\lambda, s; q_{j-1},\dots, q_1)d\lambda, \quad s, t \geqq a, \quad j \geqq 1,$$

where $q_1, q_2,\dots$ are locally integrable on $[a, \infty)$. It is easily verified that the functions

$$x_i(t) = p_0(t)I_{i-1}(t, a; p_1,\dots, p_{i-1}), \quad 1 \leqq i \leqq n, \tag{7}$$

form a fundamental system for (5) on $[a, \infty)$, and that the functions

$$y_i(t) = p_n(t)I_{n-i}(t, a; p_{n-1},\dots, p_i), \quad 1 \leqq i \leqq n, \tag{8}$$

are similarly related to the formal adjoint equation

$$L_n^* y = \frac{1}{p_0}\frac{d}{dt}\frac{1}{p_1}\cdots\frac{1}{p_{n-1}}\frac{d}{dt}\frac{y}{p_n} = 0.$$

From (3) and (7),

$$L_r x_i(t) = I_{i-r-1}(t, a; p_{r+1},\dots, p_{i-1}), \quad 0 \leqq r \leqq i-1, \tag{9}$$

and

$$L_r x_i = 0, \quad i \leqq r \leqq n.$$

Because of (2) and Lemma 2 of [11],

$$\lim_{t\to\infty}\frac{L_r x_j(t)}{L_r x_i(t)} = \infty, \quad r < i < j \leqq n, \tag{10}$$

and

$$\lim_{t\to\infty}\frac{y_i(t)}{y_j(t)} = \infty, \quad 1 \leqq i < j < n, \tag{11}$$

and the ratios in (10) and (11) are increasing on $[a, \infty)$. For reasons which will become clear below, it is also convenient to define

$$d_{ir}(t) = \begin{cases} I_{i-r-1}(t, a; p_{r+1},\dots, p_{i-1}), & 0 \leqq r \leqq i-1, \\ 1/I_{r-i-1}(t, a; p_r,\dots, p_i), & i \leqq r \leqq n-1, \end{cases} \quad 1 \leqq i \leqq n. \tag{12}$$

It is important to notice that

$$d_{ir} = L_r x_i, \quad 0 \leqq r \leqq i-1, \quad 1 \leqq i \leqq n \tag{13}$$

(cf. (9)), and that (again because of (2))

$$\lim_{t\to\infty} \frac{d_{mr}(t)}{d_{ir}(t)} = \infty, \quad 0 \leqq r \leqq n-1, \quad 1 \leqq i < m \leqq n. \tag{14}$$

Because of this, there is a $b > a$ such that

$$d_{ir}(t) \leqq d_{mr}(t), \quad 0 \leqq r \leqq n-1, \quad 1 \leqq i < m \leqq n, \quad t \geqq b. \tag{15}$$

Equation (4) is related to (5) in the same way that the equation

$$u^{(n)} + f(t, u, \ldots, u^{(n-1)}) = 0 \tag{16}$$

is related to

$$x^{(n)} = 0, \tag{17}$$

since (4) and (5) reduce to (16) and (17) if

$$p_1 = \cdots = p_n = 1. \tag{18}$$

In order to gain insight into the results given below, the reader may wish to interpret them in the case where (18) applies. We believe that our global existence theorems are new even in this case (Beesack [1] has obtained different global existence results for (16), by methods based on a generalization of Bihari's inequality). Note that

$$I_k(t, a; p_{i_1}, \ldots, p_{i_k}) = \frac{(t-a)^k}{k!} \tag{19}$$

if (18) holds.

Throughout this paper i and m are integers, with $1 \leqq i \leqq m \leqq n$, and

$$q = \textstyle\sum_{j=i}^{m} b_j x_j \tag{20}$$

(see (7)) is a given solution of (5). We give various conditions which imply that (4) has a solution $\hat{u}$ such that

$$L_r \hat{u} = L_r q + o(d_{ir}), \quad 0 \leqq r \leqq n-1 \tag{21}$$

(where we use "o" in the standard manner to indicate behavior as $t \to \infty$). In the simpler case (18), (19) implies that

$$q(t) = \textstyle\sum_{j=i}^{m} b_j \dfrac{(t-a)^{j-1}}{(j-1)!}, \tag{22}$$

and (21) becomes

$$\hat{u}^{(r)}(t) = q^{(r)}(t) + o(t^{i-r-1}), \quad 0 \leqq r \leqq n-1;$$

thus the constants $b_i, \ldots, b_m$ in (22) are all significant in describing the behavior of $\hat{u}^{(r)}$ $(0 \leqq r \leqq m-1)$ as $t \to \infty$. Because of (10), (13) and (14), a similar comment applies to the general case; i.e., $b_i, \ldots, b_m$ are all significant in describing the behavior of $L_r\hat{u}$ $(0 \leqq r \leqq m-1)$ as $t \to \infty$.

3. A fundamental lemma

All our results in Section 4 can be obtained by direct application of the Schauder-Tychonoff fixed theorem. However, to avoid repetition, we will use this theorem just once to prove the following fundamental lemma. Since the hypotheses of this lemma are easy to check in specific situations, we believe that it should be widely useful as a substitute for the direct application of the Schauder-Tychonoff theorem to problems of this kind.

LEMMA 1. *Let q be the given solution* (20) *of* (5). *Suppose that $t_0 \geqq b$ (cf.* (15)) *and there is a constant $M > 0$ such that the function $f(t, u_0, \ldots, u_{n-1})$ is continuous and satisfies the inequality*

$$|f(t, u_0, \ldots, u_{n-1})| \leqq W(t) \tag{23}$$

on the set

$$S = \{(t, u_0, \ldots, u_{n-1}) \mid |u_r - L_r q(t)| \leqq M d_{ir}(t),\ 0 \leqq r \leqq n-1,\ t \geqq t_0\}, \tag{24}$$

where W is continuous on $[b, \infty)$ and

$$\int_{t_0}^{\infty} y_i(t) W(t) dt \leqq M, \tag{25}$$

with y_i as in (8). *Let*

$$\rho(t) = \int_t^{\infty} y_i(s) W(s) ds. \tag{26}$$

Then (4) *has a solution $\hat{u}$ on $[t_0, \infty)$ such that*

$$L_r\hat{u} = L_r q + o(L_r x_i), \quad 0 \leqq r \leqq i-2, \tag{27}$$

and

$$|L_r\hat{u}(t) - L_r q(t)| \leqq \rho(t) d_{ir}(t), \quad t \geqq t_0, \quad i-1 \leqq r \leqq n-1. \tag{28}$$

The following lemma will be used to prove Lemma 1.

LEMMA 2. *Suppose $Q \in C[t_0, \infty)$ and*

$$\int_{t_0}^{\infty} y_i(s)\,|Q(s)|ds < \infty. \tag{29}$$

Then the integral

$$\hat{J}_i(t;\, Q) = \int_t^{\infty} p_n(s) I_{n-i}(t,\, s;\, p_i, \ldots, p_{n-1}) Q(s) ds \tag{30}$$

converges absolutely for $t \geqq t_0$. *Now define*

$$\sigma(t) = \int_t^{\infty} y_i(s)\,|Q(s)|ds \tag{31}$$

and

$$J_i(t,\, t_0;\, Q) = p_0(t)\hat{J}_i(t;\, Q) \quad \textit{if} \quad i = 1;$$

or

$$J_i(t,\, t_0;\, Q) = p_0(t) I_1(t,\, t_0;\, p_1 \hat{J}_i(\cdot\,;\, Q)) \quad \textit{if} \quad i = 2;$$

or

$$J_i(t,\, t_0;\, Q) = p_0(t) I_{i-1}(t,\, t_0;\, p_1, \ldots, p_{i-2},\, p_{i-1}\hat{J}_i(\cdot\,;\, Q)) \quad \textit{if} \quad 3 \leqq i \leqq n.$$

Then

$$L_r J_i(t,\, t_0;\, Q) = \int_t^{\infty} p_n(s) I_{n-r-1}(t,\, s;\, p_{r+1}, \ldots, p_{n-1}) Q(s) ds, \qquad i-1 \leqq r \leqq n-1 \tag{32}$$

(*where the integrals converge absolutely*);

$$L_{i-2} J_i(t,\, t_0;\, Q) = I_1(t,\, t_0;\, p_{i-1}\hat{J}_i(\cdot\,;\, Q)) \quad \textit{if} \quad i \geqq 2; \tag{33}$$

$$L_r J_i(t,\, t_0;\, Q) = I_{i-r-1}(t,\, t_0;\, p_{r+1}, \ldots, p_{i-2},\, p_{i-1}\hat{J}_i(\cdot\,;\, Q)), \qquad 0 \leqq r \leqq i-3 \quad \textit{if} \quad i \geqq 3; \tag{34}$$

and

$$L_n J(t,\, t_0;\, Q) = -\,Q(t). \tag{35}$$

Moreover,

$$|L_r J_i(t,\, t_0;\, Q)| \leqq \sigma(t_0) L_r x_i(t), \quad 0 \leqq r \leqq i-2, \tag{36}$$

$$L_r J_i(t,\, t_0;\, Q) = o(L_r x_i(t)), \qquad 0 \leqq r \leqq i-2, \tag{37}$$

and

$$|L_r J_i(t,\, t_0;\, Q)| \leqq \sigma(t) d_{ir}(t), \quad i-1 \leqq r \leqq n-1. \tag{38}$$

PROOF. The formal verification of (32)–(35) is straightforward from (3).

To establish the absolute convergence of the integrals in (30) and (32) and to obtain the estimates (36) and (38), we employ an argument of Fink and Kusano [4]. From Lemma 2.2 of Willett [13],

$$(39)\quad I_{n-r-1}(t, s; p_{r+1},\dots, p_{n-1}) = (-1)^{n-r-1}I_{n-r-1}(s, t; p_{n-1},\dots, p_{r+1})$$

and

$$I_{n-i}(s, a; p_{n-1},\dots, p_i) = \sum_{\nu=0}^{n-1} I_{n-i-\nu}(s, t; p_{n-1},\dots, p_{\nu+i})I_\nu(t, a; p_{\nu+i-1},\dots, p_i).$$

If $s\geqq t\geqq a$, then all the iterated integrals here are nonnegative; therefore, if $i-1\leqq r\leqq n-1$ we can single out the term $\nu=r-i+1$ on the right side and conclude that

$$I_{n-r-1}(s, t; p_{n-1},\dots, p_{r+1})I_{r-i+1}(t, a; p_r,\dots, p_i) \leqq I_{n-i}(s, a; p_{n-1},\dots, p_i), \quad s \geqq t \geqq a.$$

This and (8), (12), and (39) imply that

$$(40)\quad p_n(s)\,|I_{n-r-1}(t, s; p_{r+1},\dots, p_{n-1})| \leqq y_i(s)d_{ir}(t), \quad s \geqq t \geqq a, \quad i-1 \leqq r \leqq n-1.$$

This and (29) imply the absolute convergence of the integrals in (30) and (32). With (31) it also implies (38). With $r=i-1$, (40) implies that

$$|\hat{J}_i(t, t_0; Q)| \leqq \sigma(t)$$

(see (30) and (31)); hence, (36) follows from (9), (33), and (34). Finally, (37) follows from (2), (33), (34), and L'Hospital's rule. This completes the proof of Lemma 2.

PROOF OF LEMMA 1. Let $\mathscr{L}_{n-1}[t_0, \infty)$ be the set of functions v such that $L_0v,\dots, L_{n-1}v$ are continuous on $[t_0, \infty)$, with the topology of uniform convergence on finite intervals; i.e., if $\{v_k\}$ is a sequence of functions in $\mathscr{L}_{n-1}[t_0, \infty)$, we write $v_k\to v$ if

$$\lim_{k\to\infty} L_rv_k(t) = L_rv(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1,$$

and all limits are uniform on $[t_0, t_1]$ for every $t_1\geqq t_0$. Let $V=V(t_0, q, m, i)$ be the colsed convex subset of $\mathscr{L}_{n-1}[t_0, \infty)$ consisting of functions v such that

$$(41)\quad |L_rv(t)-L_rq(t)| \leqq Md_{ir}(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1.$$

Our assumptions on f imply that the Nemitskii function Fv defined by

$$(Fv)(t) = f(t, L_0v(t),\dots, L_{n-1}v(t))$$

is continuous on $[t_0, \infty)$, and that

$$|(Fv)(t)| \leqq W(t), \quad t \geqq t_0, \tag{42}$$

if $v \in V$. Now define the transformation $\mathscr{T}$ by

$$(\mathscr{T}v)(t) = q(t) + J_i(t, t_0; Fv). \tag{43}$$

We will show that $\mathscr{T}$ satisfies the hypotheses of the Schauder-Tychonoff theorem on V.

From (26) and (42),

$$\int_t^\infty y_i(s)\,|(Fv)(s)|\,ds \leqq \rho(t), \quad t \geqq t_0,$$

for every v in V; hence, Lemma 2 with $Q = Fv$ implies that $\mathscr{T}v \in \mathscr{L}_{n-1}[t_0, \infty)$, and that

$$|L_r(\mathscr{T}v)(t) - L_r q(t)| \leqq \begin{cases} \rho(t_0) L_r x_i(t), & 0 \leqq r \leqq i-2, \\ \rho(t) d_{ir}(t), & i-1 \leqq r \leqq n-1, \end{cases} \quad t \geqq t_0.$$

Therefore, (25) and the definition of V (see (41) and recall that $d_{ir} = L_r x_i$, $0 \leqq r \leqq i-1$) imply that $\mathscr{T}v \in V$. Therefore, we conclude that

$$\mathscr{T}(V) \subset V. \tag{44}$$

To see that $\mathscr{T}$ is continuous on V, suppose that $\{v_k\}$ is a sequence in V such that $v_k \to v$. Then $|Fv_k - Fv| \leqq 2W$ (see (42)), so (25) and Lebesgue's domainated convergence theorem imply that

$$\lim_{k\to\infty} \int_{t_0}^\infty y_i(s)\,|(Fv_k)(s) - (Fv)(s)|\,ds = 0.$$

Therefore, if $\varepsilon > 0$, there is a k_0 such that

$$\left| \int_t^\infty y_i(s)[(Fv_k)(s) - (Fv)(s)]\,ds \right| < \varepsilon, \quad t \geqq t_0, \quad k \geqq k_0.$$

Now Lemma 2 with $Q = Fv_k - Fv$ implies that

$$|L_r(\mathscr{T}v_k)(t) - L_r(\mathscr{T}v)(t)| \leqq \varepsilon d_{ir}(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1, \quad k \geqq k_0.$$

This implies that $\mathscr{T}v_k \to \mathscr{T}v$; i.e., that $\mathscr{T}$ is continuous on V.

From (44) and the definition of V, it is clear that the family of vector functions

$$\{(L_0 v, \ldots, L_{n-1} v) \mid v \in V\} \tag{45}$$

is equibounded on every $[t_0, T]$ with $T \geqq t_0$; moreover, since (35) (with $Q = Fv$)

and (43) imply that

$$L_n(\mathscr{T}v) = -Fv, \tag{46}$$

(42) also implies that the family (45) is equicontinuous on $[t_0, T]$ for every $T \geqq t_0$. Hence $\mathscr{T}(V)$ has compact closure, by the Arzela-Ascoli theorem.

Now the Schauder-Tychonoff theorem implies that there is a function $\hat{u}$ in V such that $\mathscr{T}\hat{u} = \hat{u}$. Letting $v = \hat{u}$ in (46) shows that $\hat{u}$ satisfies (4) on $[t_0, \infty)$ (recall (42)). Moreover, since $\hat{u} = \mathscr{T}\hat{u}$, (43) (with $v = \hat{u}$) and Lemma 2 (with $Q = F\hat{u}$) imply that $\hat{u}$ satisfies (27) and (28). This completes the proof of Lemma 1.

4. Specific results

Our first two theorems require the following assumption.

ASSUMPTION A. The function f: $[a, \infty) \times \boldsymbol{R}^n \to \boldsymbol{R}$, is continuous and satisfies the inequality

$$|f(t, u_0, \dots, u_{n-1})| \leqq F(t, |u_0|, \dots, |u_{n-1}|), \tag{47}$$

where $F(t, v_0, \dots, v_{n-1})$ is continuous and nonnegative for $t \geqq a$, $v_r \geqq 0$ $(0 \leqq r \leqq n-1)$, and nondecreasing with respect to each v_r. Also,

$$\int^{\infty} y_i(t) W(t; \lambda) dt < \infty$$

for some $\lambda > 0$, where

$$W(t; \lambda) = F(t, \lambda L_0 x_m(t), \dots, \lambda L_{m-1} x_m(t), \lambda d_{im}(t), \dots, \lambda d_{i,n-1}(t)). \tag{48}$$

Note: If $m = n$, then (48) becomes

$$W(t; \lambda) = F(t, \lambda L_0 x_n(t), \dots, \lambda L_{n-1} x_n(t)).$$

It is convenient to define

$$\sigma(t_0; \lambda) = \int_{t_0}^{\infty} y_i(t) W(t; \lambda) dt. \tag{49}$$

The following theorem extends a result of Fink and Kusano [4], applicable to the case where $m = i$.

THEOREM 1. *Suppose Assumption* A *holds and let q be as in* (20), *with $|b_m| < \lambda$ and* (*if $i < m$*) *$b_i, \dots, b_{m-1}$ arbitrary. Then* (4) *has a solution $\hat{u}$ on $[t_0, \infty)$ such that*

$$L_r \hat{u} - L_r q = \begin{cases} o(L_r x_i), & 0 \leqq r \leqq i-1, \\ o(d_{ir}), & i \leqq r \leqq n-1, \end{cases}$$

provided that t_0 is sufficiently large.

PROOF. Choose $\alpha > 1$ such that $\alpha|b_m| < \lambda$. Then choose $t_0 > b$ (cf. (15)) such that

$$|L_r q(t)| \leqq \alpha|b_m| L_r x_m(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq m-1,$$

and

$$\sigma(t_0; \lambda) \leqq \lambda - \alpha|b_m|.$$

Since

$$d_{ir}(t) \leqq d_{mr}(t) = L_r x_m(t), \quad t \geqq b, \quad 0 \leqq r \leqq m-1, \tag{50}$$

we can now infer the conclusion from Lemma 1, with $W(t) = W(t; \lambda)$ and $M = \lambda - \alpha|b_m|$.

Theorem 1 is "local near ∞", in that $\hat{u}$ is guaranteed to exist only for t sufficiently large. The following theorems are global, in that the desired solution is guaranteed to exist on a given interval $[t_0, \infty)$.

THEOREM 2. *In addition to Assumption* A, *suppose that*

$$\lambda^{-1}\sigma(t_0; \lambda) \leqq \gamma < 1 \tag{51}$$

for some $t_0 \geqq b$ and $\lambda > 0$. Let

$$p = x_m + \textstyle\sum_{j=i}^{m-1} b_j x_j, \tag{52}$$

where (if $i < m$) $b_i, \dots, b_{m-1}$ are arbitrary constants. Define

$$\mu = \sup_{t \geqq t_0} \max_{0 \leqq r \leqq m-1} \frac{|L_r p(t)|}{L_r x_m(t)}. \tag{53}$$

Now suppose that c is any constant such that

$$0 < |c| \leqq \frac{\lambda(1-\gamma)}{\mu}. \tag{54}$$

Then (4) *has a solution $\hat{u}$ which is defined on $[t_0, \infty)$ and has the asymptotic behavior*

$$L_r \hat{u} - c L_r p = \begin{cases} o(L_r x_i), & 0 \leqq r \leqq i-1, \\ o(d_{ir}) & i \leqq r \leqq n-1. \end{cases} \tag{55}$$

PROOF. If c satisfies (54), then we can write

$$|c| = \frac{\lambda}{\theta + \mu}, \tag{56}$$

where

$$\frac{\theta}{\mu+\theta} \geqq \gamma. \tag{57}$$

We now apply Lemma 1 with $q=cp$, $W(t)=W(t;\lambda)$, and $M=\theta|c|$: if

$$|u_r - cL_rp(t)| \leqq \theta|c|d_{ir}(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1,$$

then (50) and (53) imply that

$$|u_r| \leqq (\mu+\theta)|c|L_rx_m(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq m-1,$$

and

$$|u_r| \leqq \theta|c|d_{ir}(t) < (\mu+\theta)|c|d_{ir}(t), \quad t \geqq t_0, \quad m \leqq r \leqq n-1;$$

hence (56) implies that

$$|f(t, u_0,\dots, u_{n-1})| \leqq W(t;\lambda), \quad t \geqq t_0.$$

This verifies (23) on the set S as in (24), with $M=\theta|c|$ and $q=cp$. Now (51), (56), and (57) imply (25) with $M=\theta|c|$, and Lemma 1 implies the conclusion.

COROLLARY 1. *Suppose Assumption* A *holds, let p be as in* (52), *and let c be a given nonzero constant. Then* (4) *has a solution which is defined on* $[t_0, \infty)$ *and satisfies* (55), *provided that*

$$\text{(i)} \qquad \limsup_{\lambda\to\infty} \lambda^{-1}\sigma(t_0;\lambda) = \hat{\gamma} < 1 \tag{58}$$

or

(ii) $|c|$ *is sufficiently small and*

$$\limsup_{\lambda\to 0+} \lambda^{-1}\sigma(t_0;\lambda) = \hat{\gamma} < 1. \tag{59}$$

PROOF. Suppose that $\hat{\gamma}<\gamma<1$. If assumption (i) holds, choose λ sufficiently large so that (51) and (54) (with given c) both hold; then Theorem 2 implies the conclusion. If (59) holds, choose λ sufficiently small so that (51) holds. Then (54) holds for sufficiently small $|c|$ ($\neq 0$), and again the conclusion follows.

Corollary 1 has nontrivial applications to equations of the form

$$L_nu + \textstyle\sum_{r=0}^{n-1} P_{n-r}(t)L_ru + g(t, L_0u,\dots, L_{n-1}u) = 0, \tag{60}$$

as follows.

COROLLARY 2. *Suppose that $P_1,\dots, P_n \in C[a, \infty)$ and*

$$\int^\infty y_i(t)|P_{n-r}(t)|L_rx_m(t)dt < \infty, \quad 0 \leqq r \leqq m-1,$$

$$\int^{\infty} y_i(t)\,|P_{n-r}(t)|d_{ir}(t)dt < \infty, \quad m \leqq r \leqq n-1.$$

Let $g\colon [a, \infty) \times \boldsymbol{R}^n \to \boldsymbol{R}$ *be continuous and satisfy the inequality*

$$|g(t, u_0, \ldots, u_{n-1})| \leqq G(t, |u_0|, \ldots, |u_{n-1}|),$$

where $G(t, v_0, \ldots, v_{n-1})$ *is continuous and nonnegative for* $t \geqq a$, $v_r \geqq 0$ $(0 \leqq r \leqq n-1)$, *and nondecreasing with respect to each* v_r, *and*

$$\int^{\infty} y_i(t)U(t;\lambda)dt < \infty \tag{61}$$

for all λ, *with*

$$U(t;\lambda) = G(t, \lambda L_0 x_m(t), \ldots, \lambda L_{m-1}x_m(t), \lambda d_{im}(t), \ldots, \lambda d_{i,n-1}(t)). \tag{62}$$

Suppose also that $t_0 \geqq b$ *and*

$$\int_{t_0}^{\infty} y_i(t)\left[\textstyle\sum_{r=0}^{m-1} |P_{n-r}(t)|L_r x_m(t) + \sum_{r=m}^{n-1} |P_{n-r}(t)|d_{ir}(t)\right]dt < 1, \tag{63}$$

and let p be as in (52). *Then* (60) *has a solution* $\hat{u}$ *which is defined on* $[t_0, \infty)$ *and satisfies* (55) *if either of the following hypotheses is satisfied*:

(H_1):

$$\lim_{\lambda \to \infty} \lambda^{-1} G(t, \lambda v_0, \ldots, \lambda v_{n-1}) = 0 \tag{64}$$

for every $(t, v_0, \ldots, v_{n-1})$ *in* $[t_0, \infty) \times \boldsymbol{R}_+^n$.

(H_2): $|c|$ *is sufficiently small and*

$$\lim_{\lambda \to 0+} \lambda^{-1} G(t, \lambda v_0, \ldots, \lambda v_{n-1}) = 0 \tag{65}$$

for every $(t, v_0, \ldots, v_{n-1})$ *in* $[t_0, \infty) \times \boldsymbol{R}_+^n$.

PROOF. Equation (60) is of the form (4), with

$$f(t, u_0, \ldots, u_{n-1}) = \textstyle\sum_{r=0}^{n-1} P_{n-r}(t)u_r + g(t, u_0, \ldots, u_{n-1}),$$

which satisfies (47) with

$$F(t, v_0, \ldots, v_{n-1}) = \textstyle\sum_{r=0}^{n-1} |P_{n-r}(t)|v_r + G(t, v_0, \ldots, v_{n-1}).$$

Therefore, from (48), (49), and (62),

$$\sigma(t_0, \lambda) = \lambda I(t_0) + \int_{t_0}^{\infty} y_i(t)U(t;\lambda)dt,$$

where $I(t_0)$ is the integral in (63). From (61), (62), (64) and Lebesgue's dominated convergence theorem,

$$\lim_{\lambda\to\infty} \lambda^{-1} \int_{t_0}^{\infty} y_i(t)U(t;\lambda)dt = 0$$

if (H_1) holds, and this together with (63) implies (58). Similarly, (65) implies (59) if (H_2) holds. Therefore, Corollary 1 implies the stated conclusion.

Note: It is sufficient that (61) holds for λ sufficiently small if (H_2) holds. The prototype form for g in (60) is

$$g(t, u_0,\dots, u_{n-1}) = \sum_{r=0}^{n-1} Q_{n-r}(t)\,|u_r|^{\gamma_r} \operatorname{sgn} u_r,$$

where $Q_0,\dots, Q_{n-1} \in C[a, \infty)$ and

$$\int^{\infty} y_i(t)\,|Q_{n-r}(t)|\,(L_r x_m(t))^{\gamma_r} dt < \infty, \quad 0 \leqq r \leqq m-1,$$

and

$$\int^{\infty} y_i(t)\,|Q_{n-r}(t)|\,(d_{ir}(t))^{\gamma_r} dt < \infty, \quad m \leqq r \leqq n-1.$$

Then (H_1) holds if $0<\gamma_r<1$ $(0\leqq r\leqq n-1)$, while (H_2) holds if $\gamma_r>1$ $(0\leqq r\leqq n-1)$.

THEOREM 3. *Suppose that the function* f: $[a, \infty)\times \boldsymbol{R}^n \to \boldsymbol{R}$ *is continuous and satisfies the inequality*

$$|f(t, u_0,\dots, u_{n-1})| \leqq F(t, u_0, |u_1|,\dots, |u_{n-1}|),$$

where $F(t, v_0,\dots, v_{n-1})$ *is continuous and nonnegative for* $t\geqq a$, $-\infty<v_0<\infty$, $0\leqq v_r<\infty$ $(1\leqq r\leqq n-1)$, *and nondecreasing with respect to each* v_r, *and*

$$\lim_{v_0\to-\infty} F(t, v_0,\dots, v_{n-1}) = 0 \tag{66}$$

for each $(t, v_1,\dots, v_{n-1})$ *such that* $t\geqq a$, $v_r\geqq 0$ $(1\leqq r\leqq n-1)$. *Let*

$$\rho(t, \lambda, \alpha) = F(t, \alpha+\lambda L_0 x_m(t), \lambda L_1 x_m(t),\dots, \lambda L_{m-1} x_m(t), \lambda d_{im}(t),\dots, \lambda d_{i,n-1}(t))$$

if $m>1$, *or*

$$\rho(t, \lambda, \alpha) = F(t, \alpha+\lambda, \lambda d_{11}(t),\dots, \lambda d_{1,n-1}(t))$$

if $m=1$, *and suppose that*

$$\int^{\infty} y_i(t)\rho(t, \lambda, 0)dt < \infty \tag{67}$$

for some $\lambda>0$. *Let* p *and* μ *be as in* (52) *and* (53), *respectively, and suppose that* $0<c\mu<\lambda$. *Let* $t_0\geqq b$ *be given. Then there is an* $\alpha_0\leqq 0$ *such that if* $\alpha\leqq\alpha_0$, *then* (4) *has a solution* $\hat{u}$ *which is defined on* $[t_0, \infty)$ *and exhibits the asymptotic* *behavior*

$$L_0\hat{u} - \alpha - cL_0 p = o(L_0 x_i),$$

$$L_r\hat{u} - cL_r p = \begin{cases} o(L_r x_i), & 1 \leqq r \leqq i-1, \\ o(d_{ir}), & i \leqq r \leqq n-1. \end{cases}$$

PROOF. Choose $\theta>0$ so that $c(\mu+\theta)<\lambda$. Then choose α_0 so that

$$\int_{t_0}^{\infty} y_i(t)\rho(t, \lambda, \alpha_0)dt \leqq c\theta, \quad \alpha \leqq \alpha_0,$$

(this is possible because of (66), (67), and Lebesgue's dominated convergence theorem). Now apply Lemma 1 with $M=c\theta$, $q=\alpha x_1+cp$, and $W(t)=\rho(t, \lambda, \alpha_0)$.

Theorem 3 applies, for example, to equations of the form

$$L_n u + e^{h(L_0 u)} g(t, L_1 u, \dots, L_{n-1} u) = 0, \tag{68}$$

as follows.

COROLLARY 3. *Suppose that the function* $g\colon [a, \infty)\times \boldsymbol{R}^{n-1}\to \boldsymbol{R}$ *is continuous and satisfies the inequality*

$$|g(t, u_1, \dots, u_{n-1})| \leqq G(t, |u_1|, \dots, |u_{n-1}|),$$

where $G(t, v_1, \dots, v_{n-1})$ *is continuous and nonnegative for* $t\geqq a$, $v_r\geqq 0\ (1\leqq r\leqq n-1)$, *and nondecreasing with respect to each* v_r. *Suppose also that* h *is continuous and nondecreasing on* $(-\infty, \infty)$, *and that*

$$\lim_{u_0\to-\infty} h(u_0) = -\infty.$$

Finally, suppose that, for some $\lambda>0$,

$$\int^{\infty} y_i(t)e^{h(\lambda L_0 x_m(t))}G(t, \lambda L_1 x_m(t), \dots, \lambda L_{m-1}x_m(t), \lambda d_{im}(t), \dots, \lambda d_{i,n-1}(t))dt < \infty$$

if $m>1$, *or*

$$\int^{\infty} y_1(t)G(t, \lambda d_{11}(t), \dots, \lambda d_{1,n-1}(t))dt < \infty$$

if $m=1$. *Then the conclusions of Theorem* 3 *apply to* (68).

5. Application to semilinear elliptic equations

Here we consider the semilinear elliptic equation of order $2n$,

$$\Delta^n v + \phi(|x|, v, \Delta v, \dots, \Delta^{n-1}v) = 0, \quad x\in\Omega_\rho, \tag{69}$$

where $x \in \boldsymbol{R}^2$, Δ is the two-dimensional Laplacian, Δ^i is the i-th iteration of Δ, $n \geqq 1$, and

$$\Omega_\rho = \{x \in \boldsymbol{R}^2 \mid |x| > \rho\}, \quad \rho > 0.$$

We will use the results of Section 4 to derive conditions which imply that (69) has radially symmetric solutions on Ω_ρ which have certain prescribed types of asymptotic behavior as $|x| \to \infty$.

It is easy to see that $v(x) = u(|x|)$ is a radially symmetric solution of (69) on Ω_ρ if and only if $u(t)$ is a solution of the ordinary differential equation

$$L_{2n}u + \phi(t, L_0 u, L_2 u, \dots, L_{2n-2}u) = 0, \quad t > \rho, \tag{70}$$

where

$$L_{2k} = \left(t^{-1} \frac{d}{dt} t \frac{d}{dt}\right)^k, \quad k = 0, 1, \dots, n;$$

thus

$$L_{2n} = \frac{1}{p_{2n}} \frac{d}{dt} \frac{1}{p_{2n-1}} \cdots \frac{1}{p_1} \frac{d}{dt} \frac{\cdot}{p_0}$$

with

$$p_0(t) = 1,$$
$$p_1(t) = p_3(t) = \cdots = p_{2n-1}(t) = t^{-1},$$
$$p_2(t) = p_4(t) = \cdots = p_{2n}(t) = t.$$

Straightforward computation based on (3), (7), (8) (with n replaced by $2n$) and (13) yields

$$y_{2j}(t) = \frac{t^{2n-2j+1}[1+o(1)]}{[2^{n-j}(n-j)!]^2}, \quad 1 \leqq j \leqq n,$$

$$y_{2j-1}(t) = \frac{t^{2n-2j+1} \log t \cdot [1+o(1)]}{[2^{n-j}(n-j)!]^2}, \quad 1 \leqq j \leqq n,$$

$$d_{2j,2k}(t) = \frac{t^{2(j-k-1)} \log t \cdot [1+o(1)]}{[2^{j-k-1}(j-k-1)!]^2}, \quad 0 \leqq k \leqq j-1,$$

$$d_{2j,2k}(t) = \frac{2(k-j+1)[2^{k-j}(k-j)!]^2}{t^{2(k-j+1)}}[1+o(1)], \quad j \leqq k \leqq n-1,$$

$$d_{2j-1,2k}(t) = \frac{t^{2(j-k+1)}[1+o(1)]}{[2^{j-k-1}(j-k-1)!]^2}, \quad 0 \leqq k \leqq j-1,$$

$$d_{2j-1,2k}(t) = \frac{2(k-j+1)[2^{k-j}(k-j)!]^2}{t^{2(k-j+1)} \log t}[1+o(1)], \quad j \leqq k \leqq n-1.$$

Now let j be an integer, $1 \leqq j \leqq n$, and let c be a given nonzero constant. We will give sufficient conditions for (69) to have a radially symmetric solution $\hat{v}$ on Ω_ρ such that either

$$\lim_{|x|\to\infty} \frac{\hat{v}(x)}{|x|^{2j-2}\log|x|} = c \tag{71}$$

or

$$\lim_{|x|\to\infty} \frac{\hat{v}(x)}{|x|^{2j-2}} = c. \tag{72}$$

ASSUMPTION B. The function ϕ: $(0, \infty)\times \boldsymbol{R}^n \to \boldsymbol{R}$ is continuous and satisfies the inequality

$$|\phi(t, u_0,\dots, u_{n-1})| \leqq \Phi(t, |u_0|,\dots, |u_{n-1}|),$$

where $\Phi(t, \xi_0,\dots, \xi_{n-1})$ is continuous and nonnegative for $t>0$, $\xi_r \geqq 0$ $(0\leqq r\leqq n-1)$, and nondecreasing with respect to each ξ_r.

THEOREM 4. *Suppose that Assumption* B *holds and there is a constant* $\lambda>0$ *such that*

$$\int^\infty t^{2n-2j+1}\Phi(t, \lambda t^{2(j-1)}\log t, \lambda t^{2(j-2)}\log t,\dots, \lambda\log t, \lambda t^{-2}, \lambda t^{-4},\dots, \lambda t^{-2(n-j)})dt < \infty. \tag{73}$$

Then, if $|c|(>0)$ *is sufficiently small, there is a* ρ *sufficiently large such that* (69) *has a solution* $\hat{v}$ *on* Ω_ρ *which satisfies* (71).

The proof of this theorem is obtained by applying Theorem 1 (with $m=i=2j$) to (70). We leave the details to the reader. Similar reasoning (with $m=i=2j-1$) yields the next theorem.

THEOREM 5. *Suppose that Assumption* B *holds and there is a constant* $\lambda>0$ *such that*

$$\int^\infty t^{2n-2j+1}(\log t)\Phi(t, \lambda t^{2(j-1)}, \lambda t^{2(j-2)},\dots, \lambda, \lambda(t^2\log t)^{-1}, \lambda(t^4\log t)^{-1},\dots, \lambda(t^{2(n-j)}\log t)^{-1})dt < \infty. \tag{74}$$

Then, if $|c|$ (>0) *is sufficiently small, there is a* ρ *sufficiently large such that* (69) *has a solution* $\hat{v}$ *on* Ω_ρ *which satisfies* (72).

The last two theorems are local near ∞, in that they guarantee the existence of $\hat{v}$ only for large $|x|$. In the following theorems, it is to be understood that ρ is a *given* positive number, so the results are global. Theorems 6 and 7 are obtained by applying Corollary 2 (and Remark 1) to (70).

THEOREM 6. *In addition to Assumption* B, *suppose that*

$$\lim_{\lambda\to\infty} \lambda^{-1}\Phi(t, \lambda\xi_0,\dots, \lambda\xi_{n-1}) = 0$$

for every $(t, \xi_0,\dots, \xi_{n-1})$ *such that* $t>0$ *and* $\xi_r \geqq 0$ $(0\leqq r\leqq n-1)$. *Let* $c\neq 0$ *be an arbitrarily given constant. Suppose that* (73) [(74)] *holds for some* $\lambda>0$. *Then* (69) *has a solution* $\hat{v}$ *on* Ω_ρ *which satisfies* (71) [(72)].

THEOREM 7. *In addition to Assumption* B, *suppose that*

$$\lim_{\lambda\to 0+} \lambda^{-1}\Phi(t, \lambda\xi_0,\dots, \lambda\xi_{n-1}) = 0$$

for every $(t, \xi_0,\dots, \xi_{n-1})$ *such that* $t>0$ *and* $\xi_r\geqq 0$ $(0\leqq r\leqq n-1)$. *Suppose that* (73) [(74)] *holds for some* $\lambda>0$. *Then* (69) *has a solution* $\hat{v}$ *on* Ω_ρ *which satisfies* (71) [(72)], *provided that* $|c|$ (>0) *is sufficiently small.*

We close by applying Corollary 3 to the equation

$$\Delta^n v + \psi(|x|)e^{h(v)} = 0, \quad x\in\Omega_\rho. \tag{75}$$

We remind the reader that ρ is a given positive number.

THEOREM 8. *Suppose that* $\psi\in C(0, \infty)$, h *is nondecreasing on* $(-\infty, \infty)$, *and* $\lim_{v\to-\infty} h(v)=-\infty$.

(i) *If*

$$\int^\infty t^{2n-1}(\log t)\,|\psi(t)|dt < \infty,$$

then there is a constant β_0 *such that if* $\beta<\beta_0$, *then* (75) *has a solution* $\hat{v}$ *on* Ω_ρ *such that* $\lim_{|x|\to\infty} \hat{v}(x)=\beta$.

(ii) *If* $2\leqq j\leqq n$ *and*

$$\int^\infty t^{2n-2j+1}(\log t)\,|\psi(t)|\,[\exp h(\lambda t^{2j-2})]dt < \infty$$

for some $\lambda>0$, *then* (75) *has a solution* $\hat{v}$ *on* Ω_ρ *which satisfies* (72), *provided that* c *is a sufficiently small positive constant.*

(iii) *If* $1\leqq j\leqq n$ *and*

$$\int^\infty t^{2n-2j+1}|\psi(t)|\,[\exp h(\lambda t^{2j-2}\log t)]dt < \infty$$

for some $\lambda>0$, *then* (75) *has a solution* $\hat{v}$ *on* Ω_ρ *which satisfies* (71), *provided that* c *is a sufficiently small positive constant.*

References

[1] P. R. Beesack, Asymptotic behaviour of solutions of some general nonlinear differential equations and integral inequalities, Proc. Royal Soc. Edinburgh, **98** Pt 1/2 (1984), 49–67.

[2] T. A. Čanturija, On monotone and oscillatory solutions of higher order ordinary differential equations, Ann. Polon. Math., **37** (1980), 93–111. (Russian)

[3] A. M. Fink and T. Kusano, Nonoscillation theorems for differential equations with general diviating arguments, Lecture Notes in Mathematics 1032, pp. 224–239, Springer, 1983.

[4] A. M. Fink and T. Kusano, Nonoscillation theorems for a class of perturbed disconjugate differential equations, Japan. J. Math., **9** (1983), 277–291.

[5] A. Granata, Singular Cauchy problems and asymptotic behavior for a class of *n*th order differential equations, Funkcial. Ekvac., **20** (1977), 193–212.

[6] Y. Kitamura and T. Kusano, Nonlinear oscillation of higher-order functional differential equations with a general deviating argument, J. Math. Anal. Appl., **77** (1980), 100–119.

[7] T. Kusano and W. F. Trench, Global existence theorems for solutions of nonlinear differential equations with prescribed asymptotic behaviour, J. London Math. Soc., (2) **31** (1985), 478–486.

[8] T. Kusano and W. F. Trench, Existence of global solutions with prescribed asymptotic behavior for nonlinear ordinary differential equations, Ann. Mat. Pura Appl., (4) **152** (1985), 381–392.

[9] M. Švec, Behavior of nonoscillatory solutions of some nonlinear differential equations, Acta Math. Univ. Comenian., **39** (1980), 115–130.

[10] W. F. Trench, Canonical forms and principal systems for general disconjugate equations, Trans. Amer. Math. Soc., **189** (1974), 319–327.

[11] W. F. Trench, Asymptotic theory of perturbed general disconjugate equations, Hiroshima Math. J., **12** (1982), 43–58.

[12] W. F. Trench, Asymptotic theory of perturbed general disconjugate equations II, Hiroshima Math. J., **14** (1984), 169–187.

[13] D. Willett, Asymptotic behaviour of disconjugate *n*-th order differential equations, Canad. J. Math., **23** (1971), 293–314.

Department of Mathematics,
Faculty of Science,
Hiroshima University
and
Department of Mathematics,
Trinity University