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A form of classical Liouville theorem for polyharmonic functions

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ABSTRACT. Our aim in this paper is to propose a form of classical Liouville theorem for polyharmonic functions which is a direct generalization of our former one for harmonic functions.

1. Introduction

We denote by $\mathbf{R}^d$ the Euclidean space of dimension $d \geq 2$. The length $|x|$ of a point or a vector $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ is given by $|x| = (\sum_{i=1}^d x_i^2)^{1/2}$. A real valued function $u(x)$ is *harmonic* on $\mathbf{R}^d$ if $u \in C^2(\mathbf{R}^d)$ and $\Delta u(x) = 0$ on $\mathbf{R}^d$, where Δ is the Laplacian $\sum_{i=1}^d (\partial/\partial x_i)^2$. We denote by $H(\mathbf{R}^d)$ the real linear space of harmonic functions on $\mathbf{R}^d$. We also denote by $HB(\mathbf{R}^d)$ ($HP(\mathbf{R}^d)$, resp.) the class of bounded (nonnegative, resp.) functions $u \in H(\mathbf{R}^d)$. The Liouville theorem in the theory of harmonic functions (cf. e.g. Axler et al. [3]) consists of the following two contents: $HB(\mathbf{R}^d) = \mathbf{R}$; $HP(\mathbf{R}^d) = \mathbf{R}^+$, where $\mathbf{R}$ is the real number field and $\mathbf{R}^+ = \{t \in \mathbf{R} : t \geq 0\}$. Picard (cf. e.g. [11], [12]) essentially showed that these two statements are equivalent. We proposed ([8]) the following theorem (stated here in a slightly modified fashion beyond the original presentation) as a form of Liouville theorem for harmonic functions.

THEOREM A. *Suppose $u \in H(\mathbf{R}^d)$ and s is any real number with $s > 0$. Then u is a harmonic polynomial of degree less than s if and only if there exists an increasing divergent sequence $(r_i)_{i \geq 1}$ of positive numbers r_i $(i = 1, 2, \ldots)$ such that*

$$\liminf_{i \uparrow \infty} \left(\min_{|x| = r_i} \frac{u(x)}{|x|^s} \right) \geq 0. \tag{1.1}$$

In fact, if $u \in HB(\mathbf{R}^d)$ ($u \in HP(\mathbf{R}^d)$, resp.), then (1.1) is valid for any $0 < s < 1$ and for any increasing divergent positive sequence $(r_i)_{i \geq 1}$. Thus

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by the above theorem u is a polynomial of degree less than s so that u is constant, i.e., Theorem A implies the Liouville theorem for harmonic functions. This is the reason why we viewed Theorem A as a form of classical Liouville theorem for harmonic functions. The *purpose* of this paper is to give a polyharmonic version of Theorem A.

A real valued function u on $\mathbf{R}^d$ is *polyharmonic* of order m on $\mathbf{R}^d$, where m is a positive integer, if $u \in C^{2m}(\mathbf{R}^d)$ and $\Delta^m u(x) = 0$ on $\mathbf{R}^d$, where $\Delta^m u = \Delta^{m-1}(\Delta u)$ with the convention that Δ^0 being the identity operator. We denote by $H^m(\mathbf{R}^d)$ the real linear space of polyharmonic functions of order m on $\mathbf{R}^d$ so that $H^1(\mathbf{R}^d) = H(\mathbf{R}^d)$. We also denote by $H^mB(\mathbf{R}^d)$ ($H^mP(\mathbf{R}^d)$, resp.) the class of bounded (nonnegative, resp.) functions $u \in H^m(\mathbf{R}^d)$. A real valued function u on $\mathbf{R}^d$ belongs to $H^m(\mathbf{R}^d)$ if and only if

$$u(x) = \sum_{p=1}^{m} |x|^{2p-2} h_p(x) \qquad (h_p \in H(\mathbf{R}^d)\ (p = 1, \dots, m)) \tag{1.2}$$

for every $x \in \mathbf{R}^d$ (cf. e.g. [2, p. 4]). Here the expression (1.2) is unique in the sense that $\sum_{p=1}^{m} |x|^{2p-2} h_p(x) = 0$ on $\mathbf{R}^d$ implies that $h_p(x) = 0$ $(p = 1, \dots, m)$ on $\mathbf{R}^d$ (cf. e.g. [2, p. 4]; see also §4 below) and (1.2) is referred to as the *Almansi decomposition* of u on $\mathbf{R}^d$. Then, as the polyharmonic version of Theorem A, we have the following result.

THEOREM 1. *Suppose that m is any positive integer, $u \in H^m(\mathbf{R}^d)$, and s is any real number with $s > 2m-2$. Then u is a polyharmonic polynomial of degree less than s if and only if there exists an increasing divergent sequence $(r_i)_{i \geq 1}$ of positive numbers r_i $(i = 1, 2, \dots)$ such that*

$$\liminf_{i \uparrow \infty} \left(\min_{|x| = r_i} \frac{u(x)}{|x|^s} \right) \geq 0. \tag{1.3}$$

The $m = 1$ case of Theorem 1 is nothing but Theorem A so that Theorem 1 is a direct generalization of Theorem A. Theorem 1 may also be viewed as a form of the Liouville theorem for polyharmonic functions. The first content of the Liouville theorem $H^mB(\mathbf{R}^d) = \mathbf{R}$ as the counterpart of $HB(\mathbf{R}^d) = \mathbf{R}$ is true (Nicolesco [10], Huilgol [4]). The second content of the Liouville theorem $H^mP(\mathbf{R}^d) = \mathbf{R}^+$ as the formal counterpart of $HP(\mathbf{R}^d) = \mathbf{R}^+$ is not true for $m \geq 2$ since e.g. $|x|^{2m-2} \in H^mP(\mathbf{R}^d) \setminus \mathbf{R}^+$; the true form of the second content of the Liouville theorem corresponding to $HP(\mathbf{R}^d) = \mathbf{R}^+$ is that if $u \in H^mP(\mathbf{R}^d)$, then u is a polyharmonic polynomial of degree at most $2m-2$ $(m = 1, 2, \dots)$ (Kuran [6]), which reduces to $HP(\mathbf{R}^d) = \mathbf{R}^+$ for $m = 1$. Now, if $u \in H^mP(\mathbf{R}^d)$, then (1.3) is valid for any $2m-2 < s < 2m-1$ so that Theorem 1 implies that u is a polynomial of degree less that s, or equivalently, at most $2m-2$. If $u \in H^mB(\mathbf{R}^d)$, then (1.3) is also valid for any $2m-2 < s < 2m-1$ so that

Theorem 1 implies that u is a polynomial of degree at most $2m-2$. However a polynomial u can be bounded if and only if u is constant. Thus Theorem 1 implies two contents of the Liouville theorem for polyharmonic functions.

The proof of Theorem 1 will be given in §2 by the Fourier expansion method, one of the most powerful tools in treating objects related to harmonic functions defined on a domain rotationally invariant about a point such as $\mathbf{R}^d$, which has been constantly our claim (cf. [5], [8], [9], etc.); this proof is essentially identical with that for Theorem A in [8]. Results in the same category as Theorem 1 have already been given by Armitage [1] many years ago and also by Mizuta [7] relatively recently. We compare our condition (1.3) with those of Armitage [1] and Mizuta [7] in §3. Theorem 1 may be further generalized for functions not necessarily polyharmonic, which will be discussed in the final §4.

2. Proof of Theorem 1

We use the polar coordinate $x = r\xi$ for points $x \in \mathbf{R}^d$, where $r = |x| \geq 0$ and $\xi = x/|x| \in S^{d-1}$ for $x \neq 0$ and $\xi = (1, 0, \dots, 0) \in S^{d-1}$ for $x = 0$ just for the sake of definiteness. Here S^{d-1} is the unit sphere $\{x \in \mathbf{R}^d : |x| = 1\}$. We choose and then fix an orthonormal basis $\{S_{kj} : j = 1, \dots, N(k)\}$ of the subspace of all spherical harmonics of degree k of $L^2(S^{d-1}, d\sigma)$, where $d\sigma$ is the area element on S^{d-1}. Then $\{S_{kj} : j = 1, \dots, N(k); k = 0, 1, \dots\}$ is a complete orthonormal system in $L^2(S^{d-1}, d\sigma)$. We have, as the special case of the addition theorem (cf. [3, 5.11]),

$$\sum_{j=1}^{N(k)} S_{kj}(\xi)^2 = \frac{N(k)}{\sigma_d},$$

where σ_d is the surface area $\sigma(S^{d-1})$ of S^{d-1}. Here $N(0) = 1$ and

$$N(k) = (2k + d - 2)\Gamma(k + d - 2)/\Gamma(k + 1)\Gamma(d - 1)$$

for $k = 1, 2, \dots$ (cf. [3, 5.17]). For simplicity we set $A_k := \sqrt{N(k)/\sigma_d}$ so that

$$|S_{kj}(\xi)| \leq A_k \qquad (j = 1, \dots, N(k);\ k = 0, 1, \dots)$$

for every $\xi \in S^{d-1}$. Then we have the following Fourier expansion of any harmonic function $h(r\xi)$ on $\mathbf{R}^d$ in terms of spherical harmonics $\{S_{kj}\}$:

$$h(r\xi) = \sum_{k=0}^{\infty} \left(\sum_{j=1}^{N(k)} a_{kj} S_{kj}(\xi) \right) r^k, \tag{2.1}$$

where a_{kj} $(j = 1, \dots, N(k); k = 0, 1, \dots)$ are constants. Here the series on the right hand side of (2.1) converges uniformly in $\xi \in S^{d-1}$ for any fixed $0 < r < \infty$.

Since $2m - 2 < s$, there is a unique integer n such that

$$2m - 2 \leq n < s \leq n + 1. \tag{2.2}$$

We only have to show that any given $u \in H^m(\mathbf{R}^d)$ is a polyharmonic polynomial of degree at most n if the condition (1.3) is postulated since the converse is trivially true. We note as a direct consequence of (1.3) that for an arbitrarily fixed positive number $\varepsilon > 0$ there exists a positive integer $i(\varepsilon)$ such that

$$u(x) \geq -\varepsilon |x|^{n+1} \qquad (|x| = r_i) \tag{2.3}$$

for every $i \geq i(\varepsilon)$. In fact, (1.3) assures that for any positive number $\varepsilon > 0$ there exists a positive integer $i(\varepsilon)$ such that $u(x)/|x|^s \geq -\varepsilon$ ($|x| = r_i$) and $r_i > 1$ for every $i \geq i(\varepsilon)$. By the choice of n in (2.2), $|x|^s \leq |x|^{n+1}$ for $|x| \geq 1$, which induces (2.3).

Consider the Almansi decomposition (1.2) of $u(x)$ on $\mathbf{R}^d$:

$$u(x) = \sum_{p=1}^{m} |x|^{2p-2} h_p(x),$$

where $h_p \in H(\mathbf{R}^d)$ $(p = 1, \ldots, m)$. In terms of polar coordinate $x = r\xi$, (2.1) yields

$$h_p(r\xi) = \sum_{k=0}^{\infty} \left(\sum_{j=1}^{N(k)} a_{pkj} S_{kj}(\xi) \right) r^k \qquad (p = 1, \ldots, m),$$

where a_{pkj} $(p = 1, \ldots, m;\ k = 0, 1, \ldots;\ j = 1, \ldots, N(k))$ are constants. The series on the right hand side of the above displayed identity converges uniformly in $\xi \in S^{d-1}$ for any fixed $0 < r < \infty$. Hence we obtain

$$u(r\xi) = \sum_{p=1}^{m} r^{2p-2} \left(\sum_{k=0}^{\infty} \left(\sum_{j=1}^{N(k)} a_{pkj} S_{kj}(\xi) \right) r^k \right), \tag{2.4}$$

which converges uniformly in $\xi \in S^{d-1}$ for any fixed $0 < r < \infty$. This with (2.3) yields

$$\sum_{p=1}^{m} r_i^{2p-2} \left(\sum_{k=0}^{\infty} \left(\sum_{j=1}^{N(k)} a_{pkj} S_{kj}(\xi) \right) r_i^k \right) + \varepsilon r_i^{n+1} \geq 0 \qquad (i \geq i(\varepsilon)) \tag{2.5}$$

for any $\xi \in S^{d-1}$.

We now claim that (2.5) implies the following relations:

$$a_{pkj} = 0 \qquad ((2p - 2) + k \geq n + 1). \tag{2.6}$$

First we prove that $a_{pkj}=0$ for $(2p-2)+k>n+1$. Contrary to the assertion assume that there exist $1\leq p\leq m$, $0\leq k<\infty$, and $1\leq j\leq N(k)$ such that $(2p-2)+k>n+1$ and yet $a_{pkj}\neq 0$. Let p_1 be the greatest p satisfying $a_{pkj}\neq 0$ so that $p_1=m$ or $a_{pkj}=0$ for $p_1<p\leq m$ and $a_{p_1kj}\neq 0$. Choose a real number η with $|\eta|=1$ such that

$$\eta a_{p_1kj}>0. \tag{2.7}$$

Multiply $A_k-\eta S_{kj}(\xi)\geq 0$ to both sides of (2.5) and then integrate both sides of the resulting inequality over S^{d-1} with respect to $d\sigma$; present authors have been using this device frequently (see e.g. [5], [8], [9], etc.). Then we obtain

$$\sigma_d A_k\left(\sigma_d^{-1/2}\sum_{p=1}^{m}a_{p01}r_i^{2p-2}+\varepsilon r_i^{n+1}\right)-\sum_{p=1}^{p_1}\eta a_{pkj}r_i^{(2p-2)+k}\geq 0 \qquad (i\geq i(\varepsilon)). \tag{2.8}$$

Let $P(r):=\sigma_dA_k(\sigma_d^{-1/2}\sum_{p=1}^m a_{p01}r^{2p-2}+\varepsilon r^{n+1})$, which is a polynomial of r of degree $n+1$, and $Q(r):=\sum_{p=1}^{p_1}\eta a_{pkj}r^{(2p-2)+k}$, which is a polynomial of r of degree $(2p_1-2)+k>n+1$ by (2.7). Then $\deg P<\deg Q$, the leading coefficient of Q is strictly positive by (2.7), and yet (2.8) means that $P(r_i)\geq Q(r_i)$ $(i\geq i(\varepsilon))$. This is clearly absurd and we have proved (2.6) for $(2p-2)+k>n+1$.

Next we prove that $a_{pkj}=0$ for $(2p-2)+k=n+1$. Again contrary to the assertion assume that there exist $1\leq p_1\leq m$, $0\leq k<\infty$, and $1\leq j\leq N(k)$ such that $(2p_1-2)+k=n+1$ and $a_{p_1kj}\neq 0$. Observe that $p_1=m$ or $p_1<m$. In the latter case, $a_{pkj}=0$ for $p_1<p\leq m$ by what we have shown above since $(2p-2)+k>(2p_1-2)+k=n+1$. Similarly as above choose a real number η with $|\eta|=1$ and $\eta a_{p_1kj}>0$. We now choose $\varepsilon>0$ so small and then $i(\varepsilon)$ so large accordingly that (2.5) is valid for any $i\geq i(\varepsilon)$ and

$$\eta a_{p_1kj}-\sigma_dA_k\varepsilon>0. \tag{2.9}$$

Multiplying $A_k-\eta S_{kj}(\xi)\geq 0$ to both sides of (2.5) and then integrate both sides of the resulting inequality over S^{d-1} with respect to $d\sigma$ as we did above. Then we obtain

$$\sigma_d^{1/2}A_k\sum_{p=1}^{m}a_{p01}r_i^{2p-2}-\left\{(\eta a_{p_1kj}-\sigma_dA_k\varepsilon)r_i^{n+1}+\sum_{p=1}^{p_1-1}\eta a_{pkj}r_i^{(2p-2)+k}\right\}\geq 0. \tag{2.10}$$

Set $P(r):=\sigma_d^{1/2}A_k\sum_{p=1}^m a_{p01}r^{2p-2}$, which is a polynomial of r of degree at most $2m-2<n+1$, and $Q(r):=(\eta a_{p_1kj}-\sigma_dA_k\varepsilon)r^{n+1}+\sum_{p=1}^{p_1-1}\eta a_{pkj}r^{(2p-2)+k}$, which is a polynomial in r of degree $n+1$. Then $\deg P<\deg Q$, the leading coefficient of Q is strictly positive by (2.9), and nevertheless (2.10) means that $P(r_i)\geq Q(r_i)$ $(i\geq i(\varepsilon))$. This is clearly absurd and we have also shown (2.6) for $(2p-2)+k=n+1$. The proof of (2.6) is herewith complete.

In view of (2.6) we can rewrite (2.4) as

$$u(r\xi) = \sum_{p=1}^{m}\left(\sum_{(2p-2)+k\le n}\left(\sum_{j=1}^{N(k)} a_{pkj} r^{2p-2} r^k S_{kj}(\xi)\right)\right) \tag{2.11}$$

for every $\xi \in S^{d-1}$. Since $S_{kj}(x) := r^k S_{kj}(\xi)$ $(x = r\xi)$ is a homogeneous harmonic polynomial in x of degree k, (2.11) yields that

$$u(x) = \sum_{p=1}^{m}\left(\sum_{(2p-2)+k\le n}\left(\sum_{j=1}^{N(k)} a_{pkj} |x|^{2p-2} S_{kj}(x)\right)\right)$$

is a polynomial of x of degree at most n, which was to be shown. □

3. Equivalent conditions

By examining the proof in [8] of Theorem A and also the proof in §2 of Theorem 1, we can see that the condition (1.1) in Theorem A or the condition (1.3) in Theorem 1 can be replaced by any one of the following four conditions:

$$\liminf_{i\uparrow\infty}\left(\min_{|x|=r_i}\frac{u(x)}{|x|^s}\right) = 0; \tag{3.1}$$

$$\liminf_{|x|\uparrow\infty}\frac{u(x)}{|x|^s} \ge 0; \tag{3.2}$$

$$\liminf_{|x|\uparrow\infty}\frac{u(x)}{|x|^s} = 0; \tag{3.3}$$

$$\lim_{|x|\uparrow\infty}\frac{u(x)}{|x|^s} = 0. \tag{3.4}$$

Thus the conditions (1.1) ((1.3), resp.), (3.1), (3.2), (3.3) and (3.4) with $s > 0$ ($s > 2m-2$, resp.) are equivalent by pairs for harmonic (polyharmonic, resp.) functions u on $\mathbf{R}^d$, which are thus equivalent to that $u(x)$ is a polynomial in x of degree less than s. Needless to say the equivalences of these conditions are only for polyharmonic functions u. For not necessarily polyharmonic functions u, e.g. (3.2) is strictly stronger than (1.3). We denote by B^d the unit open ball $\{|x| < 1\}$ in $\mathbf{R}^d$ so that $rB^d = \{|x| < r\}$ and $rS^{d-1} = \{|x| = r\}$ for $r \in \mathbf{R}^+$. In 1973, Armitage [1] proved the following result.

THEOREM B. *Suppose that m is any positive integer, $u \in H^m(\mathbf{R}^d)$, and s is any real number with $s > 2m-2$. Then u is a polyharmonic polynomial of degree less than s if and only if*

$$\lim_{r\uparrow\infty}\frac{1}{r^{s+d-1}}\int_{rS^{d-1}}u^{+}d\sigma=0. \tag{3.5}$$

Here $u^{+}(x)=\max(u(x),0)$. Recently Mizuta [7] proved the following result, which is stated here in a slightly more precise form than the original one.

THEOREM C. *Suppose that m is any positive integer, $u\in H^{m}(\mathbf{R}^{d})$, and s is any real number with $s>2m-2$. Then u is a polyharmonic polynomial of degree less than s if and only if*

$$\liminf_{r\uparrow\infty}\frac{1}{r^{s+d}}\int_{rB^{d}}u^{+}(x)dx=0. \tag{3.6}$$

Hence we can make the following conclusion. We are grateful to Professor Mizuta for calling our attentions to the polyharmonic Liouville theorem and especially to the Armitage and Mizuta conditions stated above.

THEOREM 2. *The following four conditions for $u\in H^{m}(\mathbf{R}^{d})$ and $s>2m-2$ are equivalent by pairs: the N.-T. condition* (1.3)*; the Armitage condition* (3.5)*; the Mizuta condition* (3.6)*; u is a polyharmonic polynomial of degree less than s.*

4. A generalization

We consider the class $H^{*}(\mathbf{R}^{d})$ of real valued function u defined on $\mathbf{R}^{d}$ expressible as

$$u(x)=\sum_{i=0}^{n}|x|^{i}h_{i}(x) \tag{4.1}$$

on $\mathbf{R}^{d}$, where n is a nonnegative integer determined by u and $h_{i}\in H(\mathbf{R}^{d})$ $(i=0,\dots,n)$ also determined by u. Then we have $\bigcup_{m=1}^{\infty}H^{m}(\mathbf{R}^{d})\subset H^{*}(\mathbf{R}^{d})$. First we show the following result.

PROPOSITION 1. *The expression* (4.1) *is uniquely determined by u.*

PROOF. We have to show that if $\sum_{i=0}^{n}|x|^{i}h_{i}(x)=0$ on $\mathbf{R}^{d}$ for $h_{i}\in H(\mathbf{R}^{d})$ $(i=0,\dots,n)$, then $h_{i}(x)=0$ on $\mathbf{R}^{d}$ $(i=0,\dots,n)$. This is clear for $n=0$. Assuming it is true for n, we prove that

$$\sum_{i=0}^{n+1}|x|^{i}h_{i}(x)=0 \tag{4.2}$$

on $\mathbf{R}^{d}$ for $h_{i}\in H(\mathbf{R}^{d})$ $(i=0,\dots,n+1)$ implies $h_{i}(x)=0$ on $\mathbf{R}^{d}$ $(i=0,\dots,n+1)$. By considering (4.2) on S^{d-1} we have $\sum_{i=0}^{n+1}h_{i}(x)=0$ on S^{d-1}. By the maximum principle for harmonic functions, the same is true on B^{d}. Then by virtue of the uniqueness theorem for harmonic functions we must conclude that

$$\sum_{i=0}^{n+1} h_i(x) = 0 \tag{4.3}$$

on $\mathbf{R}^d$. Subtract (4.3) from (4.2) and then divide both sides of the resulting identity by $|x| - 1$. Then we obtain $\sum_{i=1}^{n+1}(\sum_{j=0}^{i-1}|x|^j)h_i(x) = 0$ on $\mathbf{R}^d$ or

$$\sum_{i=0}^{n} |x|^i \left(\sum_{j=i+1}^{n+1} h_j(x)\right) = 0 \tag{4.4}$$

on $\mathbf{R}^d$ with $\sum_{j=i+1}^{n+1} h_j \in H(\mathbf{R}^d)$ $(i = 0, \dots, n)$. By the induction hypothesis applied to (4.4), we have $\sum_{j=i+1}^{n+1} h_j = 0$ $(i = 0, \dots, n)$ on $\mathbf{R}^d$. This with (4.3) implies that

$$\sum_{j=i}^{n+1} h_j = 0 \qquad (i = 0, 1, \dots, n+1) \tag{4.5}$$

on $\mathbf{R}^d$. The relations (4.5) are equivalent to $h_i = 0$ $(i = 0, 1, \dots, n+1)$ on $\mathbf{R}^d$, and this completes the induction. $\square$

We state a generalization of Theorem 1 on replacing $H^m(\mathbf{R}^d)$ in Theorem 1 by strictly larger class $H^*(\mathbf{R}^d)$.

THEOREM 3. *Suppose that n is any nonnegative integer, $u \in H^*(\mathbf{R}^d)$ is given by* (4.1)*:*

$$u(x) = \sum_{i=0}^{n} |x|^i h_i(x) \qquad (h_i \in H(\mathbf{R}^d) \ (i = 0, \dots, n))$$

on $\mathbf{R}^d$, and s is any real number with $s > n$. Then each $h_i(x)$ in (4.1) *is a harmonic polynomial in x of degree less than $s - i$ $(i = 0, \dots, n)$ if and only if there exists an increasing divergent positive sequence $(r_j)_{j \geq 1}$ such that*

$$\liminf_{j \uparrow \infty} \left(\min_{|x| = r_j} \frac{u(x)}{|x|^s}\right) \geq 0. \tag{4.6}$$

If $u(x)$ in (4.1) takes the form $u(x) = \sum_{p=1}^{m} |x|^{2p-2} h_p(x)$, then each $h_p(x)$ is a harmonic polynomial in x of degree less than $s - (2p - 2)$ $(p = 1, \dots, m)$ by Theorem 3, which is the case if and only if $u(x)$ is a polyharmonic polynomial in x of degree less than s. Hence Theorem 3 certainly contains Theorem 1 as a special case. Since the proof for Theorem 1 can be applied to Theorem 3 mutatis mutandis, we omit here the proof of the above result.

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