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Existence and qualitative theorems for nonnegative solutions of a similinear elliptic equation

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In this paper we study a qualitative feature of positive solutions for the Dirichlet problem

$$(0.1)\qquad \begin{aligned} &\Delta u(x) + f(u(x)) = 0 && \text{in } B_R \\ &u(x) = 0 && \text{on } \partial B_R, \end{aligned}$$

where $B_R = (x \in \mathbf{R}^N; |x| < R\}$, $N \geq 2$ and f is a continuous function on $[0, \infty)$ which satisfies the following conditions:

(A1) $\limsup_{s \to +0} f(s)/s \leq -m < 0$.

(A2) There exists a unique $\zeta_0 \in (0, \infty)$ such that $F(\zeta_0) = 0$, $F(\zeta) < 0$ for $\zeta \in (0, \zeta_0)$ and $f(\zeta_0) > 0$, where $F(\zeta) = \int_0^\zeta f(s)ds$,

(A3) $\alpha = \sup\{\zeta < \zeta_0; f(\zeta) = 0\}$ and $\beta = \inf\{\zeta > \zeta_0; f(\zeta) = 0\}$ exist and $0 < \alpha < \beta < \infty$.

(A4) f is Lipschitz continuous in a neighborhood of β.

We first establish an existence of positive radially symmetric solutions of (0.1) and study their shape. Hence they satisfy the following ordinally differential equation associated to (0.1)

$$(0.2)\qquad \begin{aligned} &u'' + \frac{N-1}{r} u' + f(u) = 0 \qquad \text{for} \quad 0 < r < R, \\ &u(0) = \mu, \qquad u'(0) = u(R) = 0, \end{aligned}$$

where u is now a function of $r = |x|$ alone ($x \in \mathbf{R}^N$). Then we show the following

THEOREM 1. *Under the conditions* $(A1)$–$(A4)$ *there exists an* $R_0 > 0$ *such that for any* $R > R_0$ *the equation* (0.2) *admits a positive solution with properties*

$$\zeta_0 < u(0) < \beta \text{ and } u' < 0 \text{ on } (0, R].$$

THEOREM 2. *Let* $R = \infty$ *and define* $u(\infty)$ *by* $\lim_{r \to \infty} u(r)$. *Under the conditions* $(A1)$–$(A4)$ *for some* $\mu \in (\zeta_0, \beta)$ *there exists a nonnegative solution* u *of* (0.2). *Let*

$$R_1 = \inf\{r > 0;\, u(r) = 0\}.$$

Then $u' < 0$ *on* $(0, R_1)$ *and* $u \equiv 0$ *on* (R_1, ∞) *if* $R_1 < \infty$.

According to L. A. Peletier and J. Serrin [7, Theorem 5] the nonnegative solutions $u(r)$ of (0.2) with $R = \infty$ have compact supports if and only if $\int_0^\alpha |F(\zeta)|^{-1/2} d\zeta < \infty$. Taking this fact into account we have

COROLLARY. *Let* R_1 *be the same constant as in Theorem* 2 *and* f *satisfies the conditions* $(A1)$–$(A4)$. *If, furthermore,* $f(0) = 0$ *and* $f(s)$ *is Hölder continuous at* $s = 0$, *then the solution obtained in Theorem* 2 *has a compact support which is equal to* $[0, R_1]$.

When f is locally Lipschitz continuous on $[0, \infty)$, these theorems are known by H. Berestyski, P. L. Lions and L. A. Peletier [2] with help of B. Gidas, W. -M. Ni and L. Nireberg's theorem [5]. But in the case f is not Lipschitz continuous at $s = 0$, the situation is subtle. In [3, 4] one of the authors and N. Fukagai obtain analogous results by the "shooting method". This method is elementary but the calculus was complicated because of the lacking of regularity of f at $s = 0$. In this paper, to simplify the calculus we give different proofs under little weakened conditions than in [3, 4]. Since we adopt variational methods for existence of nonnegative solutions of (0.1), we rewrite as

$$J(u) = \Phi(u) - \Psi(u),$$

where

$$\Phi(u) = \frac{1}{2}\int_{B_R} |\nabla u|^2\, dx$$

and

$$\Psi(u) = \int_{B_R} F(u)dx.$$

If we define f as $f(s) = 0$ on $[\beta, \infty)$, the nonnegative solutions of (0.2) for this f don't exceed β by virtue of the maximum principle, and so these solutions are considered as the solutions of (0.2) for the original function f. Thus we may assume $f(s) = 0$ on $[\beta, \infty)$. Furthermore, since the solutions considered here are nonnegative, we define $f(s)$ on $(-\infty, 0)$ as $f(s) = -f(-s)$.

REMARK 1. If our problem is only the existence of solutions, the conditions (A3) and (A4) are not necessary, but we may pose only a weaker condition

$$\lim_{s\to\infty} f^{+}(s)/s^{l} = 0 \qquad \text{with} \quad l < \frac{N+2}{N-2},$$

where $f^{+}(s) = \max\{f(s), 0\}$.

§ 1. Existence of nonnegative solutions

As preliminaries for the proofs of Theorems 1 and 2 we show the existence of weak solutons in $H^1_{0,r}(B_R)$ and in $H^1_r(\mathbf{R}^N)$, and then regularity of them, where

$$H^1_{0,r}(B_R) = (u \in H^1_0(B_R);\ u(x) = u(|x|)\}$$

and

$$H^1_r(\mathbf{R}^N) = \{u \in H^1(\mathbf{R}^N);\ u(x) = u(|x|)\}.$$

We study critical points of $J(u)$ in $H^1_{0,r}(B_R)$ and of $\Phi(u)$ in $H^1_r(\mathbf{R}^N)$ under the condition $\Psi(u) = 1$. Let ζ_1 be arbitrarily chosen in (ζ_0, β). Then by virtue of (A2) and (A3) we see $F(s) < F(\zeta_1)$ for $0 \le s < \zeta_1$.

LEMMA 1. *Let $\rho \in (R_0, R)$ and put*

$$\tilde{u}(x) = \begin{cases} \zeta_1 & \text{if } 0 \le |x| < \rho - 1, \\ \zeta_1(\rho - |x|) & \text{if } \rho - 1 \le |x| < \rho, \\ 0 & \text{if } \rho \le |x|, \end{cases}$$

Then $\tilde{u} \in H^1_{0,r}(B_R)$ and if R_0 is large enough, then

$$J(\tilde{u}) < 0$$

and

$$\Psi(\tilde{u}) > 0.$$

PROOF. By simple calculation we have

$$\begin{aligned} \Phi(\tilde{u}) &= \frac{1}{2}|S^{N-1}|\zeta_1^2 \int_{\rho-1}^{\rho} r^{N-1}\,dr \\ &= \frac{1}{2}\omega_N \zeta_1^2\{\rho^N - (\rho-1)^N\}, \end{aligned}$$

where $|S^{N-1}|$ is the area of $N-1$ dimensional unit sphere and

$$\omega_N = 2\pi^{N/2}/N\Gamma(N/2)$$

with

$$\Gamma(N) = \int_0^\infty e^{-t} t^{N-1}\, dt.$$

On the other hand we have

$$\begin{aligned}
\Psi(\tilde{u}) &= \int_{B_{\rho-1}} F(\tilde{u})dx + \int_{B_\rho \setminus B_{\rho-1}} F(\tilde{u})\, dx \\
&= F(\zeta_1)\int_{B_{\rho-1}} dx + |S^{N-1}| \int_{\rho-1}^{\rho} F(\tilde{u}) r^{N-1}\, dr \\
&\geq F(\zeta_1)\omega_N(\rho-1)^N + F(\xi_2)\omega_N\{\rho^N - (\rho-1)^N\}
\end{aligned}$$

and so

$$\begin{aligned}
J(\tilde{u}) &= \Phi(\tilde{u}) - \Psi(\tilde{u}) \\
&\leq \frac{1}{2}\omega_N(\rho-1)^N\{[\zeta_1^2 - 2F(\zeta_2)]\left[\left(\frac{\rho}{\rho-1}\right)^N - 1\right] - 2F(\zeta_1)\},
\end{aligned}$$

where $F(\zeta_2) = \min_{0\leq\zeta\leq\zeta_1} F(\zeta)$. Then there exists R_0 such that $\Psi(\tilde{u}) > 0$ and $J(\tilde{u}) < 0$ for any $\rho > R_0$. The proof is complete.

LEMMA 2. *Let R_0 be the constant obtained in Lemma* 1. *Then under the conditions* $(A1)$–$(A3)$ *for any $R > R_0$ there exists a weak solution v of* (0.1) *in $H^1_{0,r}(B_R)$ such that $J(v) < 0$.*

PROOF. Since the proof is standard, we sketch a brief proof. Consider $\inf\{J(u);\, u \in H^1_{0,r}(B_R)\}$. Since $F(\zeta)$ is bounded, $J(u)$ is bounded from below. Hence we can choose $\{u_j\}$ in $H^1_{0,r}(B_R)$ such that

$$J(u_j) \longrightarrow C = \inf\{J(u);\, u \in H^1_{0,r}(B_R)\} \qquad \text{as} \quad j \longrightarrow \infty.$$

Then by an easy caculation we see that $\{u_j\}$ is bounded in $H^1_{0,r}(B_R)$, and so we may extract a subsequence–still denoted by $\{u_j\}$– such that

$$u_j \longrightarrow v \qquad \text{weakly in} \quad H^1_{0,r}(B_R)$$

and by Sobolev's imbedding theorem

$$u_j \longrightarrow v \quad \text{strongly in} \quad L^q(B_R) \qquad \text{for} \quad 2 < q < 2^*$$

and

$$u_j \longrightarrow v \quad \text{a.e.},$$

where $2^* = 2N/(N-2)$ if $N > 2$ and 2^* is any constant > 2 if $N = 2$. From these facts it follows that $J(v) \leq C$. By the definition of C we have

$$J(v) = C.$$

On the other hand, in view of Lemma 1 we see

$$J(v) \leq J(\tilde{u}) < 0,$$

which asserts Lemma 2.

LEMMA 3 (Strauss [9]). *Let $N \geq 2$. Every function $u \in H_r^1(\mathbf{R}^N)$ is almost everywhere equal to a function $U(x)$ continuous for $x \neq 0$ and such that*

$$|U(x)| \leq C_N |x|^{(1-N)/2} \|u\|_{H^1(\mathbf{R}^N)} \qquad \textit{for} \quad |x| \geq \alpha_N$$

where C_N and α_N depend only on the dimension N.

LEMMA 4 (Strauss [9]). *The injection $H_r^1(\mathbf{R}^N) \subset L^q(\mathbf{R}^N)$ is compact for $2 < q < 2^*$*

Putting $R = \infty$ in (0, 1) we interpret (0. 1) as $B_R = \mathbf{R}^N$ and $\lim_{|x| \to \infty} u(x) = 0$ instead of $u(x) = 0$ on ∂B_R. Then by the same way as in [1] we have the following Lemma. We give a brief proof to close the paper.

LEMMA 5. *Let $R = \infty$ in (0. 1). Then under the conditions $(A1)$–$(A3)$ there exists a nonnegative nontrivial weak solution w of (0. 1).*

PROOF. Let $M = \{u \in H_r^1(\mathbf{R}^N);\ \Psi(u) = 1\}$. Then $M \neq \phi$. In fact, according to Lemma 1 we have $\Psi(\tilde{u}) > 0$. Defining $\tilde{u}_\sigma$ by $\tilde{u}_\sigma(x) = \tilde{u}(x/\sigma)$ for any $\sigma \in (0, \infty)$. We see $\Psi(\tilde{u}_\sigma) = \sigma^N \Psi(\tilde{u})$. if we choose σ as $\sigma^N \Psi(\tilde{u}) = 1$, we see $M \neq \phi$. Consider $\inf\{\Phi(u);\ u \in M\}$. By the same way as in the proof of Lemma 3, taking Lemma 4 into account we have $w \in H_r^1(\mathbf{R}^N)$ such that

$$\Phi(w) = \inf\{\Phi(u);\ u \in M\}.$$

Since $w\ (\geq 0)$ attains an infimum of $\Phi(u)$ under the condition $\Psi(u) = 1$, there exists a nonzero constant θ such that

$$\Phi'(w) = \theta \Psi'(w),$$

that is

$$-\Delta w = \theta f(w) \qquad \text{weakly in} \quad H_r^1(\mathbf{R}^N), \tag{1.1}$$

or

$$\frac{d^2 w}{dr^2} + \frac{N-1}{r}\frac{dw}{dr} = -\theta f(w) \qquad \text{weakly in} \quad 0 < r < \infty. \tag{1.2}$$

Suppose $\theta < 0$. Then we see $w \leq \alpha$. In fact, from Lemma 3 it follows that w is continuous except $r \neq 0$ and $\lim_{r \to \infty} w(r) = 0$, from which we can find a bounded domain $\Omega \subset \mathbf{R}^N$ such tht $w \geq \alpha$ in Ω and for some ball $B \Subset \Omega$ we have

$$\sup_B w = \sup_\Omega w,$$

if there exists $x_0 \in \mathbf{R}^N$ such that $w(x_0) > \alpha$. Then the maximum principle [6, Theorem 8.19] leads to a contradiction, and so we see $w \leq \alpha$. Since $w \leq \alpha$, we have $F(w) \leq 0$, which contradicts

$$\int F(w)\,dx = 1.$$

Thus $\theta > 0$. If we take $w(x/\sqrt{\theta})$ as w, this w is the solution to be found. The proof is complete.

Let u be v or w. Since $u \in H^1_{0,r}(B_R)$ (or $H^1_r(\mathbf{R}^N)$), a weak derivative du/dr is locally integrable function of $(0, R]$ (when $R = \infty$, $(0, R]$ is interpreted as $(0, \infty)$). Thus it follows from Schwartz distributional arguments [10, Theorem 17] that u is locally absolutely continuous on $(0, R]$, and therefore u has derivatives $u'(r)$ at almost all $r \in (0, R]$. Since u satisfies

$$\frac{d^2u}{dr^2} + \frac{N-1}{r}\frac{du}{dr} + f(u) = 0 \qquad \text{weakly in} \quad (0, R),$$

and $f(u)$ is bounded in $(0, R)$, it follows from the same reasoning as above that u' is also locally absolutely continuous on $(0, R]$. From this we have, for any s, $r \in (0, R)$

$$u'(r) = \left(\frac{s}{r}\right)^{N-1} u'(s) - \int_s^r f(u(\zeta))\left(\frac{\zeta}{r}\right)^{N-1} d\zeta, \tag{1.3}$$

which yields $u \in C^2(0, R]$. Furthermore, since $f(u)$ is bounded in $(0, R)$, we see that $u \in W^{2,p}_{\mathrm{loc}}(B_R)$ for any $1 < p < \infty$ (c.f. [6, Theorem 9.15]). Hence $u \in C^1(B_R)$. Letting $s \to 0$ and then $r \to 0$ in (1.3) we have $u'(0) = 0$. From the equation

$$u'' + \frac{N-1}{r}u'' + f(u) = 0$$

there exists $u''(0)$ and so $u \in C^2[0, R]$. Thus we have the following

PROPOSITION 1. *Let R_0 be the constant obtained in Lemma* 1. *Then under the conditions* $(A1)$–$(A3)$ *there exists a C^2 positive solution v of* (0.2) *for some $\mu \in (0, \beta)$ such that $J(v) < 0$.*

In view of the above facts and Lemma 3 we have the following

PROPOSITION 2. *Let $R = \infty$. Then under the conditions* $(A1)$–$(A3)$ *there exists a C^2 nonnegative nontrivial solution w of* (0.2) *for some $\mu \in (0, \beta)$.*

§ 2. Qualitative lemmas for solutions

As in Section 1 let u be v or w. Then u is a C^2 solutin of

$$u'' + \frac{N-1}{r} u' + f(u) = 0, \tag{2.1}$$

and we have the following

LEMMA 6. *For any $0 \le r_1 \le r_2 \le R$ the following identity*

$$\frac{1}{2}|u'(r_2)|^2 + F(u(r_2)) + \int_{r_1}^{r_2} \frac{N-1}{r} |u'(r)|^2 \, dr = \frac{1}{2}|u'(r_1)|^2 + F(u(r_1)) \tag{2.2}$$

holds.

PROOF. Multiply the both sides of (2.1) by u' and integrate them from r_1 to r_2. Then we have (2.2) since$\{(u')^2\}' = 2u''u'$ and $\{F(u)\}' = f(u)u'$. The proof is complete.

LEMMA 7 (Pohozaev's identity [8]). *Let v be a C^2 solution of* (0.2). *Then the following identity*

$$\left(\frac{2-N}{2}\right) \int_0^R |v'(r)|^2 r^{N-1} dr + N \int_0^R F(v(r)) r^{N-1} = \frac{1}{2} R^N |v'(R)|^2. \tag{2.2}$$

holds.

PROOF. Multiply the both sides of the equation

$$r^{1-N}(r^{N-1}v')' = -f(v)$$

by $v'r'^N$ and integrate them from 0 to R. Then we have

$$-\int_0^R f(v)v'r^N dr = [-r^N F(v(r))]_0^R + \int_0^R F(v(r)) r^{N-1} dr = \int_0^R F(v(r)) r^{N-1} \, dr. \tag{2.3}$$

On the other hand

$$\int_0^R (r^{N-1}v')'v'r dr = R^N(v'(R))^2 + \int_0^R \{(v')^2 r^{N-1} + v'v''r^N\} dr.$$

Since

$$\int_0^R v'v''r^N dr = \frac{1}{2}R^N(v'(R))^2 - \frac{N}{2}\int_0^R (v')^2 r^{N-1}\,dr,$$

it follows that

$$\int_0^R (r^{N-1}v')v'r dr = \frac{1}{2}R^N(v'(R))^2 - \left(\frac{2-N}{2}\right)\int_0^R (v')^2 r^{N-1}\,dr. \tag{2.4}$$

From (2.3) and (2.4) we obtain (2.2). The proof is complete.

LEMMA 8. $v'(R) < 0$.

PROOF. By Pohozaev's identity we have

$$\left(\frac{2-N}{2}\right)\int_0^R (v')^2 r^{N-1} dr + N\int_0^R F(v) r^{N-1} dr = R^N(v'(R))^2.$$

On the other hand, since v satisfies $J(v) < 0$ or

$$\frac{1}{|S^{N-1}|}J(v) = \frac{1}{2}\int_0^R (v')r^{N-1} dr - \int_0^R F(v)r^{N-1} dr < 0,$$

it follows that

$$R^N(v'(R))^2 = \int_0^R (v')^2 r^{N-1} dr - \frac{1}{|S^{N-1}|}J(v) > 0,$$

which together with the fact $v'(R) \leq 0$ yields $v'(R) < 0$.

LEMMA 9. *Suppose there exists $r_0 \in [0, R)$ such that $u'(r_0) = 0$. Then one of the following statements holds*:

(i) $u(r_0) > \zeta_0$.

(ii) $u \equiv 0$ *on* $[r_0, R]$.

PROOF. Use Lemma 6 with $r_1 = r_0$ and $r_2 = R$. Then,

$$\frac{1}{2}|u'(R)|^2 + \int_{r_0}^R \frac{N-1}{r}|u'(r)|^2 dr = F(u(r_0)).$$

Hence we obtain $F(u(r_0)) \geq 0$, from which together with (A3) it follows that

$$u(r_0) \geq \zeta_0 \tag{2.5}$$

or

$$u(r_0) = 0. \tag{2.6}$$

First consider the case of (2.5). If $u(r_0) = \zeta_0$, then $u' \equiv 0$ on $[r_0, R]$, and so $u \equiv \zeta_0$ on $[r_0, R]$, which is a contradiction, since $u(R) = 0$. Thus

$u(r_0) \neq \zeta_0$. By the same reasoning as is mensioned above we see $u \equiv 0$ on $[r_0, R]$ in the case of (2.6). Thus the proof is complete.

LEMMA 10. *Suppose that there exists an* $r_0 \in [0, R)$ *such that* $u(r_0) = 0$. *Then* $u \equiv 0$ *on* $[r_0, R]$.

PROOF. Since u is C^2 and nonnegative on $[0, R]$, we obtain $u'(r_0) = 0$. Hence it follows from Lemma 9 that $u \equiv 0$ on $[r_0, R]$, since $u(r_0) = 0$. The proof is complete.

§3. Proofs of Theorems 1 and 2

As for the proof of Theorem 1 taking Lemmas 8, 9 and 10 into account we have only to prove $v' < 0$ on $(0, R)$. On the other hand, as for the proof of Theorem 2, let

$$R_1 = \inf\{r > 0; w(r) = 0\}.$$

Since $w \neq 0$, we have, from Lemma 10,

$$R_1 > 0,\ w > 0 \qquad \text{on } [0, R).$$

and

$$w \equiv 0 \text{ on } [R_1, \infty) \text{ if } R_1 < \infty.$$

Since $w'(0) = 0$, it follows from Lemma 9 that $w(0) > \zeta_0$. Thus we also have only to show $w' < 0$ on $(0, R_1)$. Since the proof of Theorem 1 is the same as in Theorem 2, we prove only Theorem 2. Suppose there exists $r' \in (0, R_1)$ such that $w'(r') = 0$. Then we may assume $w''(r') \leq 0$, since $w > 0$ on $[0, R_1)$ and $w(R_1) = 0$. From Lemma 9 it follows that

$$w(r') > \zeta_0. \tag{3.1}$$

Consider the case $w''(r') = 0$. Since w satisfies the equation (2.1), we have

$$f(w(r')) = 0.$$

This together with (3.1) leads to

$$w(r') = \beta.$$

Then from the uniqueness of solutions of the equation (2.1) with $u(r') = \beta$ and $u'(r') = 0$ it follows that $w \equiv \beta$ on $(0, R_1]$, which contradicts $w(R_1) = 0$. As for the case $w''(r') < 0$, since $w'(0) = 0$, there exists a $r'' \in [0, r)$ such that

$$w'(r'') = 0 \qquad \text{and} \quad w''(r'') \geq 0. \tag{3.2}$$

Then it follows from Lemma 9 that

$$w(r'') > \zeta_0. \tag{3.3}$$

On the other hand, since w is a solution of the equation (2.1), we see $f(w(r'')) \leq 0$, which yields

$$0 \leq w(r'') \leq \alpha \tag{3.4}$$

or

$$w(r'') = \beta. \tag{3.5}$$

The inequality (3.4) contradicts (3.3). On the other hand, (3.5) doesn't occur by the same reasoning as is mentioned above. Thus we have $w' < 0$ on $(0, R_1)$. The proof is complete.

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