THE MINIMUM NUMBER OF POINTS OF INFLEXION OF CLOSED CURVES IN THE PROJECTIVE PLANE

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(Received July 17, 1957)

1. Let us consider a smooth closed curve Γ in a projective plane P_2. By a point of inflexion of Γ we mean a point M of Γ such that the tangent to Γ at M is stationary. We shall prove the following theorems:

THEOREM 1. *Any smooth closed curve in the projective plane which is not homotopic to zero has at least a point of inflexion.*

THEOREM 2. *Any simple smooth closed curve Γ in the projective plane which is not homotopic to zero has at least three points of inflexion.*

Many theorems in the large are known for Euclidean differential geometry and Riemannian geometry and some for affine differenfial geometry too. However, it seems to the author that few theorems are known for projective differential geometry in the large. The above theorems give simple examples of theorems for projective differential geometry in the large.

2. We take a unit sphere S_2 in Euclidean space E_3 and consider it as the universal covering space of P_2. We shall denote the natural projection of S_2 onto P_2 by ϕ.

Take a point A on Γ and denote the points on S_2 over A by A_1 and A_2. If we draw the curve over Γ on S_2 starting from A_1, then the curve Γ_1 will end at A_2 by the assumption that Γ is not homotopic to zero. If we draw again the curve over Γ on S_2 starting from A_2, then the curve Γ_2 will end at A_1 and these two curves over Γ are congruent on S_2 and lie symmetric with respect to the center of S_2. We shall call any closed curve such that the antipodal point of any point of the curve lies on the curve as an *antipodal curve*. Then the curve $\Gamma^* = \Gamma_1 + \Gamma_2$ is a smooth antipodal curve.

Now, by a point of inflexion of a curve on S_2 we mean a point M on the curve such that the tangent great circle to the curve at M is stationary. Then, we can easily see that any point of inflexion of the curve Γ on P_2 corresponds to an antipodal pair of points of inflexion of the curve Γ^* on S_2 as straight lines on P_2 correspond to great circles on S_2. Hence, the problem turns to prove the existence of points of inflexion on antipodal curves on S_2.

First, let us study analytical condition for a point of inflexion. In ordinary Euclidean plane E_2, the condition for a point of inflexion of a curve $y = f(x)$ referred to a Cartesian coordinate system is $y'' = 0$. If all points of the curve are points of inflexion, then the curve must be a solution of the differential equation $y'' = 0$. The differential equation $y'' = 0$ is that of straight lines in E_2. In the same way, the condition for a point of inflexion of a curve on S_2 is given by the differential equation of geodesics on S_2. However, it is necessary and sufficient that the equation holds only for the point of inflexion.

We shall represent S_2 by

$$x = \cos u \cos v, \qquad y = \sin u \cos v, \qquad z = \sin v \tag{1}$$

and the antipodal curve Γ^* on S_2 by periodic functions

$$u = u(s), \quad v = v(s) \tag{2}$$

of period $2l$ where s is the arc length of Γ^* and $2l$ is the length of Γ^*, so the values s and $s + l$ correspond to antipodal pair of points on Γ^*. Then the necessary and sufficient condition that a point $s = s_0$ on Γ^* is a point of inflexion is that the following relation holds good at $s = s_0$:

$$\cos v(\dot{v}\ddot{u} - \ddot{u}\dot{v}) + 2 \sin v\, \dot{u}\dot{v}^2 + \sin v \cos^2 v\, \dot{u}^3 = 0. \tag{3}$$

The last equation multiplied by $\cos u$ is nothing but the differential equation of geodesics in Weierstrass' form. We shall remark that the geodesic curvature k_g of Γ^* is given by

$$k_g = \cos v(\dot{u}\ddot{v} - \ddot{u}\dot{v}) + 2 \sin v\, \dot{u}\dot{v}^2 + \sin v \cos^2 v\, \dot{u}^3. \tag{4}$$

3. To prove Theorem 1 it is sufficient to prove the following one for antipodal curves on S_2:

THEOREM 1'. *Any smooth antipodal closed curve on S_2 has at least an antipodal pair of points of inflexion.*

PROOF. Let A_1 and A_2 be an antipodal pair of points on Γ^* which are not points of inflexion and consider the great circle perpendicular to the radius OA_1. Then we may assume that on the great circle there exist at least a point C_1 and its antipodal point C_2 which do not lie on Γ^*, for if every point of the great circle lies on Γ^*, then Γ^* has infinitely many points of inflexion. We take a rectangular coordinate system such that OA_1 and OC_1 coincide with x-axis and z-axis respectively. We can easily see that in some neighborhoods of C_1 and its antipodal point C_2, there do not lie points of the curve Γ^* any more. And we assume that our curve Γ^* is given by the equations (1) with (2) with respect to this coordinate system.

We shall denote $k_g(s)$ given by (4) by $k(s)$. As the correspondence $M_1(s)$ of Γ^* to its antipodal point $M_2(s + l)$ is given by

$$(u, v) \rightarrow (u + m\pi, \ -v)\ , \tag{5}$$

where m is a fixed odd integer, we can easily see that

$$k(s + l) = -k(s). \tag{6}$$

By assumption A_1 and A_2 are not points of inflexion. If we assume that A_1 and A_2 correspond to parameter values 0 and l, then $k(0) \neq 0$, $k(l) \neq 0$ and $k(l) \lesseqgtr 0$ according as $k(0) \gtreqless 0$. Hence there exists at least a value s_0 in $(0, l)$ such that $k(s_0) = k(s_0 + l) = 0$. Such values s_0 and $s_0 + l$ correspond to points of inflexion on Γ^*. Accordingly Γ^* must have at least an antipodal pair of points of inflexion. Q. E. D.

Theorem 1′ can be stated also as follows:

THEOREM 1″. *Any smooth antipodal closed curve on S_2 has at least an antipodal pair of points such that the geodesic curvatures at them are zero.*

Fig. 1 and 2 give examples of closed curves with a node in P_2 which have only one point of inflexion. In Fig. 1, the straight line g is asymptotic to both branches of the curve Γ and the points at infinity on g is a point of inflexion of Γ. In Fig. 2, the straight line g is asymptotic for both branches

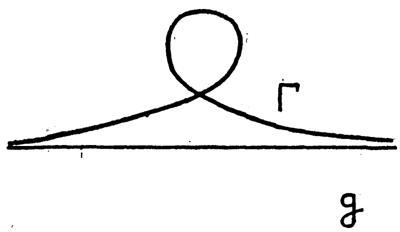

Fig. 1

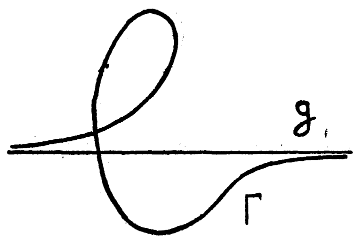

Fig. 2

of the curve Γ, but the point at infinity on g is not a point of inflexion of Γ. The point of inflexion lies in finite reigion of the plane.

4. To prove Theorem 2 it is sufficient to prove the following one for antipodal curves on S_2:

THEOREM 2′. *Any smooth, simple, antipodal closed curve on S_2 has at least three antipodal pairs of points of inflexion.*

PROOF. By Theorem 1′, there exists on Γ^* a pair of points of inflexion. We shall denote these points by C_1 and C_2 and take rectangular coordinate system such that the radius OC_1 coincides with the positive z-axis and the tangent great circle to Γ^* at C_1 lies in xz-plane. We may assume that C_1 and C_2 have parameter values 0 and l respectively.

We may assume also, performing a reflexion with respect to a plane passing through O if it is necessary, that the arc corresponding to parameter values $0 \leqq s \leqq s_1$ for s_1 sufficiently small lies in the positive side of xz-plane. If we take sufficiently small neighborhood of C_1 and C_2, then it is clear that there does not enter other part of Γ^* than the parts which correspond parameter values sufficiently near to 0 and l.

Now, we map the sphere S_2 onto a plane E_2 by the Mercator projection ψ defined by

$$\xi = u, \qquad \eta = \log\tan\left(\frac{v}{2} + \frac{\pi}{4}\right), \tag{7}$$

where (ξ, η) is rectangular coordinate system in the plane E_2. Then the part of Γ^* for $0 \leqq s \leqq s_1$ is mapped into E_2 as a branch of a curve $\psi(\Gamma^*)$ which has the positive ξ-axis as its asymptote for $s \to 0$.

As s runs in $l \leqq s \leqq l + s_1$, the corresponding arc of Γ lies antipodally to the arc corresponding to $0 \leqq s \leqq s_1$, so the half arc Γ_1 of Γ^* from C_1 to C_2 corresponding to $0 \leqq s \leqq l$ arrives at C_2 from the direction of the oriented great circle $C_1A_1C_2$ where A_1 is the point $(1, 0, 0)$. Accordingly, if we map the arc $s_2 \leqq s \leqq l$ for s_2 sufficiently near to l by the Mercator projection ψ, then the part of $\psi(\Gamma^*)$ for $s_2 \leqq s \leqq l$ has a straight line

$$\xi = m\pi \qquad (m: \text{a fixed even integer}) \tag{8}$$

as its asymptote for $s \to l$. We shall show that $m = 0$.

To show it, we consider the antipodal part Γ_2 of the arc Γ_1 corresponding to $0 \leqq s \leqq l$. Then Γ_2 can be represented both by parameters $(u(s) + \pi, -v(s))$ and by parameters $(u(s) - \pi, -v(s))$ for $0 \leqq s \leqq l$. Hence, if $m \neq 0$ in (8), we consider that Γ_2 is represented by $(u(s) + \pi, -v(s))$ for $0 \leqq s \leqq l$. Then we can see that $\psi(\Gamma_1)$ intersects with the curve $\psi(\Gamma_2)$, which shows that Γ^* intersects with itself contrary to the assumption that Γ^* is simple. So m can not be greater than zero. In the same way m can not be smaller than zero. Hence $m = 0$.

Now, consider the point M_1 on the arc Γ_1 $(0 \leqq s \leqq l)$ such that the parameter value $\xi(s)$ is the absolute maximum. At the point M_1 $\dot{u} = 0$, $\ddot{u} \leqq 0$, but $\dot{v}$ can not be greater than zero, for if so the curve $\psi(\Gamma_1)$ must intersect to itself which contradicts to the assumption that Γ^* is simple. If we assume that M_1 corresponds to s_0, then we can easily see that

$$k(s_0) = -\cos v(s_0)\, \ddot{u}\dot{v} \leqq 0.$$

So, if the point M_1 is not a point of inflexion, then $k(s_0) < 0$ at M_1. Even if M_1 is a point of inflexion, if Γ_1 does not contain a subarc of a great circle $u = u(s_0)$, $k(s) < 0$ near s_0.

(i) The case where the arc corresponding to $s_2 \leqq s \leqq l$ mentioned above lies in the positive side of xz-plane. In this case $k(s)$ is greater than zero for $s > 0$ sufficiently near to zero and for $s < l$ sufficiently near to l. Hence there exist two values a and b such that $k(a) = k(b) = 0$ and $0 < a < s_0$, $s_0 < b < l$. The pairs of points on Γ^* corresponding to $a, a + l$ and b, $b + l$ are antipodal pairs of points of inflexion of Γ^*.

(ii) The case where the arc corresponding to $s_2 \leqq s \leqq l$ mentioned above lies in the negative side of xz-plane. We consider the point N_1 on the arc corresponding to $s_0 \leqq s \leqq l$ such that the parameter value $\xi(s)$ is the absolute minimum. At the point N_1 $\dot{u} = 0$, $\ddot{u} \geqq 0$, but $\dot{v}$ can not be greater than zero, for if so the curve $\psi(\Gamma_1)$ must intersect with itself. If we assume that N_1 corresponds to s_0', then we can easily see that

$$k(s_0') = -\cos v(s_0')\, \ddot{u}\dot{v} \geqq 0.$$

So, if the point N_1 is not a point of inflexion, then $k(s_0') > 0$ at N_1. Even if N_1 is a point of inflexion, if Γ_1 does not contain a subarc of a great circle $u = u(s_0')$, $k(s) > 0$ near s_0'. Hence, in this case, there exist two values a and b such that $k(a) = k(b) = 0$ and $0 < a < s_0$, $s_0 < b < s_0'$. The pairs of points on Γ^* corresponding to $a, a + l$ and b, $b + l$ are antipodal pairs of points of inflexion of Γ^*.

Consequently, we can see that Γ^* in consideration has at least three antipodal pairs of points of inflexion. Q. E. D.

THEOREM 2′ can be stated also as follows:

THEOREM 2″. *Any smooth, simple, antipodal closed curve on S_2 has at least three antipodal pairs of points such that the geodesic curvatures at them are zero.*

Fig. 3 and 4 give examples of simple closed curves in P_2 which have just

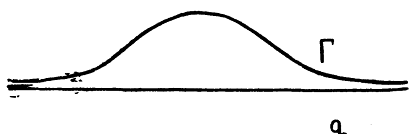

Fig. 3

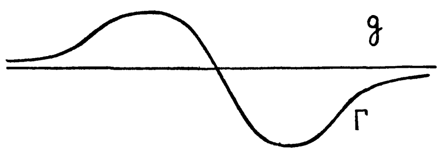

Fig. 4

three points of inflexion. In Fig. 3 the straight line g is asymptotic to both branches of the curve Γ and the point at infinity on g is a point of inflexion. In Fig. 4, the straight line g is asymptotic to both branches of the curve Γ and the point at infinity on g is not a point of inflexion.

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