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On Orevkov's rational cuspidal plane curves

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Abstract. In this note, we consider rational cuspidal plane curves having exactly one cusp whose complements have logarithmic Kodaira dimension two. We classify such curves with the property that the strict transforms of them via the minimal embedded resolution of the cusp have the maximal self-intersection number. We show that the curves given by the classification coincide with those constructed by Orevkov.

1. Introduction.

Let C be a plane curve on $\boldsymbol{P}^2 = \boldsymbol{P}^2(\boldsymbol{C})$. A singular point of C is said to be a *cusp* if it is a locally irreducible singular point. We say that C is *cuspidal* (resp. *unicuspidal*) if C has only cusps (resp. one cusp) as its singular points. We denote by $\bar{\kappa} = \bar{\kappa}(\boldsymbol{P}^2 \setminus C)$ the logarithmic Kodaira dimension of the complement $\boldsymbol{P}^2 \setminus C$. Let C' denote the strict transform of a rational unicuspidal plane curve C via the minimal embedded resolution of the cusp of C. By [**Y**], $\bar{\kappa} = -\infty$ if and only if $(C')^2 > -2$. By [**Ts**, Proposition 2], there exist no rational cuspidal plane curves with $\bar{\kappa} = 0$. See also [**K1**], [**O**]. Thus $\bar{\kappa} \geq 1$ if and only if $(C')^2 \leq -2$. In [**To**], rational unicuspidal plane curves with $\bar{\kappa} = 1$ have already been classified. It was Orevkov [**O**] who constructed two sequences C_{4k}, C_{4k}^* $(k = 1, 2, \ldots)$ of rational unicuspidal plane curves with $\bar{\kappa} = 2$. See Section 3 for details. The purpose of this note is to classify rational unicuspidal plane curves C with $\bar{\kappa} = 2$ and $(C')^2 = -2$. The main result of this note is the following:

Theorem 1. *Let C be a rational unicuspidal plane curve with $\bar{\kappa} = 2$. Then C is projectively equivalent to one of the Orevkov's curves if and only if $(C')^2 = -2$.*

For a plane curve C, we denote by $\overline{P}_m = \overline{P}_m(\boldsymbol{P}^2 \setminus C)$ the logarithmic m-genus of the complement $\boldsymbol{P}^2 \setminus C$. In [**K3**], the curves C_4 and C_4^* were characterized by $\bar{\kappa}$ and $\overline{P}_3$. With the help of our Theorem 1, it was proved that a reduced plane curve C can be constructed as C_4 or C_4^* if and only if $\bar{\kappa}(\boldsymbol{P}^2 \setminus C) \geq 0$ and $\overline{P}_3(\boldsymbol{P}^2 \setminus C) = 0$.

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2. Preliminaries.

In this section, we prepare some preliminaries.

2.1. Linear chains.

Let D be a divisor on a smooth surface V, $\varphi : V' \to V$ a composite of successive blowing-ups and $B \subset V'$ a divisor. We say that φ *contracts* B to D, or simply that B *shrinks to* D if $\varphi(\operatorname{Supp} B) = \operatorname{Supp} D$ and each center of blowing-ups of φ is on D or one of its preimages. Let $D_1, \dots, D_r$ be the irreducible components of D. We call D an *SNC-divisor* if D is a reduced effective divisor, each D_i is smooth, $D_i D_j \le 1$ for distinct D_i, D_j, and $D_i \cap D_j \cap D_k = \emptyset$ for distinct D_i, D_j, D_k. Assume that D is an SNC-divisor and that each D_i is projective. Let $\Gamma = \Gamma(D)$ denote the dual graph of D. We give the vertex corresponding to a component D_i the weight D_i^2. We sometimes do not distinguish between D and its weighted dual graph Γ. We use the following notation and terminology (cf. [**F**, Section 3] and [**MT1**, Chapter 1]). A blowing-up at a point $P \in D$ is said to be *sprouting* (resp. *subdivisional*) *with respect to* D if P is a smooth point (resp. node) of D. A component D_i is called a *branching component* of D if $D_i(D - D_i) \ge 3$.

Assume that Γ is connected and linear. In cases where $r > 1$, the weighted linear graph Γ together with a direction from an endpoint to the other is called a *linear chain*. By definition, the empty graph $\emptyset$ and a weighted graph consisting of a single vertex without edges are linear chains. If necessary, renumber $D_1, \dots, D_r$ so that the direction of the linear chain Γ is from D_1 to D_r and $D_i D_{i+1} = 1$ for $i = 1, \dots, r-1$. We denote Γ by $[-D_1^2, \dots, -D_r^2]$. We sometimes write Γ as $[D_1, \dots, D_r]$. The linear chain is called *rational* if every D_i is rational. In this note, we always assume that every linear chain is rational. The linear chain Γ is called *admissible* if it is not empty and $D_i^2 \le -2$ for each i. Set $r(\Gamma) = r$. We define the *discriminant* $d(\Gamma)$ of Γ as the determinant of the $r \times r$ matrix $(-D_i D_j)$. We set $d(\emptyset) = 1$.

Let $A = [a_1, \dots, a_r]$ be a linear chain. We use the following notation if $A \ne \emptyset$:

$$ {}^tA := [a_r, \dots, a_1], \quad \overline{A} := [a_2, \dots, a_r], \quad \underline{A} := [a_1, \dots, a_{r-1}]. $$

The discriminant $d(A)$ has the following properties ([**F**, Lemma 3.6]).

Lemma 2. *Let $A = [a_1, \dots, a_r]$ be a linear chain.*

(i) *If $r > 1$, then $d(A) = a_1 d(\overline{A}) - d(\overline{\overline{A}}) = d({}^tA) = a_r d(\underline{A}) - d(\underline{\underline{A}})$.*
(ii) *If $r > 1$, then $d(\overline{A}) d(\underline{A}) - d(A) d(\overline{\underline{A}}) = 1$.*
(iii) *If A is admissible, then $\gcd(d(A), d(\overline{A})) = 1$ and $d(A) > d(\overline{A}) > 0$.*

Let $A = [a_1, \dots, a_r]$ be an admissible linear chain. The rational number

$e(A) := d(\overline{A})/d(A)$ is called the *inductance* of A. By [**F**, Corollary 3.8], the function e defines a one-to-one correspondence between the set of all the admissible linear chains and the set of rational numbers in the interval $(0, 1)$. For a given admissible linear chain A, the admissible linear chain $A^* := e^{-1}(1 - e({}^tA))$ is called the *adjoint* of A ([**F**, 3.9]). Admissible linear chains and their adjoints have the following properties ([**F**, Corollary 3.7, Proposition 4.7]).

Lemma 3. *Let A and B be admissible linear chains.*

(i) *If $e(A) + e(B) = 1$, then $d(A) = d(B)$ and $e({}^tA) + e({}^tB) = 1$.*
(ii) *We have $A^{**} = A$, ${}^t(A^*) = ({}^tA)^*$ and $d(A) = d(A^*) = d(\overline{A^*}) + d(\underline{A})$.*
(iii) *The linear chain $[A, 1, B]$ shrinks to $[0]$ if and only if $A = B^*$.*

For integers m, n with $n \geq 0$, we define $[m_n] = [\overbrace{m, \ldots, m}^{n}]$, $t_n = [2_n]$. For non-empty linear chains $A = [a_1, \ldots, a_r]$, $B = [b_1, \ldots, b_s]$, we write $A * B = [\underline{A}, a_r + b_1 - 1, \overline{B}]$, $A^{*n} = \overbrace{A * \cdots * A}^{n}$, where $n \geq 1$. We remark that $(A*B)*C = A*(B*C)$ for non-empty linear chains A, B and C. By using Lemma 2 and Lemma 3, we can show the following lemma.

Lemma 4. *Let $A = [a_1, \ldots, a_r]$ be an admissible linear chain.*

(i) *For a positive integer n, we have $[A, n+1]^* = t_n * A^*$.*
(ii) *We have $A^* = t_{a_r-1} * \cdots * t_{a_1-1}$.*
(iii) *If there exist positive integers m, n such that $[A, m+1] = [n+1, A]$ (resp. $A * t_m = t_n * A$), then $m = n$, $a_1 = \cdots = a_r = n + 1$ (resp. $A = t_n^{*\,r(A^*)}$).*

The following two lemmas describe the processes of contractions of special linear chains. The first one can be proved easily. We prove the second one.

Lemma 5. *Let A be an admissible linear chain and B a non-empty linear chain. Suppose that a composite π of blowing-downs contracts $[A, 1]$ to B.*

(i) *The linear chain B is the image of the first $r(B)$ curves of A. We have $A = B * t_n$, where $n = r(A) + 1 - r(B)$.*
(ii) *Every blowing-up of π is sprouting with respect to B or its preimage.*
(iii) *The exceptional curve of each blowing-up of π is a unique (-1)-curve in the preimage of B.*

Conversely, $[B * t_n, 1]$ shrinks to B for a given positive integer n and a non-empty linear chain B.

Lemma 6. *Let A, B be admissible linear chains and c a positive integer. Suppose that a composite π of blowing-downs contracts $[A, 1, B]$ to $[c, 1]$.*

(i) *The first curve of* $[c,1]$ *is the image of the first curve of* A. *We have* $n := r(A) - r(B^*) \geq 0$ *and* $A = [c, t_n] * B^*$. *In particular,* $n = 0$ *if* $c = 1$.

(ii) *The first* n *blowing-ups of* π *are sprouting and the remaining ones are subdivisional with respect to* $[c,1]$ *or its preimages. The composite of the subdivisional blowing-ups contracts* $[A,1,B]$ *to* $[c, t_n, 1]$.

(iii) *The exceptional curve of each blowing-up of* π *is a unique* (-1)*-curve in the preimage of* $[c,1]$.

PROOF. Write $A = [a_1, \dots, a_r]$, $B = [b_1, \dots, b_s]$. We prove the assertions by induction on $r + s \geq 2$. After the first blowing-down of π, $[A,1,B]$ becomes $T := [\underline{A}, a_r - 1, b_1 - 1, \overline{B}]$. The last blowing-up of π satisfies (iii) and is subdivisional with respect to T. Suppose $r + s = 2$. We have $T = [c,1]$, $\underline{A} = \overline{B} = \emptyset$, $b_1 = 2$ and $c = a_r - 1$. By Lemma 4, we obtain $B^* = [2]$ and $n = 0$. Hence $A = [c] * t_1 = [c, t_n] * B^*$. The remaining assertions are clear in this case. Assume $r + s \geq 3$. We have $T \neq [c,1]$. Since A and B are admissible, a_r or b_1 must be equal to 2. If $a_r = b_1 = 2$, then $T = [\underline{A}, 1, 1, \overline{B}]$, which is contracted to $[\dots, 0, \dots]$ by the second blowing-down. But the latter linear chain cannot shrink to $[c,1]$. Hence either a_r or b_1 must be greater than 2.

Case (1): $a_r = 2$, $b_1 > 2$. If $r = 1$, then $[b_s, \dots, b_2, b_1 - 1, 1]$ shrinks to $[1, c]$. By Lemma 5, $[b_s, \dots, b_2, b_1 - 1] = [1, c] * t_{s-1}$. Thus $b_s = 1$, which is a contradiction. Hence $r > 1$. Since $\underline{A}$ is admissible, we have $\underline{A} = [c, t_{n'}] * [b_1 - 1, \overline{B}]^*$ by the induction hypothesis, where $n' = r - r([b_1 - 1, \overline{B}]^*) - 1$. Hence $A = [c, t_{n'}] * [[b_1 - 1, \overline{B}]^*, 2]$. By Lemma 4, we obtain $[[b_1 - 1, \overline{B}]^*, 2] = (t_1 * [b_1 - 1, \overline{B}])^* = B^*$ and $r([b_1 - 1, \overline{B}]^*) = r(B^*) - 1$. The remaining assertions follow from the induction hypothesis.

Case (2): $a_r > 2$, $b_1 = 2$. If $s = 1$, then $[\underline{A}, a_r - 1, 1]$ shrinks to $[c,1]$. By Lemma 5, $[\underline{A}, a_r - 1] = [c,1] * t_{r-1} = [c, t_{r-1}]$. Hence $A = [c, t_{r-1}] * t_1 = [c, t_{r-1}] * B^*$. The remaining assertions also follow from Lemma 5 in this case. If $s > 1$, then we have $[\underline{A}, a_r - 1] = [c, t_{n'}] * (\overline{B})^*$ by the induction hypothesis, where $n' = r - r((\overline{B})^*)$. By Lemma 4, we obtain $A = [c, t_{n'}] * (\overline{B})^* * t_1 = [c, t_{n'}] * [2, \overline{B}]^* = [c, t_{n'}] * B^*$ and $r((\overline{B})^*) = r(B^*)$. The remaining assertions follow from the induction hypothesis. □

The following corollary to Lemma 6 describes the process of the contractions of linear chains in Lemma 3 (iii).

COROLLARY 7. *Let* A *and* B *be admissible linear chains. Suppose that a composite* π *of blowing-downs contracts* $[A,1,B]$ *to* $[0]$.

(i) *The first blowing-up of* π *is sprouting with respect to* $[0]$ *and the remaining ones are subdivisional with respect to preimages of* $[0]$.

(ii) *The exceptional curve of each blowing-up of* π *except the first one is a unique*

(-1)-*curve in the preimage of* $[0]$.

The next one is a corollary to Lemma 3 (iii), Lemma 5 and Lemma 6. It will be used to describe the process of the resolutions of cusps.

COROLLARY 8. *Let a be a positive integer and A an admissible linear chain. Let B be a linear chain which is empty or admissible. Assume that a composite π of blowing-downs contracts $[A, 1, B]$ to $[a]$ and that $[a]$ is the image of A under π.*

- (i) *The linear chain $[a]$ is the image of the first curve of A. There exits a positive integer n such that $A^* = [B, n+1, t_{a-1}]$. Moreover, $A = [a] * t_n * B^*$ if $B \neq \emptyset$.*
- (ii) *The first n blowing-ups of π are sprouting and the remaining ones are subdivisional with respect to $[a]$ or its preimages. The composite of the subdivisional blowing-ups contracts $[A, 1, B]$ to $[[a] * t_n, 1]$.*
- (iii) *The exceptional curve of each blowing-up of π is a unique (-1)-curve in the preimage of $[a]$.*

Conversely, $[[a] * t_n * B^*, 1, B]$ shrinks to $[a]$ for given positive integers a, n and an admissible linear chain B.

The following corollary follows from Corollary 8 (ii).

COROLLARY 9. *Let the notation and the assumption be as in Corollary 8 and b an integer. Then π contracts $[A, 1, B, b]$ to $[a, b-n]$. The second curve of $[a, b-n]$ is the image of the last curve of $[A, 1, B, b]$.*

2.2. Resolution of a cusp.

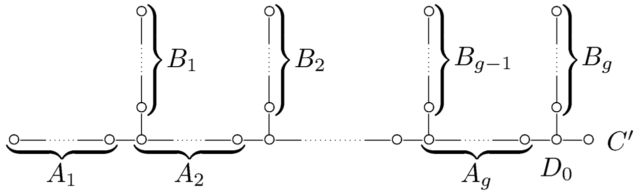
Let C be a curve on a smooth surface V. Suppose that C has a cusp P. Let $\sigma : V' \to V$ be the minimal embedded resolution of the cusp. That is, σ is the composite of the shortest sequence of blowing-ups such that the strict transform C' of C intersects $\sigma^{-1}(P)$ transversally. Let $V' = V_n \xrightarrow{\sigma_{n-1}} V_{n-1} \longrightarrow \cdots \longrightarrow V_2 \xrightarrow{\sigma_1} V_1 \xrightarrow{\sigma_0} V_0 = V$ be the blowing-ups of σ. The following lemma follows from the assumptions that P is a cusp and σ is minimal.

LEMMA 10. *For $i \geq 1$, the strict transform of C on V_i intersects $(\sigma_0 \circ \cdots \circ \sigma_{i-1})^{-1}(P)$ in one point, which is on the exceptional curve of σ_{i-1}. The point of intersection is the center of σ_i if $i < n$.*

We prove the following lemma.

LEMMA 11. *The following assertions hold* (*cf.* [**BK**], [**MaSa**]).

(i) *The dual graph of $\sigma^{-1}(C)$ has the following shape, where $g \geq 1$, D_0 is the* *exceptional curve of* σ_{n-1} *and* A_1 *contains the exceptional curve of* σ_0 *by definition.*



We number the irreducible components $A_{i,j}$ *of* A_i (*resp.* $B_{i,j}$ *of* B_i) *from the left-hand side to the right* (*resp. the bottom to the top*) *in the above figure. With these directions and the weights* $A_{i,j}^2, B_{i,j}^2$, *we regard* A_i, B_i *as linear chains.*

(ii) *The morphism* σ *can be written as* $\sigma = \sigma_0 \circ \rho_1' \circ \rho_1'' \circ \cdots \circ \rho_g' \circ \rho_g''$, *where each* ρ_i' (*resp.* ρ_i'') *consists of sprouting* (*resp. subdivisional*) *blowing-ups of* σ *with respect to preimages of* P.

(iii) *The morphisms* $\rho_i := \rho_i' \circ \rho_i''$ *have the following properties.*

(a) *For* $j < i$, ρ_i *does not change the linear chains* A_j, B_j.

(b) *For each* i, $\rho_i \circ \cdots \circ \rho_g$ *maps* $A_{i,1}$ *to a* (-1)*-curve.*

(c) ρ_g *contracts the linear chain* $A_g + D_0 + B_g$ *to the* (-1)*-curve* $\rho_g(A_{g,1})$. *For* $i < g$, ρ_i *contracts the linear chain* $(\rho_{i+1} \circ \cdots \circ \rho_g)(A_i + A_{i+1,1} + B_i)$ *to the* (-1)*-curve* $(\rho_i \circ \cdots \circ \rho_g)(A_{i,1})$.

Proof. For the sake of simplicity, we do not distinguish between a curve and its strict transforms via blowing-ups. The second blowing-up of σ is sprouting with respect to the exceptional curve of σ_0. Since P is a cusp and σ is minimal, the last blowing-up of σ must be subdivisional with respect to the preimage of P. These facts show the assertion (ii). Let $E_{0,0}$ denote the exceptional curve of σ_0 and $E_{i,0}$ the exceptional curve of the last blowing-up of ρ_i'' for each i. Put $E_0 = \emptyset$. Let E_i denote the exceptional curve of ρ_i. By Lemma 10, we infer that the dual graph of the sum of $E_{i-1,0}$ and the exceptional curve of ρ_i' is linear. Hence the dual graph of $E_{i-1,0} + E_i$ is linear. It follows that $E_{1,0}, \ldots, E_{g-1,0}, E_{g,0} = D_0$ are all the branching components of $\sigma^{-1}(C)$. The divisor $E_{i-1,0} + E_i - E_{i,0}$ consists of two connected components. Let A_i denote the one containing $E_{i-1,0}$ and B_i the remaining one. Then A_i, B_i and ρ_i have the desired properties. $\square$

We regard A_i and B_i as linear chains in the same way as in Lemma 11 (i). By Lemma 10, these linear chains are admissible. Let o_i denote the number of the blowing-ups in ρ_i'. The following proposition follows from Corollary 8 and Lemma 11 (iii).

PROPOSITION 12. *The following assertions hold for $i = 1, \ldots, g$.*

(i) *We have $A_i = t_{o_i} * B_i^*$, $A_i^* = [B_i, o_i + 1]$.*
(ii) *The linear chain A_i contains an irreducible component E with $E^2 \leq -3$.*

We will use the next lemma to prove some properties of the Orevkov's curves.

LEMMA 13. *Let D' be an SNC-divisor on a smooth surface V'. Suppose the following conditions are satisfied.*

(i) *The weighted dual graph of D' consists of a (-1)-curve D_0 and admissible rational linear chains $A_1, B_1, \ldots, A_g, B_g$, $g \geq 1$. They meet each other in the way described in Lemma 11 (i).*
(ii) *For $i = 1, \ldots, g$, there exists a positive integer o_i such that $A_i = t_{o_i} * B_i^*$, or equivalently $A_i^* = [B_i, o_i + 1]$.*

Then the following assertions hold.

(a) *The divisor D' shrinks to a point P by blowing-downs $\sigma : V' \to V$. The way of blowing-downs to contract D' to a point is unique.*
(b) *Let C' be a smooth curve on V'. If C' intersects only D_0 at one point transversally among the irreducible components of D', then $\sigma(C')$ is smooth outside of P and has a cusp at P, whose minimal embedded resolution coincides with σ.*

PROOF. (a) By Corollary 8, $[A_g, D_0, B_g]$ shrinks to a (-1)-curve, which is the image of the first curve $A_{g,1}$ of A_g. The image of $[A_{g-1}, A_{g,1}, B_{g-1}]$ under the above contraction shrinks to a (-1)-curve, which is the image of the first curve of A_{g-1}. Continuing in this way, we get blowing-downs $\sigma : V' \to V$ which contracts D' to a point P. The uniqueness follows from Corollary 8 (iii).

(b) Since C' is smooth, $\sigma(C')$ is also smooth outside of P. If the center of a blowing-up of σ is not on the image of C', then those of the remaining blowing-ups are not on the images of C' by Corollary 8 (iii). This contradicts the assumption that C' intersects D_0. Hence the center of each blowing-up of σ is on the image of C'. The remaining assertions of (b) follow from this fact. □

3. Orevkov's curves and proof of the "only if" part of Theorem 1.

In this section, we prove some properties of Orevkov's curves, from which the "only if" part of Theorem 1 follows. In [**O**], Orevkov constructed two sequences C_{4k}, C_{4k}^* ($k = 1, 2, \ldots$) of rational unicuspidal plane curves with $\bar{\kappa} = 2$ in the following way. Let N be a nodal cubic. Let Γ_1, Γ_2 denote the two analytic branches of N at the node. Let $\phi : W \to \boldsymbol{P}^2$ denote the composite of 7-times of blowing-ups such that the center of the first one is the node and every center of the remaining ones is the point of intersection of the strict transform of Γ_1 and the exceptional curve of the previous blowing-up. The dual graph of the exceptional curve E of ϕ is connected and linear. The curve E consists of 6-pieces of (-2)-curves and one (-1)-curve E' as an endpoint and intersects the strict transform of N at its two endpoints.

Let $\phi' : W \to \boldsymbol{P}^2$ denote the contraction of the strict transform of N and the 6-pieces of (-2)-curves in E. Put $f = \phi' \circ \phi^{-1}$. The curve $\phi'(E')$ is a nodal cubic. Let Γ denote one of the two analytic branches of $\phi'(E')$ at the node such that the center of the second blowing-up of ϕ' is not on its strict transform. We may assume $\phi'(E') = N$ and $\Gamma = \Gamma_1$ by composing a suitable projective transformation to f. Let C_0 be the tangent line at a flex of N and C_0^* an irreducible conic meeting with N only at one smooth point. See [**O**], [**AT**] or the appendix for the existence of C_0^*. Orevkov defined C_{4k}, C_{4k}^* as $C_{4k} = f(C_{4k-4})$, $C_{4k}^* = f(C_{4k-4}^*)$ $(k = 1, 2, \ldots)$. They have a cusp at the node and tangent to Γ_2 at the node.

Lemma 14. *Let C be a rational unicuspidal plane curve, $\sigma : V \to \boldsymbol{P}^2$ the minimal embedded resolution of the cusp and C' the strict transform of C via σ. Put $D = \sigma^{-1}(C)$. Let A_1, $B_1, \ldots, A_g, B_g, D_0$ denote the linear chains given for the cusp by Lemma* 11.

(i) *The curve C can be constructed in the same way as C_4 (resp. C_4^*) if and only if C satisfies the following conditions.*
  (a) *$g = 1$, $A_1 = [t_6, 4]$, $B_1 = t_2$ (resp. $A_1 = [t_6, 7]$, $B_1 = t_5$).*
  (b) *There exists a (-1)-curve E_0 such that it meets with D at two points transversally and intersects only the first curve and the last curve of A_1 among the irreducible components of D.*

(ii) *The curve C can be constructed in the same way as C_{4k+4} (resp. C_{4k+4}^*) for some $k \geq 1$ if and only if C satisfies the following conditions.*
  (a) *$g = 2$, $A_1 = t_6^{*k+1}$, $B_1 = [7_k]$, $A_2 = [4]$, $B_2 = t_2$ (resp. $A_2 = [7]$, $B_2 = t_5$).*
  (b) *There exists a (-1)-curve E_0 such that it meets with D at two points transversally and intersects only the first curve of A_1 and the last curve of B_1 among the irreducible components of D.*

(iii) *If C can be constructed in the same way as C_{4k} or C_{4k}^* for some $k \geq 1$, then $(C')^2 = -2$.*

Proof. The assertions for C_4 and C_4^* follow from their definition. We prove (ii) and (iii) for C_{4k+4}, $k \geq 1$. We can similarly deal with C_{4k+4}^*. We first show the "if" part of (ii) by induction on k. Let a_i and b_i denote the i-th curves of the linear chains A_1 and B_1, respectively. For the sake of simplicity, we sometimes use the same symbols for the strict transforms of them via a rational map which does not contract them.

Let $\sigma_2 : V_1 \to \boldsymbol{P}^2$ denote the composite of the first seven blowing-ups of σ and $\sigma_1 : V \to V_1$ the composite of the remaining ones. By Corollary 8 (ii), the last six blowing-ups of σ_2 are sprouting with respect to the preimages of the cusp. The weighted dual graph of the preimage of the cusp under σ_2 is the linear chain $[t_6, 1]$. By Corollary 8 (iii), the blowing-ups of σ_1 are done over the point of intersection of t_6 and the (-1)-curve. From these facts, we see $[t_6, 1] = [\sigma_1(a_1), \dots, \sigma_1(a_6), \sigma_1(b_k)]$. The dual graph of $\sigma_1(E_0 + a_1 + \cdots + a_6 + b_k)$ is a loop. We have $[1, t_6, 1] = [\sigma_1(E_0), \sigma_1(a_1), \dots, \sigma_1(a_6), \sigma_1(b_k)]$. Let $\varphi_1 : V_1 \to V_0$ denote the contraction of $\sigma_1(E_0 + a_1 + \cdots + a_5)$ and $\varphi_0 : V_0 \to \boldsymbol{P}^2$ the contraction of $\varphi_1(\sigma_1(a_6))$. Put $\varphi = \varphi_0 \circ \varphi_1$.

We arrange the order of blowing-downs of $\varphi \circ \sigma_1$ in the following way. We first perform six blowing-downs $\varphi_1' : V \to V'$ in the same way as φ_1. It contracts $E_0 + a_1 + \cdots + a_5$ to a point. Then we perform blowing-downs $\sigma_1' : V' \to V_0'$ in the same way as σ_1. It contracts $\varphi_1'(D - (C' + a_1 + \cdots + a_6 + b_k))$ to a point. Finally we perform the blowing-down $\varphi_0' : V_0' \to \boldsymbol{P}^2$ which contracts $\sigma_1'(\varphi_1'(a_6))$. The rational map $\varphi_0' \circ \sigma_1' \circ \varphi_1' \circ (\varphi \circ \sigma_1)^{-1}$ is a projective transformation since it does not have exceptional curves. By Corollary 9, $\varphi_1'(a_6)$ (resp. $\varphi_1'(b_k)$) is a (-2)-curve (resp. (-1)-curve). The weighted dual graph of $D - (a_1 + \cdots + a_6 + b_k)$ is unchanged by φ_1'.

We decompose the exceptional curve $\varphi_1'(D - (C' + a_1 + \cdots + a_5 + b_k))$ of $\varphi_0' \circ \sigma_1'$ into linear chains $A_1', B_1', \dots, A_{g'}', B_{g'}', \varphi_1'(D_0)$. If $k = 1$, then we set $g' = 1$, $A_1' = [\varphi_1'(a_6), \dots, \varphi_1'(a_{11}), \varphi_1'(A_2)]$ and $B_1' = \varphi_1'(B_2)$. We have $(A_1')^* = [t_6, 4]^* = [B_1', 8]$. If $k > 1$, then we set $g' = 2$, $A_1' = [\varphi_1'(a_6), \dots, \varphi_1'(a_{5k+6})]$, $B_1' = [\varphi_1'(b_1), \dots, \varphi_1'(b_{k-1})]$, $A_2' = \varphi_1'(A_2)$ and $B_2' = \varphi_1'(B_2)$. We have $(A_1')^* = [7_k] = [B_1', 7]$. It follows from Lemma 13 that $\hat{C} := \varphi(\sigma_1(C'))$ is unicuspidal and that $\varphi_0' \circ \sigma_1'$ is the minimal embedded resolution of the cusp. The linear chains $A_1', B_1', \dots, A_{g'}', B_{g'}'$ coincide with those given for $\hat{C}$ by Lemma 11. By the induction hypothesis ($k > 1$) and the assertion (i) ($k = 1$), $\hat{C}$ can be constructed in the same way as C_{4k}. The curve $\varphi_1(\sigma_1(a_6))$ intersects $\varphi_1(\sigma_1(b_k))$ only at two points transversally. This shows that $\varphi(\sigma_1(b_k))$ is a nodal cubic. The morphism φ (resp. σ_2) performs blowing-ups in the same way as ϕ (resp. ϕ'). Thus C can be constructed in the same way as C_{4k+4}.

We next show the "only if" part of (ii) and the assertion (iii) for C_{4k+4} by induction on k. The curve C is the strict transform of an Orevkov's curve C_{4k} via $f = \phi' \circ \phi^{-1}$. We denote by N_i (resp. $\phi_i : W_i \to \boldsymbol{P}^2$, $\phi_i' : W_i \to \boldsymbol{P}^2$, $\Gamma_{i,1}$, $\Gamma_{i,2}$) the nodal cubic N (resp. the birational morphisms ϕ, ϕ', the branches Γ_1, Γ_2 at the node) which is used to make C_{4i+4} from C_{4i}. Let $\sigma : V \to \boldsymbol{P}^2$ (resp. $\sigma_k : V_k \to \boldsymbol{P}^2$) denote the minimal embedded resolution of the cusp of C (resp. C_{4k}). From the definition of the Orevkov's curves, we infer that the centers of blowing-ups of ϕ_k' (resp. ϕ_{k-1}') are the cusp of C (resp. C_{4k}) or its strict transforms. This shows that $\sigma : V \to \boldsymbol{P}^2$ (resp. $\sigma_k : V_k \to \boldsymbol{P}^2$) can be written as $\sigma = \phi'_k \circ \sigma'$ (resp. $\sigma_k = \phi'_{k-1} \circ \sigma'_k$), where σ' (resp. σ'_k) consists of blowing-ups.

Let $\phi_{k,0} : W_{k,0} \to \boldsymbol{P}^2$ denote the first blowing-up of ϕ_k, which coincides with that of ϕ'_{k-1}. Let $\phi_{k,1}$ (resp. $\phi'_{k-1,1}$) denote the composite of the remaining blowing-ups of ϕ_k (resp. ϕ'_{k-1}). Let $A'_1, B'_1, \ldots, A'_{g'}, B'_{g'}, D'_0$ denote the linear chains given by Lemma 11 for C_{4k}. Let e_i denote the exceptional curve of the i-th blowing-up of ϕ'_{k-1}. On V_k, e_1 coincides with the first curve of A'_1. On $W_{k,0}$, N_k meets with e_1 in two points. The blowing-ups of $\phi_{k,1}$ are done over the one point $\Gamma_{k,1} \cap e_1$, while that of $\phi'_{k-1,1} \circ \sigma'_k$ are done over the other point $\Gamma_{k,2} \cap e_1$. By the definition of ϕ', the first blowing-up of $\phi'_{k-1,1}$ is done at $e_1 \cap \Gamma_{k,2}$. Each of the remaining ones is done at the point of intersection of N_k and the exceptional curve of the previous blowing-up. On W_{k-1}, N_k is a (-1)-curve.

Suppose $k = 1$. On W_0, C_4 (resp. e_7) coincides with the strict transform of C_0 (resp. N_0) via ϕ_0. This means that C_4 intersects only e_7 among $N_1, e_1, \ldots, e_7$. The blowing-ups of σ'_1 are done over $C_4 \cap e_7$. It follows from the assertion (i) that $A'_1 = [e_1, \ldots, e_7]$ on V_1. The curve N_1 is a (-1)-curve on V_1. It intersects only the first curve e_1 and the last curve e_7 of A'_1 among the irreducible components of $\sigma_1^{-1}(C_4)$. Suppose $k > 1$. By the definition of the Orevkov's curve, we infer that C_{4k} meets with $N_1 + e_1 + \cdots + e_7$ only at $e_6 \cap e_7$ on W_{k-1}. The blowing-ups of σ'_k are done over $e_6 \cap e_7$. It follows that on V_k, $e_1, \ldots, e_6$ are the first six curves of A'_1 and that e_7 is the last curve of B'_1. The curve N_k is a (-1)-curve on V_k. It intersects only the first curve e_1 of A'_1 and the last curve e_7 of B'_1 among the irreducible components of $\sigma_1^{-1}(C_{4k})$.

We return to the situation of the paragraph before the previous one. Recall that the blowing-ups of $\phi_{k,1}$ are done over $e_1 \cap \Gamma_{k,1}$, while that of $\phi'_{k-1,1} \circ \sigma'_k$ are done over $e_1 \cap \Gamma_{k,2}$. Since C_{4k} passes through $e_1 \cap \Gamma_{k,2}$ and does not through $e_1 \cap \Gamma_{k,1}$, σ' performs blowing-ups in the same way as $\phi'_{k-1,1} \circ \sigma'_k$. It follows that the weighted dual graph of D can be obtained from that of $\sigma_1^{-1}(C_{4k})$ by performing six times of blowing-ups $\psi : V'_k \to V_k$ in the same way as $\phi_{k,1}$. The first blowing-up of ψ is done at $e_1 \cap N_k$. Each of the remaining blowing-ups is done at the point of intersection of N_k and the exceptional curve of the previous blowing-up. Let E_i denote the exceptional curve of the i-th blowing-up of ψ. The dual graph of $N_k + E_6 + E_5 + \cdots + E_1 + e_1$ is linear. We have $[N_k, E_6, E_5, \ldots, E_1, e_1] = [7, 1, t_5, 3]$. Except e_1, ψ does not change $\sigma_1^{-1}(C_{4k})$ as a weighted graph. We have $(C')^2 = (C'_{4k})^2 = -2$.

The linear chains $A_1, B_1, \ldots, A_g, B_g$ are given in the following way. If $k = 1$, then $g = 2$, $A_1 = [E_5, \ldots, E_1, e_1, \overline{A'_1}] = t_6^{*2}$, $B_1 = N_1 = [7]$, $A_2 = e_7 = [4]$ and $B_2 = B'_1$. If $k > 1$, then $g = 2$, $A_1 = [E_5, \ldots, E_1, e_1, \overline{A'_1}] = t_6^{*k+1}$, $B_1 = [B'_1, N_k] = [7_k]$, $A_2 = A'_2$ and $B_2 = B'_2$. The strict transform of N_{k+1} via σ meets with D in the same way as E_6. It satisfies the condition that E_0 must satisfy. $\square$

By Proposition 15 below, each C_{4k} (resp. C^*_{4k}) does not depend on the choice of N and C_0 (resp. C^*_0) up to the projective equivalence. The "only if" part of Theorem 1 follows from this fact and Lemma 14 (iii).

PROPOSITION 15. *Let $C^{(1)}$ and $C^{(2)}$ be plane curves. If there exists a positive integer k such that $C^{(1)}$ and $C^{(2)}$ can be constructed in the same way as C_{4k}, or they can be constructed in the same way as C^*_{4k}, then $C^{(1)}$ is projectively equivalent to $C^{(2)}$.*

PROOF. We only show the assertion for the case in which there exists $k \geq 2$ such that $C^{(1)}$ and $C^{(2)}$ can be constructed in the same way as C_{4k}. We can similarly deal with the remaining cases. For each i, let $\sigma^{(i)} : V^{(i)} \to \boldsymbol{P}^2$ denote the minimal embedded resolution of the cusp of $C^{(i)}$. Write $A_1, B_1, \ldots, A_g, B_g$, D_0, etc. given by Lemma 11 for $C^{(i)}$ as $A_1^{(i)}, B_1^{(i)}, \ldots, A_{g_i}^{(i)}, B_{g_i}^{(i)}$, $D_0^{(i)}$, etc. Let $E_0^{(i)}$ denote the (-1)-curve E_0 given for $C^{(i)}$ in Lemma 14 (ii). We define a birational morphism $\psi^{(i)} : V^{(i)} \to \boldsymbol{P}^2$ in the following way. It first contracts $D_0^{(i)} + B_2^{(i)}$ to a point. Then it contracts the image of $A_1^{(i)} + E_0^{(i)} + B_1^{(i)}$ to a point. The last blowing-down of $\psi^{(i)}$ contracts the image $a_1^{(i)}$ of the last curve of $A_1^{(i)}$ to a point. We infer that $a_1^{(i)}$ intersects the image of $A_2^{(i)}$ at two points transversally. It follows that $\psi^{(i)}(A_2^{(i)})$ is a nodal cubic and that $\psi^{(i)}(C^{(i)\prime})$ is the tangent line at a flex of $\psi^{(i)}(A_2^{(i)})$. We may assume that each nodal cubic $\psi^{(i)}(A_2^{(i)})$ is defined by the equation given in the appendix. We denote $\psi^{(i)}(A_2^{(i)})$ by N. Let O_1, O_2 and O_3 be the flexes of N defined in the appendix. There exists a positive integer $a \leq 3$ such that $\psi^{(1)}(C^{(1)\prime})$ is the tangent line at O_a. Furthermore, there exists a projective transformation h such that $h(N) = N$ and $h(\psi^{(1)}(C^{(1)\prime})) = \psi^{(2)}(C^{(2)\prime})$.

Let $\psi_j^{(i)} : V_j^{(i)} \to V_{j-1}^{(i)}$ denote the j-th blowing-up of $\psi^{(i)}$, where $V_0^{(i)} = \boldsymbol{P}^2$. Since h maps the center of $\psi_1^{(1)}$ to that of $\psi_1^{(2)}$, the rational map $h_1 = \psi_1^{(2)-1} \circ h \circ \psi_1^{(1)} : V_1^{(1)} \to V_1^{(2)}$ is an isomorphism. The center of $\psi_2^{(1)}$ is one of the two points of intersection of N and the exceptional curve of $\psi_1^{(1)}$. By replacing h with the composite of h and the projective transformation φ_a given in the appendix, if necessary, we may assume that h_1 maps the center of $\psi_2^{(1)}$ to that of $\psi_2^{(2)}$. Thus $\psi_2^{(2)-1} \circ h_1 \circ \psi_2^{(1)} : V_2^{(1)} \to V_2^{(2)}$ is an isomorphism. For the remaining blowing-ups, there are no ambiguities in choices of centers. It follows that $h' = \psi^{(2)-1} \circ h \circ \psi^{(1)} : V^{(1)} \to V^{(2)}$ is an isomorphism. Since h' maps the exceptional curve of $\sigma^{(1)}$ to that of $\sigma^{(2)}$, the rational map $\sigma^{(2)} \circ h' \circ \sigma^{(1)-1}$ is a projective transformation such that $\sigma^{(2)} \circ h' \circ \sigma^{(1)-1}(C^{(1)}) = C^{(2)}$. $\square$

4. Structure of $\boldsymbol{C}^{**}$-fibration.

Let C be a rational unicuspidal plane curve and P the cusp of C. As in Section 2.2, let $\sigma : V \to \boldsymbol{P}^2$ denote the minimal embedded resolution of the cusp, σ_0 the first blowing-up of σ and C' the strict transform of C via σ. Put $D = \sigma^{-1}(C)$. Let D_0 denote the exceptional curve of the last blowing-up of σ. We decompose the dual graph of $\sigma^{-1}(P)$ into linear chains $A_1, B_1, \ldots, A_g, B_g, D_0$ in the same way as in Section 2.2. By Lemma 11, there exists a decomposition $\sigma = \sigma_0 \circ \rho_1' \circ \rho_1'' \circ \cdots \circ \rho_g' \circ \rho_g''$, where each ρ_i' (resp. ρ_i'') consists of sprouting (resp. subdivisional) blowing-ups with respect to preimages of P. Let o_i denote the number of the blowing-ups in ρ_i'.

Assume that the rational unicuspidal plane curve C satisfies the conditions that $(C')^2 = -2$ and $\bar{\kappa}(\boldsymbol{P}^2 \setminus C) = 2$. We see that one and only one of the two irreducible components of $D - D_0 - C'$ meeting with D_0 must be a (-2)-curve. Let F_0' denote the (-2)-curve and S_2 the remaining one. Let $\varphi_0 : V \to V'$ be the contraction of D_0 and C'. Since $(F_0')^2 = 0$ on V', there exists a $\boldsymbol{P}^1$-fibration $p' : V' \to \boldsymbol{P}^1$ such that F_0' is a nonsingular fiber. Put $p = p' \circ \varphi_0 : V \to \boldsymbol{P}^1$. Since $\bar{\kappa}(\boldsymbol{P}^2 \setminus C) = 2$, there exists an irreducible component S_1 of $D - D_0 - F_0'$ meeting with F_0' on V. Put $F_0 = F_0' + D_0 + C'$. The surface $X := V \setminus D$ is a $\boldsymbol{Q}$*-homology plane*. That is, X satisfies $H_i(X, \boldsymbol{Q}) = \{0\}$ for $i > 0$. A general fiber of $p|_X$ is a curve $\boldsymbol{C}^{**} = \boldsymbol{P}^1 \setminus \{3 \text{ points}\}$. Such fibrations have already been classified in [**MiSu**]. We will use their result to prove our theorem.

There exists a birational morphism $\varphi : V \to \Sigma_n$ from V onto the Hirzebruch surface Σ_n of degree n for some n such that $p \circ \varphi^{-1} : \Sigma_n \to \boldsymbol{P}^1$ is a $\boldsymbol{P}^1$-bundle. The morphism φ is the composite of the successive contractions of the (-1)-curves in the singular fibers of p. The curve S_1 (resp. S_2) is a 1-section (resp. 2-section) of p. The divisor D contains no other sections of p.

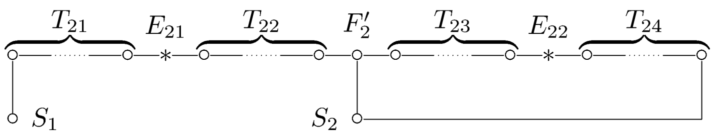
Lemma 16. *We may assume that $\varphi(S_1 + S_2)$ is smooth. We have $\varphi(S_1)^2 = -1$ and $\varphi(S_2)^2 = 4$.*

Proof. We only prove the first assertion. Suppose $\varphi(S_1 + S_2)$ has a singular point P. Let ϕ_1 be the blowing-up at P. Since $S_1 + S_2$ is smooth on V, we can choose the order of the blowing-ups of φ such that $\varphi = \phi_1 \circ \varphi'$. Let F' be the strict transform via ϕ_1 of the fiber of $p \circ \varphi^{-1}$ passing through P. Let ϕ_2 be the contraction of F'. Since F' is an irreducible component of a singular fiber of $p \circ \varphi'^{-1}$, we can replace φ with $\phi_2 \circ \varphi'$. We infer that P can be resolved by repeating the above process. Hence we may assume that $\varphi(S_1 + S_2)$ is smooth. $\square$

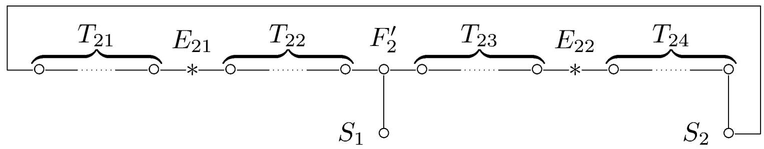
Each singular fiber of p intersects S_2 in at most two points. Suppose that there exists a singular fiber F_2 of p meeting with S_2 in two points. Let E_2 be the sum of the irreducible components of F_2 which are not components of D. Because D contains no loop, E_2 is not empty. Since $\bar{\kappa}(V \setminus D) = 2$, $V \setminus D$ does not contain contractible algebraic curves by [**MT2**, Main Theorem]. This means that each irreducible component of E_2 meets with D in at least two points.

In [**MiSu**, Lemma 1.5 and 1.6], singular fibers of a $\boldsymbol{C}^{**}$-fibration with a 2-section were classified into several types. Among them, only singular fibers of type (I_1) and (III_1) satisfy the conditions that they meet with the 2-section in two points and that each irreducible component of E_2 meets with D in at least two points. From the fact that D contains no loop, we infer that F_2 is of type (III_1). The dual graph of $F_2 + S_1 + S_2$ coincides with one of those in the following figure, where $*$ denotes a (-1)-curve and $E_2 = E_{21} + E_{22}$. The divisor $T_{2,i}$ may be empty for each i.

($\mathrm{III}_{1\mathrm{a}}$)



($\mathrm{III}_{1\mathrm{b}}$)

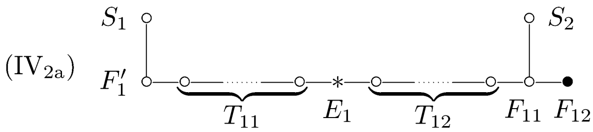


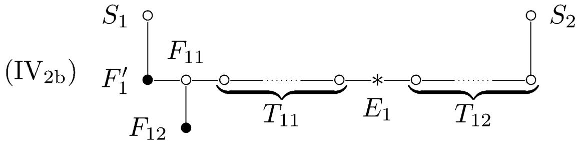
Lemma 17. *We have $\varphi(F_2) = \varphi(F_2')$, where F_2' is the irreducible component of F_2 whose position in F_2 is illustrated in the above figure.*

Proof. Suppose that φ contracts F_2'. Write $\varphi = \phi_3 \circ \phi_2 \circ \phi_1$, where ϕ_2 is the contraction of F_2'. If F_2 is of type ($\mathrm{III}_{1\mathrm{a}}$), then $\phi_1(F_2')\phi_1(S_1) = 0$ and $\phi_1(F_2')\phi_1(S_2) = 1$ by Lemma 16. Since $\phi_1(F_2 - F_2')\phi_1(F_2') \geq 2$, we have $\phi_2(\phi_1(F_2))\phi_2(\phi_1(S_2)) \geq 3$, which is a contradiction. If F_2 is of type ($\mathrm{III}_{1\mathrm{b}}$), then $\phi_1(F_2')\phi_1(S_2) = 0$ by Lemma 16. We have $\phi_2(\phi_1(F_2))\phi_2(\phi_1(S_1)) \geq 2$, which is absurd. $\square$

Suppose that there exists a singular fiber F_1 of p which intersects S_2 in one point. Let E_1 be the sum of the irreducible components of F_1 which are not components of D. By the same reasoning as for F_2, we deduce that F_1 is of type (IV_2). See [**MiSu**, Lemma 1.6]. The dual graph of $F_1 + S_1 + S_2$ coincides with one of those in the following figure, where $\bullet$ denotes a (-2)-curve. The divisor $T_{1,i}$ may be empty for each i.

We can choose the order of the blowing-downs of φ such that $\varphi = \varphi' \circ \varphi_1 \circ \varphi''$,





where φ_1 is the composite of all the contractions of irreducible components of F_1.

Lemma 18. *The morphism φ_1 contracts $\varphi''(T_{11}+E_1+T_{12}+F_{11})$ to a (-1)-curve, which is the image of F_{11}, and then contracts the (-1)-curve and the image of F_{12} in this order. We have $\varphi(F_1)=\varphi(F_1')$. Moreover, $(F_1')^2=F_{12}^2=-2$ if F_1 is of type $(\mathrm{IV}_{2\mathrm{b}})$.*

Proof. Suppose that F_1 is of type $(\mathrm{IV}_{2\mathrm{b}})$. Since $(F_1')^2 \le -2$, $F_{12}^2 \le -2$, φ contracts F_{11} before the contractions of F_1' and F_{12}. Since $\varphi(F_1)$ is smooth, $T_{11}+E_1+T_{12}$ must be contracted to a point before the contraction of F_{11}. It follows that $(F_1')^2=F_{12}^2=-2$. By Lemma 16, φ does not contract F_1'.

Suppose that F_1 is of type $(\mathrm{IV}_{2\mathrm{a}})$. Assume φ contracts F_1'. By Corollary 7, F_1' is the exceptional curve of the first blowing-up of φ_1. The remaining blowing-ups are subdivisional with respect to the preimages of $\varphi_1(\varphi''(F_1))$. By Lemma 16, the center of the first blowing-up is not on $\varphi_1(\varphi''(S_2))$. This means that $F_{11}S_2=2$, which is a contradiction. Thus φ does not contract F_1'. By Corollary 7, F_{12} is the exceptional curve of the first blowing-up of φ_1. Since the remaining blowing-ups are subdivisional with respect to the preimages of $\varphi_1(\varphi''(F_1))$, we infer that the exceptional curve of the second blowing-up of φ_1 coincides with the image of F_{11}. □

By the Riemann-Hurwitz formula, p has no more than two singular fibers which meet with S_2 in one point. Since the base curve of the fibration $p|_X$ is $\boldsymbol{C}$, p has one singular fiber of type (III_1) by [**MiSu**, Lemma 2.3]. It follows that the dual graph of D must be one of those in Figure 1.

5. Proof of the "if" part of Theorem 1.

Let the notation be as in the previous section. We determine which graphs in Figure 1 can be realized. With the direction from the left-hand side to the right of Figure 1, we regard T_{ij}'s as linear chains. Put $s_i=-S_i^2$ and $f_i=-(F_i')^2$ for

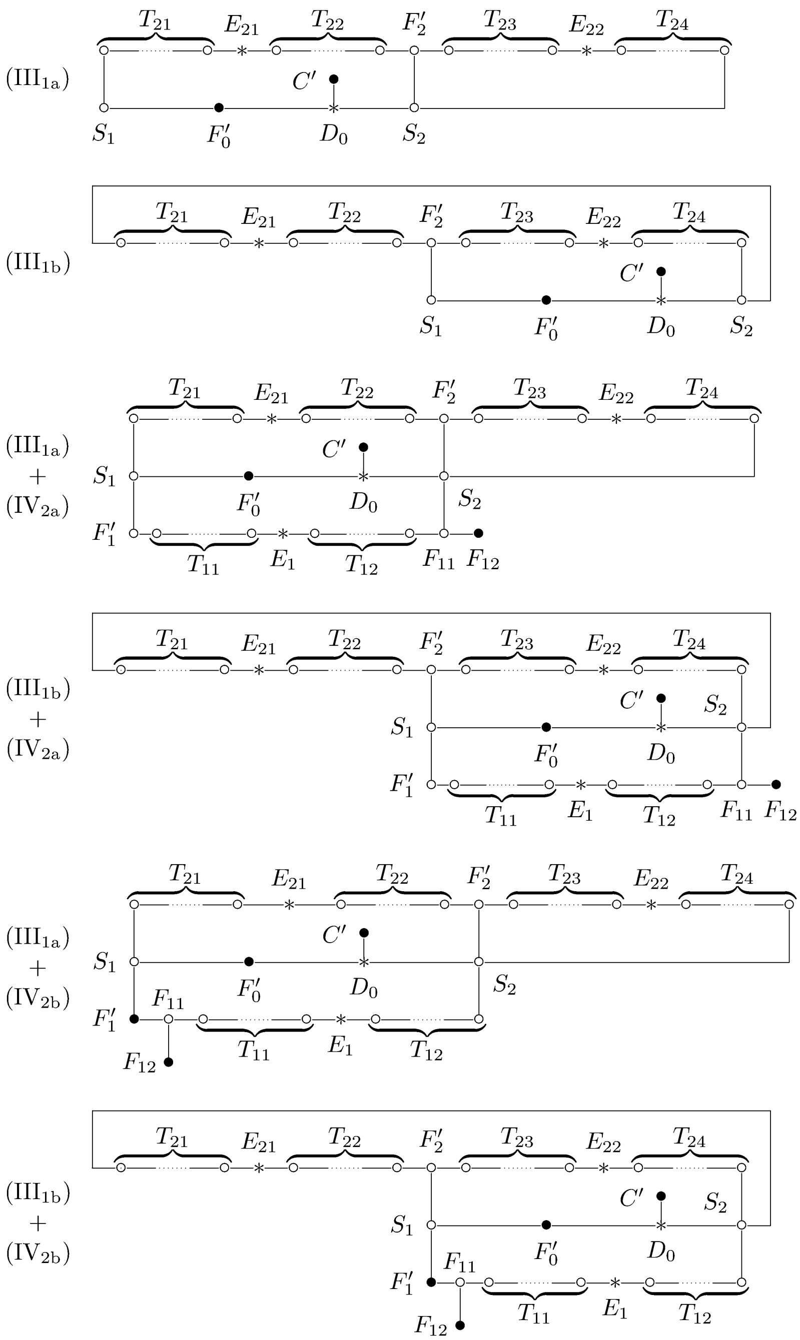


Figure 1. Dual graphs of $S_1 + S_2 + F_0 + F_1 + F_2$.

each i. We have $s_2 \geq 3$, $s_1 \geq 2$ and $f_i \geq 2$ for each i.

(III$_{1a}$). We may assume $\varphi = \varphi_0 \circ \varphi_{21} \circ \varphi_{22}$, where φ_{22} (resp. φ_{21}, φ_0) contracts $T_{23} + E_{22} + T_{24}$ (resp. $\varphi_{22}(T_{21} + E_{21} + T_{22})$, $\varphi_{21}(\varphi_{22}(C' + D_0))$) to a point. We first show the following lemma.

Lemma 19. *There exist positive integers k_{12} and k_{34} such that $[S_1, T_{21}]^* = [T_{22}, k_{12} + 1]$ and $[F_2', T_{23}]^* = [T_{24}, k_{34} + 1, t_{k_{12}-1}]$. We have $k_{34} = s_2 + 2 \geq 5$, $T_{23} \neq \emptyset$, $B_g = [F_0', S_1, T_{21}]$ and $A_g = t_{o_g} * [T_{22}, k_{12} + 2]$.*

Proof. By Lemma 16, $\varphi_{21}(\varphi_{22}(S_1))$ is a (-1)-curve. The morphism φ_{22} does not change the linear chain $[S_1, T_{21}, E_{21}, T_{22}]$. We apply Corollary 8 to $[S_1, T_{21}, E_{21}, T_{22}]$ and φ_{21}. There exists a positive integer k_{12} such that $[S_1, T_{21}]^* = [T_{22}, k_{12} + 1]$. Since $\varphi_{21}(\varphi_{22}(F_2'))$ is a 0-curve, $\varphi_{22}(F_2')$ must be a $(-k_{12})$-curve by Corollary 9. Again by Corollary 8, there exists a positive integer k_{34} such that $[F_2', T_{23}]^* = [T_{24}, k_{34} + 1, t_{k_{12}-1}]$. Since $\varphi(S_2)^2 = 4$, we have $4 = -s_2 + k_{34} + 2$ by Corollary 9. If $T_{23} = \emptyset$, then $[T_{24}, k_{34} + 1, t_{k_{12}-1}] = t_{f_2-1}$ by Lemma 4. We have $k_{34} = 1$. Thus $s_2 = -1$, which is absurd. Hence $T_{23} \neq \emptyset$. Either $A_g = {}^t[F_0', S_1, T_{21}]$ or $B_g = [F_0', S_1, T_{21}]$ by Lemma 11 (i). Suppose the former case holds. We have $g = 1$. Since $T_{23} \neq \emptyset$, we see $B_1 = [S_2, F_2', T_{23}]$ and $T_{22} = \emptyset$. By Proposition 12 (i) and Lemma 4, $[o_1 + 1, {}^tB_1] = {}^tA_1{}^* = [F_0', S_1, T_{21}]^* = [S_1, T_{21}]^* * t_1 = [k_{12} + 2]$, which is a contradiction. Thus $B_g = [F_0', S_1, T_{21}]$. By Proposition 12 (i) and Lemma 4, $A_g = t_{o_g} * B_g^* = t_{o_g} * [S_1, T_{21}]^* * t_1 = t_{o_g} * [T_{22}, k_{12} + 2]$. $\square$

Case (i): $T_{24} = \emptyset$. By Lemma 19, $[F_2', T_{23}] = [k_{34} + 1, t_{k_{12}-1}]^*$. By Lemma 4, $[k_{34} + 1, t_{k_{12}-1}]^* = [k_{12} + 1, t_{k_{34}-1}]$. Thus $f_2 = k_{12} + 1$ and $T_{23} = t_{k_{34}-1}$. Suppose $T_{22} \neq \emptyset$. We have $g = 2$ and $A_2 = [F_2', S_2]$ by Lemma 11 (i). By Lemma 19, we obtain $o_2 = 1$, $[f_2 - 1] = T_{22}$, $s_2 = k_{12} + 2$ and $k_{34} = k_{12} + 4$. Either $T_{23} = {}^tA_1$ or $T_{23} = B_1$. Since T_{23} consists of (-2)-curves, it follows from Proposition 12 (ii) that $T_{23} = B_1$ and $T_{22} = A_1$. By Proposition 12 (i), $T_{22} = A_1 = t_{o_1} * B_1^* = t_{o_1} * T_{23}^* = t_{o_1} * [k_{12} + 4]$. Thus $o_1 = 1$ and $f_2 = k_{12} + 6$, which contradicts $f_2 = k_{12} + 1$. Hence $T_{22} = \emptyset$. We have $g = 1$ and $A_1 = {}^t[S_2, F_2', T_{23}]$. By Lemma 19, $[t_{k_{34}-1}, f_2, s_2] = t_{o_1} * [k_{12} + 2]$. We see $s_2 = k_{12} + 3$, $f_2 = 2$ and $o_1 = k_{34} + 1$. It follows that $k_{12} = 1$, $s_2 = 4$, $k_{34} = 6$ and $o_1 = 7$. We have $A_1 = [t_6, 4]$ and $[B_1, o_1 + 1] = A_1^* = [t_2, 8]$. The curve E_{22} intersects only the first and the last curve of A_1 among the irreducible components of D. By Lemma 14, C can be constructed as C_4.

Case (ii): $T_{24} \neq \emptyset$. Since S_2 is a branching component of D, we infer $A_g = S_2$ by Lemma 11 (i). By Lemma 19, we obtain $o_g = 1$, $T_{22} = \emptyset$, $s_2 = k_{12} + 3$ and $k_{34} = k_{12} + 5$. We have $g = 2$. Either $B_1 = [F_2', T_{23}]$ or $B_1 = {}^tT_{24}$. If $B_1 = [F_2', T_{23}]$, then $T_{24} = A_1 = t_{o_1} * [F_2', T_{23}]^* = t_{o_1} * [T_{24}, k_{34} + 1, t_{k_{12}-1}]$, which is impossible. Thus $B_1 = {}^tT_{24}$ and $A_1 = {}^t[F_2', T_{23}]$. By Proposition 12 (i), $[o_1+1, T_{24}] = {}^tA_1^* = [F_2', T_{23}]^*$. By Lemma 19, $[o_1+1, T_{24}] = [T_{24}, k_{12}+6, t_{k_{12}-1}]$. Hence $k_{12} = 1$, $[o_1+1, T_{24}] = [T_{24}, 7]$. It follows from Lemma 4 that $o_1 = 6$ and $T_{24} = [7_k]$, where $k = r(T_{24}) \geq 1$. We have $B_1 = [7_k]$, $A_1 = t_{o_1} * B_1^* = t_6^{*k+1}$ and $A_2 = [4]$. Since $[B_2, o_2+1] = A_2^* = t_3$, we obtain $B_2 = t_2$. The curve E_{22} intersects only the first curve of A_1 and the last curve of B_1 among the irreducible components of D. By Lemma 14, C can be constructed as C_{4k+4}.

$(\mathrm{III}_{1a}) + (\mathrm{IV}_{2a})$. We have $A_g = S_2$ and $B_g = [F_0', S_1, F_1', T_{11}]$ because S_2 is a branching component of D. By Proposition 12 (i), $[B_g, o_g+1] = A_g^* = t_{s_2-1}$. Thus $[F_1', T_{11}] = t_{s_2-4}$. By Lemma 18, φ contracts F_1 to a 0-curve, which is the image of F_1'. By Lemma 3 (iii), $[T_{12}, F_{11}, F_{12}] = [F_1', T_{11}]^* = t_{s_2-4}^* = [s_2-3]$, which is absurd. Hence this case does not occur.

$(\mathrm{III}_{1a}) + (\mathrm{IV}_{2b})$. We may assume $\varphi = \varphi_0 \circ \varphi_1 \circ \varphi_{21} \circ \varphi_{22}$, where φ_{22} (resp. φ_{21}, φ_1, φ_0) contracts $T_{23}+E_{22}+T_{24}$ (resp. $\varphi_{22}(T_{21}+E_{21}+T_{22})$, $\varphi_{21}(\varphi_{22}(F_{11}+F_{12}+T_{11}+E_{11}+T_{12}))$, $\varphi_1(\varphi_{21}(\varphi_{22}(C'+D_0))))$ to a point. We show the following three lemmas.

Lemma 20. *There exist positive integers k_{12} and k_{34} such that $[S_1, T_{21}]^* = [T_{22}, k_{12}+1]$ and $[F_2', T_{23}]^* = [T_{24}, k_{34}+1, t_{k_{12}-1}]$. We have $[F_{11}, T_{11}]^* = [T_{12}, s_2 - k_{34}+1]$ and $s_2 \geq k_{34}+1$.*

Proof. By the same arguments as in the proof of Lemma 19, there exist positive integers k_{12}, k_{34} such that $[S_1, T_{21}]^* = [T_{22}, k_{12}+1]$ and $[F_2', T_{23}]^* = [T_{24}, k_{34}+1, t_{k_{12}-1}]$. By Lemma 18 and Corollary 8, there exists a positive integer l such that $[F_{11}, T_{11}]^* = [T_{12}, l+1]$. Since $\varphi(S_2)^2 = 4$, we infer $4 = -s_2 + k_{34} + 2 + l + 2$. Thus $1 \leq l = s_2 - k_{34}$. $\square$

Lemma 21. *We have $T_{21} = \emptyset$, $T_{22} = t_{s_1-2}$ and $k_{12} = 1$.*

Proof. Suppose that S_1 is a branching component of D. We have $A_g = [S_1, F_0']$, $T_{12} = T_{24} = \emptyset$ and $B_g = [S_2, F_2', \ldots]$. By Lemma 20, $[F_{11}, T_{11}] = t_{s_2-k_{34}}$ and $[F_2', T_{23}] = [k_{12}+1, t_{k_{34}-1}]$. By Proposition 12 (i), $[B_g, o_g+1] = A_g^* = t_1 * t_{s_1-1} = [3, t_{s_1-2}]$. Thus $o_g = 1$, $f_2 = 2$ and $s_2 = 3$. Since $f_2 = k_{12}+1$, we obtain $k_{12} = 1$. Because $\emptyset \neq [F_{11}, T_{11}] = t_{3-k_{34}}$, we have $k_{34} \leq 2$. If $k_{34} = 1$, then $T_{23} = t_{k_{34}-1} = \emptyset$. Thus $B_g = [S_2, F_2', {}^tT_{22}]$. By Proposition 12 (i), $A_g = t_{o_g} * B_g^* = t_1 * [3, 2, {}^tT_{22}]^* = t_1 * [2, {}^tT_{22}]^* * t_2$. By Lemma 20, $t_1 * [2, {}^tT_{22}]^* * t_2 = t_1 * [{}^tT_{21}, S_1] * t_2$. This means that $S_1 = t_1 * [{}^tT_{21}, S_1] * t_1$, which is impossible. Hence $k_{34} = 2$. Since $T_{23} = [2] \neq \emptyset$, we infer $B_g = [S_2, F_2', T_{23}]$ and $T_{22} = \emptyset$. By Lemma 20, $[S_1, T_{21}] = t_{k_{12}} = [2]$, which is absurd. Hence S_1 is not a branching component of D. We have $T_{21} = \emptyset$. By Lemma 20, $[T_{22}, k_{12}+1] = t_{s_1-1}$. From this, we obtain

$k_{12} = 1$ and $T_{22} = t_{s_1 - 2}$. □

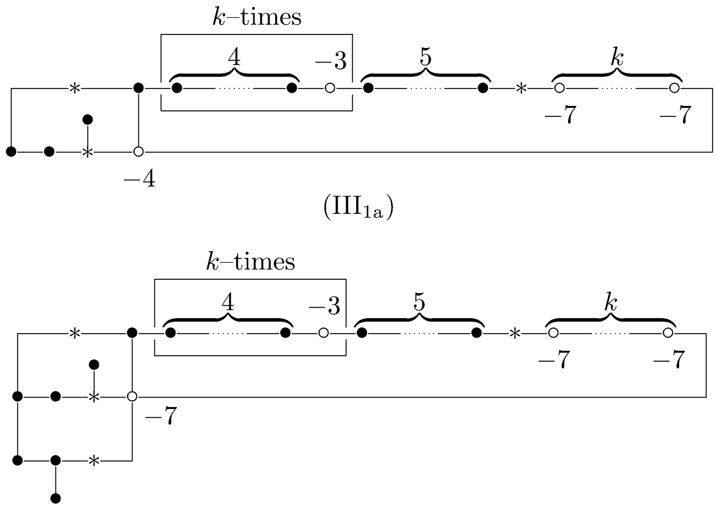
LEMMA 22. *We have* $T_{11} = T_{12} = \emptyset$, $B_g = [F_0', S_1, F_1', F_{11}, F_{12}]$, $s_2 = k_{34} + 1$ *and* $F_{11} = [2]$.

PROOF. Either $S_2 \subset A_g$ or $S_2 \subset B_g$. Suppose $S_2 \subset B_g$. We have $T_{24} = T_{12} = \emptyset$. By Lemma 20, $[F_2', T_{23}] = [k_{34} + 1]^* = t_{k_{34}}$. Thus $f_2 = 2$, $T_{23} = t_{k_{34} - 1}$. Since $[F_{11}, T_{11}] = t_{s_2 - k_{34}}$, we get $F_{11} = [2]$ and $T_{11} = t_{s_2 - k_{34} - 1}$. If $T_{11} \neq \emptyset$, then $A_1 = F_{12}$ or $A_1 = {}^t T_{11}$ since F_{11} is a branching component of D. Thus A_1 consists of (-2)-curves, which contradicts Proposition 12 (ii). Hence $T_{11} = \emptyset$. We have $s_2 = k_{34} + 1$, $g = 1$ and $A_1 = [F_{12}, F_{11}, F_1', S_1, F_0'] = [t_3, S_1, 2]$. We infer $s_1 \geq 3$. By Proposition 12 (i), $[B_1, o_1 + 1] = A_1^* = [3, t_{s_1 - 3}, 5]$. This means that $s_2 = 3$ and $k_{34} = 2$. Since $T_{23} = [2] \neq \emptyset$, we have $B_1 = [S_2, F_2', T_{23}]$ and $T_{22} = \emptyset$. By Lemma 21, $s_1 = 2$, which is a contradiction. Hence $S_2 \subset A_g$. We have $B_g = [F_0', S_1, F_1', F_{11}, F_{12}]$ and $T_{11} = \emptyset$. By Lemma 20, $[T_{12}, s_2 - k_{34} + 1] = t_{-F_{11}^2 - 1}$. This shows $s_2 = k_{34} + 1$ and $T_{12} = t_{-F_{11}^2 - 2}$. If $T_{12} \neq \emptyset$, then $F_{11}^2 < -2$ and $A_g = S_2$. By Proposition 12 (i), $[B_g, o_g + 1] = A_g^* = t_{s_2 - 1}$, which is absurd. Hence $T_{12} = \emptyset$ and $F_{11} = [2]$. □

Case (i): $T_{24} = \emptyset$. By Lemma 20, $[F_2', T_{23}] = t_{k_{34}}$. We have $f_2 = 2$ and $T_{23} = t_{k_{34} - 1} = t_{s_2 - 2} \neq \emptyset$. If $T_{22} \neq \emptyset$, then $A_1 = T_{22}$ or $A_1 = {}^t T_{23}$. Thus A_1 consists of (-2)-curves, which contradicts Proposition 12 (ii). Hence $T_{22} = \emptyset$. We infer $g = 1$ and $A_1 = {}^t[S_2, F_2', T_{23}] = [t_{k_{34}}, k_{34} + 1]$. By Lemma 21, we have $S_1 = [2]$ and $B_1 = t_5$. By Proposition 12 (i), $A_1 = t_{o_1} * [6] = [t_{o_1 - 1}, 7]$. Hence $k_{34} = 6$, $A_1 = [t_6, 7]$. The curve E_{22} intersects only the first and the last curve of A_1 among the irreducible components of D. By Lemma 14, C can be constructed as C_4^*.

Case (ii): $T_{24} \neq \emptyset$. We have $A_g = S_2$. By Proposition 12 (i), we get $[B_g, o_g + 1] = A_g^* = t_{s_2 - 1}$. We see $S_1 = [2]$, $B_g = t_5$, $s_2 = 7$ and $k_{34} = 6$ by Lemma 22. By Lemma 21, $T_{22} = \emptyset$. We infer $g = 2$. Either $B_1 = {}^t T_{24}$ or $A_1 = T_{24}$. If $A_1 = T_{24}$, then $B_1 = [F_2', T_{23}]$. By Proposition 12 (i) and Lemma 20, $T_{24} = t_{o_1} * [F_2', T_{23}]^* = t_{o_1} * [T_{24}, 7]$, which is absurd. Hence $B_1 = {}^t T_{24}$ and $A_1 = {}^t[F_2', T_{23}]$. By Proposition 12 (i) and Lemma 20, $[o_1 + 1, T_{24}] = [F_2', T_{23}]^* = [T_{24}, 7]$. It follows from Lemma 4 that $o_1 = 6$, $T_{24} = [7_k]$, where $k = r(T_{24}) \geq 1$. We have $B_2 = t_5$, $A_2 = [7]$, $B_1 = [7_k]$ and $A_1 = t_6^{*k+1}$. The curve E_{22} intersects only the first curve of A_1 and the last curve of B_1 among the irreducible components of D. By Lemma 14, C can be constructed as C_{4k+4}^*.

(III_{1b}), $(\mathrm{III}_{1b}) + (\mathrm{IV}_{2a})$ or $(\mathrm{III}_{1b}) + (\mathrm{IV}_{2b})$. In each case, we have $-2 \geq \varphi(S_1)^2$ because S_1 meets with only F_i' among the irreducible components of F_i for each i. Hence all the cases do not occur.



$(\mathrm{III}_{1a}) + (\mathrm{IV}_{2b})$

Figure 2. The dual graphs of $D + E_1 + E_2$.

We list the weighted dual graphs of $D + E_1 + E_2$ in Figure 2, where $k = 0$ if $T_{24} = \emptyset$. We proved that if a rational unicuspidal plane curve C satisfies the conditions $(C')^2 = -2$, $\bar{\kappa} = 2$, then C can be constructed in the same way as C_{4k} or C^*_{4k} for some k. By Proposition 15, C is projectively equivalent to C_{4k} or C^*_{4k}. We have thus proved Theorem 1.

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Appendix by Fumio Sakai.

Let N be the nodal cubic $x^3 + y^3 - xyz = 0$. Let O denote the node $(0, 0, 1)$. It is well known that the set $N \setminus \{O\}$ has a group structure, which is isomorphic to the multiplicative group $\boldsymbol{C}^*$. The group isomorphism is given by $\phi : \boldsymbol{C}^* \ni t \mapsto (t, -t^2, t^3 - 1) \in N \setminus \{O\}$. Geometrically, we have $t_1 t_2 t_3 = 1$ if and only if $\phi(t_1)$, $\phi(t_2)$ and $\phi(t_3)$ are collinear. We see easily that N has three flexes $O_1 = (1, -1, 0) = \phi(1)$, $O_2 = (1, -\omega, 0) = \phi(\omega)$ and $O_3 = (1, -\omega^2, 0) = \phi(\omega^2)$, where $\omega = e^{2\pi i/3}$. There exist three projective transformations

$$\varphi_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 & \omega^2 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} 0 & \omega & 0 \\ \omega^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that $\varphi_i(O_i) = O_i$, $\varphi_i(O_j) = O_k$ for distinct i, j, k among $\{1, 2, 3\}$.

THEOREM 23. *Define three conics*

$$\begin{aligned} Q_1 &: 21(x^2 + y^2) - 22xy - 6(x + y)z + z^2 = 0, \\ Q_2 &: 21(\omega x^2 + \omega^2 y^2) - 22xy - 6(\omega^2 x + \omega y)z + z^2 = 0, \\ Q_3 &: 21(\omega^2 x^2 + \omega y^2) - 22xy - 6(\omega x + \omega^2 y)z + z^2 = 0. \end{aligned}$$

Then the conic Q_1 (resp. Q_2, Q_3) intersects N only at the point $P_1 = \phi(-1)$ (resp. $P_2 = \phi(-\omega)$, $P_3 = \phi(-\omega^2)$).

Conversely, if Q is an irreducible conic with the property that Q intersects N only at a point $P \in N \setminus \{O\}$, then Q is one of the above three conics.

Note that the tangent line to Q_i at P_i passes through O_i for each i and that $\varphi_i(Q_i) = Q_i$, $\varphi_i(Q_j) = Q_k$ for distinct i, j, k among $\{1, 2, 3\}$.

PROOF. Let Q be a conic defined by the general equation:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0.$$

Suppose that Q intersects N only at a point $P = \phi(\alpha) \in N \setminus \{O\}$, where $\alpha \in \boldsymbol{C}^*$. Then we have

$$at^2 + bt^4 + c(t^3 - 1)^2 - dt^3 + et(t^3 - 1) - ft^2(t^3 - 1) = 0.$$

It follows that

$$ct^6 - ft^5 + (b + e)t^4 - (2c + d)t^3 + (a + f)t^2 - et + c = 0.$$

Since Q does not pass through O, we infer that $c \neq 0$. So we may assume that $c = 1$. Thus, we have

$$t^6 - ft^5 + (b + e)t^4 - (2 + d)t^3 + (a + f)t^2 - et + 1 = 0.$$

By our hypothesis, this equation must have only one multiple root α of order six. We see that $\alpha^6 = 1$, $f = 6\alpha$, $b + e = 15\alpha^2$, $2 + d = 20\alpha^3$, $a + f = 15\alpha^4$, $e = 6\alpha^5$.

In particular, α is a 6-th root of unity. We then obtain the equations of the conics Q_1, Q_2, Q_3 for $\alpha = -1, -\omega, -\omega^2$, respectively. For the cases in which $\alpha = 1$, ω, ω^2, the conic Q is reduced to a double tangent line at the flex O_1, O_2, O_3, respectively. $\square$

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