Banach J. Math. Anal. 7 (2013), no. 2, 225–243

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

ON HIGH DIMENSIONAL MAXIMAL OPERATORS

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Communicated by M. S. Moslehian

Abstract. In this note we describe some recent advances in the area of maximal function inequalities. We also study the behaviour of the centered Hardy–Littlewood maximal operator associated to certain families of doubling, radial decreasing measures, and acting on radial functions. In fact, we precisely determine when the weak type $(1,1)$ bounds are uniform in the dimension.

1. Introduction

Given a Borel measure μ on a metric space X and a locally integrable function g, the centered Hardy–Littlewood maximal operator M_μ is given by

$$M_\mu g(x) := \sup_{\{r>0:0<\mu(B(x,r))\}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |g| d\mu, \tag{1.1}$$

where $B(x,r)$ denotes the open ball of radius $r>0$ centered at x. Recall that g is locally integrable if for every $x \in X$ there exists an $r>0$ such that $\int_{B(x,r)} |g| d\mu < \infty$. For instance, $g(x) := 1/x$ is locally integrable on $(0,\infty)$, but not on $\mathbb{R}$, regardless of how it is extended to $(-\infty, 0]$.

We allow measures that assign infinite size to some balls. Of course, if μ assigns infinite measure to all balls, then it is of no interest in this context, since then $M_\mu g \equiv 0$ for every locally integrable g (we adopt the convention $\infty/\infty = \infty \cdot 0 = 0$). Note that if all balls (with finite radii) have finite measure, then it does not matter whether one uses open or closed balls in the definition of M_μ. It follows from countable additivity that this does not alter the value of

Date: Received: 6 February 2013; Accepted: 26 February 2013.

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2010 *Mathematics Subject Classification.* Primary 42B25; Secondary 47B38.

Key words and phrases. Maximal operators, weak type bounds, radial measures.

$M_\mu g(x)$, since closed (resp. open) balls can be obtained as countable intersections (resp. unions) of open (resp. closed) balls with the same center. When $\mu = \lambda^d$, the d-dimensional Lebesgue measure, we often simplify notation, by writing M rather than M_{λ^d} and dx instead of $d\lambda^d(x)$.

It is well known that M_μ is a positive, sublinear operator, acting on the cone of positive, locally integrable functions (M_μ is defined by using $|g|$ rather than g). The Hardy–Littlewood maximal operator admits many variants: Instead of averaging $|g|$ over balls centered at x (the centered operator) as in (1.1), it is possible to consider all balls containing x (the uncentered operator) or average over convex bodies more general than euclidean balls (and even over more general sets, for instance, star-shaped, lower dimensional, etc.). It can also be applied to locally finite measures ν (rather than just functions) by setting (say, in the centered case)

$$M_\mu \nu(x) := \sup_{\{r>0:\mu(B(x,r))>0\}} \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$

The Hardy–Littlewood maximal operator is an often used tool in Real and Harmonic Analysis, mainly (but not exclusively) due to the fact that while $|g| \le M_\mu g$ a.e., $M_\mu g$ is not too large (in an L^p sense) since for every Borel measure μ defined on $\mathbb{R}^d$, it satisfies the following strong type (p,p) inequality: $\|M_\mu g\|_p \le C_p \|g\|_p$ for $1 < p \le \infty$. Thus, $M_\mu g$ is often used to replace g, or some average of g, in chains of inequalities, without leaving L^p $(p > 1)$.

The situation when $p = 1$ is different. Taking $g = \chi_{[0,1]}$, we see that Mg (on the real line with Lebesgue measure) behaves essentially like $1/x$ near infinity, so Mg is not integrable. However, it follows from the Besicovitch Covering Theorem that M_μ satisfies the weak type $(1,1)$ inequality $\sup_{\alpha>0} \alpha\mu(\{M_\mu g \ge \alpha\}) \le c_1 \|g\|_1$ for every Borel measure μ on $\mathbb{R}^d$. This is a very important fact, as it implies the L^p bounds for $1 < p < \infty$ via interpolation (the Marcinkiewicz Interpolation Theorem generalizes this result). From now on we shall use $c_{1,d}$ to denote the lowest possible constant in the weak type (1,1) inequality when the dimension is d, and likewise, $C_{p,d}$ will denote the lowest strong (p,p) constant in dimension d.

2. Weak bounds, strong bounds, and dimensions

An aspect of the Hardy–Littlewood maximal operator that is receiving increasing attention, but which will not be touched upon here, is that of its regularity properties (cf. for instance [6, 7, 8, 5] and the references contained therein). In this paper we restrict our attention to results regarding weak and strong type bounds. Since, as mentioned above, maximal operators are often used in chains of inequalities, improvements in these bounds lead to improvements in several other inequalities.

Considerable efforts have gone into determining how changing the dimension of $\mathbb{R}^d$ modifies the best constants $C_{p,d}$ and $c_{1,d}$ in the case of Lebesgue measure. When $p = \infty$, we can take $C_{p,d} = 1$ in every dimension, since averages never exceed a supremum. At the other endpoint $p = 1$, the first boundedness arguments used the Vitali covering lemma, which leads to exponential bounds of the type

$c_{1,d} \le 3^d$, and by interpolation, to exponential bounds for $C_{p,d}$. So it is natural to try to improve on these bounds, and in particular, to seek bounds independent of the dimension, with a view towards infinite dimensional generalizations of Harmonic Analysis.

In the Vitali covering lemma one obtains a disjoint subfamily from a finite family of balls by a greedy algorithm and enlarging radii: Choose first the ball B_1 with largest radius. Then remove from the collection all the balls that intersect it. Observe that the union of these balls is contained in the ball $3B_1$ with the same center and three times the radius as B_1. Then choose B_2 as the ball with the largest radius among the balls left, and repeat. This argument works well whenever the measure of balls with large radii is controlled by the measure of balls with the same center and smaller radii, in the following sense: There exists a constant K such that for all balls B, $\mu 2B \le K \mu B$. Such measures μ are called doubling because we double the radius of B, but in fact any other constant $t > 1$ could be used in place of 2. For instance, doubling with 2 implies doubling with 4, with constant K^2, and doubling with 4 implies doubling with 2, trivially.

In his Princeton Ph. D. thesis, motivated by Fritz John's solution of the wave equation via spherical means, Prof. Antonio Cordoba (personal communication) considered what nowadays is called Bourgain's circular maximal function, where averages are taken over circumferences centered at a point, in dimension $d = 2$ (there is a small subtlety in the definition; since circumferences have area zero, one needs to work first with functions defined everywhere, for instance, continuous functions, or C^∞ functions, and then, if one manages to prove strong type bounds of some sort, the operator can be defined over measurable functions via approximation arguments). However, A. Cordoba was unable to obtain L^p bounds for this maximal operator. As it turns out, these bounds were easier to establish in higher dimensions. E. M. Stein showed that for $d \ge 3$ the (Stein's) spherical maximal operator (where averages are taken over centered spheres) was bounded in L^p if and only if $p > d/(d-1)$, cf. [31]. It took about ten years, and the efforts of Bourgain, to extend Stein's result to $d = 2$, cf. [12]. So the moral here seems to be that one should not start with the hardest case. Of course, a priori it may not be obvious what is easy and what is difficult. For instance, in $d = 1$ a simple covering argument yields, for the uncentered operator and essentially all measures, $c_{1,1} \le 2$ (cf. Theorem 3.5 below) and often $c_{1,1} = 2$ is sharp (example: Lebesgue measure). However, if we ask the same question for the centered operator and (just) Lebesgue measure, then even proving that the constant is different from 2 is difficult. This was done in [2], where the then commonly accepted conjecture $c_{1,1} = 3/2$ was also refuted. The exact value $c_{1,1} = (11 + \sqrt{61})/12$ was obtained by Melas by a rather involved argument, in the two papers [26, 27].

Returning to the spherical maximal operator, it is more or less intuitively clear that it controls the Hardy–Littlewood maximal operator M associated to euclidean balls (but this requires some argument). By proving dimension independent bounds for the spherical maximal operator, E. M. Stein showed that for M, there exist bounds for C_p that are independent of d ([32, 33], [35], see also [34]). Stein's result was generalized to the maximal function defined using an arbitrary norm by Bourgain ([13, 14, 15]) and Carbery ([17]) when $p > 3/2$. For ℓ_q balls, $1 \le q < \infty$, Müller [28] showed that uniform bounds again hold for every $p > 1$ (given $1 \le q < \infty$, the ℓ_q balls are defined using the norm $\|x\|_q := (|x_1|^q + |x_2|^q + \cdots + |x_d|^q)^{1/q}$).

Regarding weak type $(1,1)$ inequalities, in [35] Stein and Strömberg proved that the smallest constants in the weak type (1,1) inequality satisfied by M grow at most like $O(d)$ for euclidean balls, using the heat semigroup, and at most like $O(d \log d)$ for more general balls, by a difficult covering lemma argument. They also asked if uniform bounds could be found, a *question still open* for euclidean balls.

Semigroup theory enters maximal function estimates via the Hopf maximal ergodic theorem for semigroups of operators, applied to the heat semigroup. Here the supremum is taken over time (one dimensional) so the bound is independent of dimension. Now the maximal function bound Cd (C a constant) appears as follows: It is possible to express the centered maximal operator in terms of convolutions:

$$Mf(x) = \sup_{r>0} |f| * \frac{\chi_{B(0,r)}}{\lambda^d(B(0,r))}(x).$$

The argument then proceeds by showing that there exists a constant $C > 0$ and $s = s(d)$ such that

$$\frac{\chi_{B(0,r)}}{\lambda^d(B(0,r))}(x) \le \frac{Cd}{s} \int_0^s \frac{1}{(4\pi t)^{d/2}} e^{-\frac{\|x\|_2^2}{4t}} dt.$$

These results about the Hardy–Littlewood maximal operator were obtained during the eighties, after which activity in this area slowed down. But recently, it seems to have picked up steam. In 2008 the note [3] was posted in the Math ArXiv (but was published in 2011, so some papers that cite it have earlier publication dates). It is shown there that if one considers cubes with sides parallel to the coordinate axes (that is, ℓ_∞ balls) instead of euclidean balls, then the best constants $c_{1,d}$ must diverge to infinity with d, and thus the answer to the Stein-Strömberg question is negative for cubes. This was proven by elementary means, basically calculus and first year probability (the normal approximation to the binomial distribution). More advanced probabilistic techniques (the theory of stochastic processes and in particular, the brownian bridge) quickly lead to an improvement: Aubrun showed shortly after that $c_{1,d} \ge \Theta(\log^{1-\varepsilon} d)$, where Θ denotes the exact order and $\varepsilon > 0$ is arbitrary, cf. [11]. Finally, the question whether the maximal operator associated to cubes and Lebesgue measure is uniformly bounded in d, for each $1 < p \le 3/2$, has recently received a positive answer by Bourgain [16] (Math. ArXiv, December 11th, 2012). So, save for refinements on the size of the constants, the situation is now well understood for cubes (and Lebesgue measure).

These results suggest (at least to us) that uniform bounds for $c_{1,d}$ may fail to exist if one uses euclidean balls (the original question of Stein and Strömbeg) since there seems to be no reason to believe that the maximal operator associated to euclidean balls is substantially smaller than the maximal operator associated to cubes.

A very significant extension of the Stein and Strömberg's $O(d \log d)$ theorem, beyond $\mathbb{R}^d$, has recently been obtained by Naor and Tao, cf. [29]. At the level of generality these authors work, the order of growth $O(d \log d)$ cannot be lowered, as they show by constructing the appropriate counterexample.

In the Vitali covering lemma one covers balls by expanding the radius of an intersecting ball, which may have only slightly larger radius than the others. It was already noted in [35] that engulfing balls by expanding the radius of a much larger ball can be more efficient. This idea leads Naor and Tao to define the Microdoubling and Strong Microdoubling properties on metric measure spaces.

A metric measure space (X, d, μ) is a separable metric space (X, d), equipped with a Radon measure μ. Naor and Tao also assume that $0 < \mu(B(x, r)) < \infty$ for all $r > 0$. Now (X, d, μ) is defined to be d-Microdoubling with constant K if for all $x \in X$ and all $r > 0$, we have

$$\mu B\left(x, \left(1 + \frac{1}{d}\right) r\right) \leq K \mu B(x, r).$$

Note that the case $n = 1$ is just doubling. And (X, d, μ) is Strong d-Microdoubling with constant K if for all x, all $r > 0$ and all $y \in B(x, r)$,

$$\mu B\left(y, \left(1 + \frac{1}{d}\right) r\right) \leq K \mu B(x, r).$$

Naor and Tao prove a localization result for microdoubling spaces: One does not need to consider the supremum over all $r > 0$ when proving weak type bounds, provided the averaging operators are well behaved. And this is implied by strong n-microdoubling. In the specific case of $\mathbb{R}^d$ with Lebesgue measure, their localization result entails that it is enough to consider radii r satisfying $1 \leq r \leq d$. It is clear that localized maximal operators with $c \leq r \leq (1 + 1/d)c$, are bounded by the averaging operator with radius $r = c$ times the microdoubling constant. Since $(1 + 1/d)^{d \log d} \approx d$, it follows that we need roughly $d \log d$ steps to go from 1 to to d by using $c_0 = 1$, $c_1 = (1 + 1/d)$, $c_2 = (1 + 1/d)^2$, etc. Thus the maximal operator M with $1 \leq r \leq d$ is controlled by the sum of $O(d \log d)$ maximal operators with $c_i \leq r \leq (1 + 1/d)c_i$, which yields the result by Stein and Strömberg mentioned above. Localization is proved by approximating in a certain sense metric spaces by ultrametric spaces via "random partitioning methods"; certain modified Doob's maximal inequalities for sublinear operators are proved and applied in their arguments. A second proof of the $O(d \log d)$ bound is given via the "Random Vitali Covering Lemma" of Lindenstrauss.

Another setting where it is natural to explore these issues is that of d-dimensional Riemannian or sub-Riemannian manifolds, or spaces not as general as metric measure spaces. In [23], Hong-Quan Li extends to the Heisenberg groups the $O(d)$ estimate of Stein and Strömberg for euclidean balls on $\mathbb{R}^d$, by semigroup methods. And in [24], Li and Lohoué give an $O(d \log d)$ upper bound for the weak type (1,1) inequalities, when working with the Riemannian volume in hyperbolic spaces. This is quite remarkable, as the volume of balls in hyperbolic spaces grows exponentially, so no doubling or microdoubling condition is satisfied (in fact, no doubling measure can be defined in the hyperbolic spaces). Again the result is obtained by semigroup methods. In a recent preprint (personal communication) Hong-Quan Li obtains L^p bounds independent of the dimension ($p > 1$) for the centered maximal operator in hyperbolic spaces (once more by semigroup methods).

Curiously, the analogous question for area on the d-dimensional sphere appears not to have been answered. Of course, one would expect the same result to hold, that is, the existence of L^p bounds ($p > 1$) independent of the dimension, for the centered maximal operator defined by geodesic balls (spherical caps).

A different line of research explores what happens in $\mathbb{R}^d$ under measures that may be different from Lebesgue measure, restricted to some special class of functions (something which of course, simplifies arguments). From now on we always refer to the centered maximal function defined by euclidean balls. It is shown in [25, Theorem 3] that considering only radial functions (with Lebesgue measure) leads to $c_{1,d} \le 4$ in all dimensions, and the same happens if Lebesgue measure is replaced by a radial, radially increasing measure, cf. [22, Theorem 2.1]. Besides, for Lebesgue measure and radial *decreasing* functions, it is shown in [9, Theorem 2.7] that the sharp constant is $c_{1,d} = 1$.

If instead of radial, radially increasing measures one considers radial, radially *decreasing* measures, the situation changes radically. Typically, one has exponential increase in the dimension for $c_{1,d}$, and some times even for the strong type constants $C_{p,d}$. Furthermore it is enough to consider characteristic functions of balls centered at zero (hence, radial and decreasing) to prove exponential increase. The weak type $(1,1)$ case for integrable radial densities defined via bounded decreasing functions was studied in [2]. It was shown there that the best constants $c_{1,d}$ satisfy $c_{1,d} \ge \Theta(1)\left(2/\sqrt{3}\right)^{d/6}$, in strong contrast with the linear $O(d)$ upper bounds known for Lebesgue measure. Exponential increase was also shown for the same measures and small values of $p > 1$ in [19]; shortly after (and independently) these results were improved in [10], as they applied to larger exponents p and to a wider class of measures. It was also shown in [10] that exponential increase could occur for arbitrarily large values of p and suitably chosen doubling measures. Together with the results for hyperbolic spaces mentioned before, this shows that the doubling condition is neither necessary nor sufficient to have "good bounds" for maximal inequalities in terms of the dimension. Finally, it is proven in [20] that for the standard gaussian measure in $\mathbb{R}^d$, one has exponential increase in the constants *for all* $p \in (1,\infty)$. So from this viewpoint, the most important measures in $\mathbb{R}^d$, Lebesgue and Gaussian, behave in a completely opposite manner.

In the next section we consider the following question about the maximal operator acting on radial functions: As we have seen, uniform bounds hold for radial non-decreasing measures, and we have exponential increase for several classes of radial decreasing measures. So it is natural to ask whether Lebesgue measure is the borderline case which separates uniform from non-uniform behavior in the constants. We shall show in the next section that the answer to this question is negative: For the the radial decreasing measures μ_d on $\mathbb{R}^d$, defined by $d\mu_d(y) = \frac{dy}{\|y\|_2^\alpha}$, $\alpha > 0$, and the maximal operator acting on radial integrable functions, the constants $c_{1,d}$ are bounded uniformly in d; of course, the bounds we find increase with α, as was to be expected. In fact, if the exponents α_d are allowed to increase to infinity with d, then so do the constants $c_{1,d}$.

3. Uniform bounds for some radial measures and radial functions

Recall that $\|x\|_2 := (x_1^2 + x_2^2 + \cdots + x_d^2)^{1/2}$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is radial if there is a second function $f_0 : (0, \infty) \to \mathbb{R}$ such that

$$f(x) = f_0(\|x\|_2) \tag{3.1}$$

on $\mathbb{R}^d \setminus \{0\}$, i.e., $f(x)$ depends only on the distance from x to the origin, and not on x itself (no restriction is placed on $f(0)$). Thus, f is rotation invariant. Since f depends only on one parameter (the distance to the origin) it is not surprising that uniform bounds can be found (at least for some measures) by reduction to the 1-dimensional case. All functions considered in this section are radial. Next, radial measures are defined as follows. Fix $d \in \mathbb{N} \setminus \{0\}$, and let $\mu_0 : (0, \infty) \to [0, \infty)$ be a (possibly unbounded) function, not zero almost everywhere, such that $\mu_0(t)t^{d-1} \in L^1_{\mathrm{loc}}[(0, \infty), dt]$. Then the function μ_0 defines a rotationally invariant measure μ on $\mathbb{R}^d$ via

$$\mu(A) := \int_A \mu_0(\|y\|_2) d\lambda^d(y). \tag{3.2}$$

Here μ_0 is allowed to depend on d, and the local integrability of $\mu_0(t)t^{d-1}$ is assumed for each *fixed* d. Furthermore, μ may fail to be locally finite, even if $\mu_0(t)t^{d-1} \in L^1_{\mathrm{loc}}[(0, \infty), dt]$. This happens, for instance, if $d = 1$ and $\mu_0(t) = t^{-1}$: In this case $\mu(-h, h) = \infty$ for every $h > 0$. For convenience, we assume in this section that maximal operators are defined using closed balls, which we denote also by $B(x, r)$, to keep the notation simple.

We shall show next that uniform weak type (1,1) bounds hold for the radial measures with densities given $d\mu(y) = \frac{dy}{\|y\|_2^\alpha}$, where α is a fixed constant, independent of the dimension. However, as soon as we allow the exponents to grow to infinity with the dimension, this result fails. So the measures $d\mu(y) = \frac{dy}{\|y\|_2^\alpha}$ represent the borderline case between uniform and non-uniform weak type (1,1) bounds. Finally, if the exponents are allowed to grow like αd, where $\alpha \in (1/2, 1)$ is fixed, then there is exponential increase of the constants $C_{p,d}$ *for all* $p < \infty$.

Theorem 3.1. *For $d \geq 1$, let μ_{α_d} be the measure on $\mathbb{R}^d$ defined by $d\mu_{\alpha_d}(x) = \|x\|_2^{-\alpha_d} dx$. We consider the centered maximal operator defined by μ_{α_d} and euclidean balls, acting on radial functions.*

1) If the fixed constant $\alpha > 0$ satisfies $1/2 < \alpha < 1$ and $\alpha_d := \alpha d$, then for every $p \in [1, \infty)$ there exists a $b = b(p) > 1$ such that $c_{p,d} \geq \Theta(b^d)$. That is, we have exponential increase in the weak type (p, p) bounds for all $p < \infty$.

2) For $\alpha_d \leq d/2$, we have $c_{1,d} \geq \Theta((5^{1/2}/2)^{\alpha_d})$. In particular, if $\limsup_d \alpha_d = \infty$, *then we always have* $\limsup_d c_{1,d} = \infty$.

3) If $\sup_d \alpha_d \leq \alpha < \infty$, *then there exists a $C = C(\alpha)$ such that for every $d \geq 1$, $c_{1,d} \leq C$. Thus, there are bounds, uniform in the dimension, for the weak* *type (1,1) constants, and hence, by interpolation, for the strong* (p, p) *constants, whenever* $1 < p < \infty$.

Remark 3.2. If $\alpha_d \leq 0$, then we are in the case of radial non-decreasing measures, so $c_{1,d} \leq 4$, as we noted above.

Remark 3.3. Parts 1) and 3) of the preceding theorem have been independently discovered by Criado in his Ph. D. Thesis, cf. [21]. Remarkably, it is also shown there that Stein's result regarding strong L^p bounds uniform in d, for euclidean balls and Lebesgue measure, extends to the measures $d\mu_\alpha(x) = \|x\|_2^{-\alpha} dx$, $\alpha > 0$ (without restricting the action of the operator to radial functions, as we do here).

Proof of part 1) We follow the same steps as in the proof of [20, Theorem 2.8], with the appropriate modifications. Let $B_r := B(0, r)$, and denote by $\omega_{d-1} = \sigma_{d-1}(\mathbb{S}^{d-1})$ the area of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$.

Lemma 3.4. [20, Lemma 3.1] *Let* μ *be a rotation-invariant locally finite Borel measure in* $\mathbb{R}^d$. *For all* $x \in \mathbb{R}^d$ *and all* $r, R > 0$ *such that* $\mu(B_r), \mu(B(x, R)) > 0$, *we have*

$$c_{\mu,p} \geq M_\mu \chi_{B_r}(x) \left(\frac{\mu(B_{|x|})}{\mu(B_r)} \right)^{1/p} \geq \frac{\mu(B(x, R) \cap B_r)}{\mu(B(x, R))} \left(\frac{\mu(B_{|x|})}{\mu(B_r)} \right)^{1/p}.$$

Let μ_d be the Radon measure $d\mu_d(x) = \|x\|_2^{-\alpha d} dx$ in $\mathbb{R}^d$. Assume $1/2 < \alpha < 1$. We point out that the arguments below also work if instead of a constant α we use variables β_d, provided they belong to a compact subinterval of $(1/2, 1)$. That is, if β_d tends to $1/2$, then the base of exponentiation tends to 1. And if β_d tends to 1, some "constants" appearing below may explode.

In view of the preceding lemma, it is enough to show that for each fixed $\alpha \in (1/2, 1)$, there exist $r \equiv r(\alpha), R \equiv R(\alpha), c \equiv c(\alpha), C \equiv C(\alpha) > 0$ with $r, R < 1$, and $a \equiv a(\alpha) > 1$, such that

$$\frac{\mu_d(B(e_1, R) \cap B_r)}{\mu_d(B(e_1, R))} \geq \frac{c}{\sqrt{d}},$$

and

$$\frac{\mu_d(B_1)}{\mu_d(B_r)} \geq C a^d. \tag{3.3}$$

Integration in spherical coordinates shows that for all $\rho > 0$,

$$\mu(B_\rho) = \frac{\omega_{d-1}}{d(1-\alpha)} \rho^{d(1-\alpha)}.$$

Thus,

$$\frac{\mu_d(B_1)}{\mu_d(B_r)} \geq \left(\frac{1}{r} \right)^{(1-\alpha)d},$$

and (3.3) follows with $C = 1$ and $a = (1/r)^{1-\alpha}$.

Next we bound $\mu_d(B(e_1, R))$ from above, by changing to spherical coordinates:

$$\mu_d(B(e_1, R)) = \int_{1-R}^{1+R} |\partial B_s \cap B(e_1, R)|_{d-1} s^{-\alpha d} ds, \tag{3.4}$$

where $|\cdot|_{d-1}$ denotes the $n-1$ dimensional Hausdorff measure. Call β_s the angle determined by the segment that joins the origin with e_1 and the one that connects the origin to any point of intersection of ∂B_s with $\partial B(e_1, R)$. Then $0 \leq \beta_s < \pi/2$, since $R < 1$. Thus,

$$|\partial B_s \cap B(e_1, R)|_{d-1} = \int_0^{\beta_s} \omega_{d-2}(s \sin\theta)^{d-2} s d\theta = \omega_{d-2} s^{d-1} \int_0^{\beta_s} (\sin\theta)^{d-2} d\theta. \quad (3.5)$$

By the cosine law, applied to the triangle $T(1, s, R)$ with side lengths 1, s, and R, and the angle β_s facing the R-side, we have

$$\cos\beta_s = \frac{1 + s^2 - R^2}{2s}, \quad (3.6)$$

so

$$\sin\beta_s = \left[1 - \left(\frac{1 + s^2 - R^2}{2s}\right)^2\right]^{1/2}. \quad (3.7)$$

Note that the maximum value of β_s occurs when the ray starting at 0 is tangent to $B(e_1, R)$, so the triangle $T(1, s, R)$ has a right angle, and hence $s = \sqrt{1 - R^2}$. Since $\sin\beta_s$ increases with β_s and $\cos\beta_s$ decreases, from (3.6) and (3.7) we obtain $\cos\beta_s \geq \sqrt{1 - R^2}$ and $\sin\beta_s \leq R$.

Using (3.5) we conclude that

$$\frac{\omega_{d-2}}{d-1}(s \sin\beta_s)^{d-1} \leq |\partial B_s \cap B(e_1, R)|_{d-1} = \omega_{d-2} s^{d-1} \int_0^{\beta_s} (\sin\theta)^{d-2} d\theta \quad (3.8)$$

$$\leq \frac{\omega_{d-2} s^{d-1}}{\sqrt{1 - R^2}} \int_0^{\beta_s} \cos\theta (\sin\theta)^{d-2} d\theta \leq \frac{1}{\sqrt{1 - R^2}} \frac{\omega_{d-2}}{d-1}(s \sin\beta_s)^{d-1}.$$

Define

$$F_R(s) := (s \sin\beta_s)^2 s^{-2\alpha} = \frac{1}{4}\left[4s^2 - \left(1 + s^2 - R^2\right)^2\right] s^{-2\alpha}.$$

By (3.8) and (3.4),

$$\mu_d(B(e_1, R)) \leq \frac{1}{\sqrt{1 - R^2}} \frac{\omega_{d-2}}{d-1} \int_{1-R}^{1+R} (s \sin\beta_s)^{d-1} s^{-\alpha d} ds$$

$$= \frac{1}{\sqrt{1 - R^2}} \frac{\omega_{d-2}}{d-1} \int_{1-R}^{1+R} (s \sin\beta_s)^{d-1} s^{\alpha(1-d)} \frac{ds}{s^\alpha}$$

$$= \frac{1}{\sqrt{1 - R^2}} \frac{\omega_{d-2}}{d-1} \int_{1-R}^{1+R} F_R(s)^{\frac{d-1}{2}} \frac{ds}{s^\alpha}.$$

Clearly, $F_R(1 - R) = F_R(1 + R) = 0$. Furthermore, F_R is increasing on $[1 - R, \sqrt{1 - R^2}]$ since it is the product of two increasing functions there ($(\sin\beta_s)^2$ and $s^{2-2\alpha}$)).

Claim (to be proven later): Choosing $R = \sqrt{1 - 4(1-\alpha)^2}$, the function F_R achieves its unique maximum on $[1 - R, 1 + R]$ at a point $s_0 < 1$.

Assuming the claim, if we replace $F_R(s)$ and $s^{-\alpha}$ in the preceding integral by their maximum values, we obtain

$$\mu_d(B(e_1,R)) \leq \frac{2R}{(1-R)^\alpha\sqrt{1-R^2}}\frac{\omega_{d-2}}{d-1}F_R(s_0)^{\frac{d-1}{2}}. \tag{3.9}$$

Next we set $r := s_0$. To bound $\mu_d(B(e_1,R)\cap B_r)$ from below, we change to spherical coordinates and use (3.8):

$$\mu_d(B(e_1,R)\cap B_{s_0}) = \int_{1-R}^{s_0} |\partial B_s \cap B(e_1,R)|_{d-1} s^{-\alpha d} ds \geq$$

$$\frac{\omega_{d-2}}{d-1}\int_{1-R}^{s_0}(s\sin\beta_s)^{d-1}s^{-\alpha d}ds = \frac{\omega_{d-2}}{d-1}\int_{1-R}^{s_0}F_R(s)^{\frac{d-1}{2}}\frac{ds}{s^\alpha}. \tag{3.10}$$

By Taylor's approximation, for every $s\in[1-R,1+R]$ there exists a τ_s between s and s_0 such that

$$F_R(s) = F_R(s_0) + \frac{F_R''(\tau_s)}{2}(s-s_0)^2.$$

Denote by $M \equiv M(\alpha)$ the maximum value of $|F_R''|$ on $[1-R,1+R]$. We assume that $d >> 1$ is so large that

$$0 < \delta := \sqrt{4F_R(s_0)/M(d-1)} < s_0 - 1 + R$$

(we can do this since neither R nor F_R depend on d). Then, for all $s\in(s_0-\delta,s_0)$,

$$F_R(s) \geq F_R(s_0) - \frac{M}{2}\delta^2 = F_R(s_0)\left(1-\frac{2}{(d-1)}\right).$$

Since $(1-t)^{1/t}$ increases to $1/e$ as $t\downarrow 0$, for all $d\geq 4$.

$$F_R(s)^{\frac{d-1}{2}} \geq F_R(s_0)^{\frac{d-1}{2}}\left(1-\frac{2}{(d-1)}\right)^{\frac{d-1}{2}} \geq F_R(s_0)^{\frac{d-1}{2}}\left(\frac{1}{3}\right)^{\frac{3}{2}}.$$

Thus, by (3.10)

$$\begin{aligned}\mu_d(B(e_1,R)\cap B_{s_0}) &\geq \frac{\omega_{d-2}}{d-1}\int_{1-R}^{s_0}F_R(s)^{\frac{d-1}{2}}\frac{ds}{s^\alpha}\\ &\geq \frac{\omega_{d-2}}{d-1}\int_{s_0-\delta}^{s_0}F_R(s)^{\frac{d-1}{2}}\frac{ds}{s^\alpha}\\ &\geq \frac{\omega_{d-2}}{d-1}F_R(s_0)^{\frac{d-1}{2}}\left(\frac{1}{3}\right)^{\frac{3}{2}}\int_{s_0-\delta}^{s_0}\frac{ds}{s^\alpha}\\ &\geq \frac{\omega_{d-2}}{d-1}F_R(s_0)^{\frac{d-1}{2}}\left(\frac{1}{3}\right)^{\frac{3}{2}}s_0^{-\alpha}\delta. \end{aligned}\tag{3.11}$$

Finally, using (3.9) and (3.11), we get

$$\frac{\mu_n(B(e_1,R)\cap B_{s_0})}{\mu_n(B(e_1,R))} \geq \frac{\left(\frac{1}{3}\right)^{\frac{3}{2}}(1-R)^\alpha\sqrt{1-R^2}s_0^{-\alpha}\delta}{2R} \geq \frac{c}{\sqrt{d}},$$

where $c = c(\alpha) > 0$ (c depends on R, but recall that $R = \sqrt{1-4(1-\alpha)^2}$).

Proof of the claim. For simplicity, we make the change of variables $t = s^2$, and write

$$g(t) := 4F_R\left(t^{1/2}\right) = \left[4t - \left(1 + t - R^2\right)^2\right] t^{-\alpha}. \tag{3.12}$$

Clearly it is enough to show that g has a unique maximum $t_0 \in [(1-R)^2, (1+R)^2]$ such that $t_0 < 1$. It then follows that F_R has a unique maximum $s_0 \in [1-R, 1+R]$ with $s_0 = t_0^{1/2} < 1$.

Replacing R^2 by its value $1 - 4(1-\alpha)^2$ in (3.12) and simplifying we obtain

$$g(t) = \left[-16(\alpha-1)^4 + (-4 + 16\alpha - 8\alpha^2)t - t^2\right] t^{-\alpha}.$$

To find the local extrema we differentiate and rearrange:

$$g'(t) = \left[16(\alpha-1)^4\alpha + (-4 + 20\alpha - 24\alpha^2 + 8\alpha^3)t + (\alpha - 2)t^2\right] / t^{1+\alpha}.$$

Note that the zeroes of g' are the same as the zeroes of its numerator, so by solving a second degree equation, we get

$$t_0 = 4(\alpha - \alpha^2) \quad \text{and} \quad t_1 = \frac{4(\alpha-1)^3}{2-\alpha}.$$

Now at least one root belongs to $[(1-R)^2, (1+R)^2]$, since g vanishes at the endpoints and it must have a global maximum. But $t_1 < 0$, so the only solution in $[(1-R)^2, (1+R)^2]$ is t_0, and thus the global maximum of g occurs there. Furthermore, on $(1/2, 1)$, $f(\alpha) := \alpha - \alpha^2 < 1/4$, whence $t_0 = t_0(\alpha) < 1$.

This finishes the proof of Part 1). □

Proof of part 2). Assume that $0 < \alpha_d \le d/2$. It is shown next that if $d \ge 12$, then

$$c_{1,d} \ge \frac{1}{2e}\left(\frac{5}{4}\right)^{\frac{\alpha_d}{2}}.$$

The proof we present below illustrates the discretization technique, valid only for $p = 1$. In this particular application, a radial decreasing function is replaced by one Dirac delta at the origin. Clearly, any lower bound obtained using δ_0 can be approximated as much as we want, by considering instead the function $\chi_{B(0,r)}/\mu_{\alpha_d}(B(0,r))$, where $0 < r << 1$. In fact, by the 1-homogeneity of the operator, we can just take $\chi_{B(0,r)}$, since constants cancel out. We note that the proofs of exponential growth of the weak and strong type constants in the papers [2, 10, 19, 20], all use this method of considering δ_0 or $\chi_{B(0,r)}$, and then estimating how shifting balls away from the origin reduces their measure (the differences between these papers lie in the values of $r > 0$ selected, the shifted balls chosen, and how their sizes are controlled).

We utilize the following special case of [2, Proposition 2.1]:

$$c_{1,d} \ge \frac{\mu_{\alpha_d}(B(0,1))}{\mu_{\alpha_d}(B(e_1,1))}, \tag{3.13}$$

where e_1 is the first vector in the standard basis of $\mathbb{R}^d$ (any vector of length one will do, by rotational invariance). This lower bound is obtained by noticing that

$M_{\mu_{\alpha_d}}\delta_0(x) = 1/\mu_{\alpha_d}(B(x, \|x\|_2)$ (recall that balls can be taken to be closed) and that

$$B(0,1) \subset \left\{ M_{\mu_{\alpha_d}}\delta_0 \geq \frac{1}{\mu_{\alpha_d}(B(e_1,1))} \right\}.$$

So, all we need to do is to estimate from below the quotient appearing in (3.13). Writing σ_{d-1} for the $(d-1)$-dimensional Hausdorff measure on $\mathbb{S}^{d-1}$ (the unit sphere in $\mathbb{R}^d$) integration in polar coordinates yields

$$\mu_{\alpha_d}(B(0,1)) = \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{d-\alpha_d}. \tag{3.14}$$

Next, note that $B(e_1,1)$ can be decomposed in vertical sections as follows:

$$B(e_1,1) = \{x = (x_1,\ldots,x_d) \in \mathbb{R}^d : \|x - e_1\|_2 \leq 1\} =$$

$$\{x : 0 \leq x_1 \leq 2, (x_2,\ldots,x_d) \in \mathbb{R}^{d-1}, x_2^2 + \ldots + x_d^2 \leq 2x_1 - x_1^2\}.$$

Thus, by Fubini's theorem,

$$\mu_{\alpha_d}(B(e_1,1)) = \int_{B(e_1,1)} \frac{dx}{\|x\|_2^{\alpha_d}} =$$

$$= \int_0^2 \left(\int_{\{(x_2,\ldots,x_d)\in\mathbb{R}^{d-1}, x_2^2+\ldots+x_d^2 \leq 2x_1-x_1^2\}} \frac{1}{(x_1^2+x_2^2+\ldots+x_d^2)^{\alpha_d/2}} dx_2\cdots dx_d \right) dx_1$$

$$=: \int_0^2 F(x_1)dx_1,$$

where $F(x_1)$ denotes the inner integral. Using a spherical change of coordinates we get

$$F(x_1) = \sigma^{d-2}(\mathbb{S}^{d-2}) \int_0^{\sqrt{2x_1-x_1^2}} \frac{t^{d-2}dt}{(x_1^2+t^2)^{\alpha_d/2}}.$$

Thus

$$\mu_{\alpha_d}(B(e_1,1)) = \sigma^{d-2}(\mathbb{S}^{d-2}) \int_0^2 \left(\int_0^{\sqrt{2x_1-x_1^2}} \frac{t^{d-2}dt}{(x_1^2+t^2)^{\alpha_d/2}} \right) dx_1.$$

Note that the region of integration in the above expression is the upper semicircle centered at $x_1 = 1$, $t = 0$, in the x_1t-plane.

Hence, by changing to polar coordinates we obtain

$$\mu_{\alpha_d}(B(e_1,1)) = \sigma^{d-2}(\mathbb{S}^{d-2}) \int_0^{\pi/2} \left(\int_0^{2\cos\theta} \frac{(\rho\sin\theta)^{d-2}\rho}{\rho^{\alpha_d}} d\rho \right) d\theta =$$

$$= \frac{\sigma^{d-2}(\mathbb{S}^{d-2})}{d-\alpha_d} \int_0^{\pi/2} (\sin\theta)^{d-2}(2\cos\theta)^{d-\alpha_d} d\theta$$

$$= \frac{2^{d-\alpha_d-1}\sigma_{d-2}(\mathbb{S}^{d-2})\beta(\frac{d-\alpha_d+1}{2}, \frac{d-1}{2})}{d-\alpha_d} \tag{3.15}$$

By (3.13), (3.14) and (3.15),

$$c_{1,d} \geq \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{2^{d-\alpha_d-1}\sigma_{d-2}(\mathbb{S}^{d-2})\beta(\frac{d-\alpha_d+1}{2},\frac{d-1}{2})} = \sqrt{\pi}\frac{\Gamma(\frac{2d-\alpha_d}{2})}{2^{d-\alpha_d-1}\Gamma(\frac{d}{2})\Gamma(\frac{d-\alpha_d+1}{2})} \tag{3.16}$$

Now we use the Stirling representation of the Gamma function [1, p.257, 6.1.38]: For every $x > 0$, there exists a $\theta \equiv \theta(x) \in [0,1]$ such that

$$\Gamma(x+1) = \sqrt{2\pi}x^{x+1/2}e^{-x+\theta/(12x)}.$$

Thus, for $d \geq 3$, we have

$$\Gamma\left(\frac{d}{2}\right) \leq e^{1/6}\sqrt{2\pi}\left(\frac{d-2}{2}\right)^{\frac{d-1}{2}} e^{-\frac{d-2}{2}}. \tag{3.17}$$

and

$$\Gamma\left(\frac{d-\alpha_d+1}{2}\right) \leq e^{1/3}\sqrt{2\pi}\left(\frac{d-\alpha_d-1}{2}\right)^{\frac{d-\alpha_d}{2}} e^{-\frac{d-\alpha_d-1}{2}}. \tag{3.18}$$

We also obtain

$$\Gamma\left(\frac{2d-\alpha_d}{2}\right) \geq \sqrt{2\pi}\left(\frac{2d-\alpha_d-2}{2}\right)^{\frac{2d-\alpha_d-1}{2}} e^{-\frac{2d-\alpha_d-2}{2}}. \tag{3.19}$$

Using (3.16), (3.17), (3.18) and (3.19), we get

$$c_{1,d} \geq \frac{\sqrt{2}}{e}\frac{(2d-\alpha_d-2)^{\frac{2d-\alpha_d-1}{2}}}{2^{d-\alpha_d}(d-2)^{\frac{d-1}{2}}(d-\alpha_d-1)^{\frac{d-\alpha_d}{2}}}.$$

Finally, since $d \geq 12$ and $\alpha_d \leq d/2$,

$$4[4(d-\alpha_d-1)] \geq 5(2d-\alpha_d-2),$$

and

$$(2d-\alpha_d-2)^2 \geq 4(d-2)(d-\alpha_d-1).$$

Thus

$$c_{1,d} \geq \left(\frac{1}{2e}\right)\frac{(2d-\alpha_d-2)^{\frac{2d-\alpha_d}{2}}}{2^{d-\alpha_d}(d-2)^{\frac{d}{2}}(d-\alpha_d-1)^{\frac{d-\alpha_d}{2}}}$$

$$= \frac{1}{2e}\left(\frac{(2d-\alpha_d-2)^2}{4(d-2)(d-\alpha_d-1)}\right)^{d/2}\left(\frac{4(d-\alpha_d-1)}{2d-\alpha_d-2}\right)^{\frac{\alpha_d}{2}} \geq \frac{1}{2e}\left(\frac{5}{4}\right)^{\frac{\alpha_d}{2}}.$$

□

Regarding part 3), the rest of this paper presents its proof in detail. Since the upper bounds we obtain increase with the constant α (cf. Corollary 3.8 below) the case where $\alpha_d = \alpha$ for all $d \geq 1$ entails the case $\alpha_d \leq \alpha$, so from now on we suppose that $\alpha_d = \alpha$ for all d.

Note that if $d \leq \alpha$, then μ_{α_d} is not locally finite at the origin, so we want to allow this possibility in the definitions. Since below 2α there are only finitely many dimensions $1, \ldots, [2\alpha]$, to obtain a uniform bound, it is enough to prove that it exists for $d \geq 2\alpha$, and then take the largest of these (at most) $1 + [2\alpha]$

constants. The case $d \geq 2\alpha$ is considered in Corollary 3.8 at the end of this paper. This corollary follows from Theorem 3.6, which is obtained by isolating the property that makes the proofs of [22, Theorem 2.1] and [25, Theorem 3] work: To each ball, the argument associates a second ball with the same radius, and center nearer to the origin (perhaps the origin itself). It is enough to assume that this second ball is not much larger than the first.

The following (uniform in the dimension) weak type (1,1) inequality was proven in [22, Theorem 2.1] (cf. [25, Theorem 3] for Lebesgue measure): If M_μ is the maximal operator associated to centered euclidean balls in $\mathbb{R}^d$ with a radial non-decreasing measure μ, then for every $t > 0$ and every radial $f \in L^1$,

$$t\mu\{Mf > t\} \leq 4\|f\|_1.$$

Even though this proof has already appeared in print (save for some trivial modifications) we include it here because of its didactic value, as it illustrates two basic techniques in the subject: 1) Control a maximal operator in terms of another operator with known bounds. 2) Instead of integrating over a ball, integrate over a larger (but not much larger) set (perhaps, just a larger ball).

Regarding 1), the controlling operator will be the one-dimensional, uncentered Hardy–Littlewood maximal operator. Its boundedness (cf. the next result) hinges upon the fact that from a finite collection of intervals, two disjoint subcollections can be extracted, so that their union is the same as the union of the original collection (as far as we know, this was published first in [30]; it seems to have been rediscovered, as some authors attribute it to Young). To see why this is true, first throw away unnecessary intervals, those contained in the union of the others, so no point belongs to three of them; then label the intervals in increasing order, say, of the left endpoints, and notice that the subcollections of intervals with even and with odd indices are disjoint. As a consequence, one immediately obtains the next theorem, cf., for instance, [18] (which makes the unnecessary assumption that compact sets have finite measure) or [4]. The result is valid for completely arbitrary Borel measures (countably additive, non-negative and not identically 0).

Given a Borel measure ν, we always assume that it has been completed, i.e., that it has been extended to the σ-algebra generated by the Borel sets and the sets of ν-outer measure zero; we also use ν to denote this extension. While the next result is usually stated for the real line, the same proof works for subintervals. Alternatively, one can consider ν defined on a subinterval $I \subset \mathbb{R}$, and extend it to $\mathbb{R}$ by setting $\nu(I^c) = 0$, thus reducing the case of an arbitrary interval I to the case $I = \mathbb{R}$. In fact, we will only need the particular interval $I = (0, \infty)$.

Theorem 3.5. *Let μ be a Borel measure on an interval $I \subset \mathbb{R}$, let $f \in L^1(\mu)$, and let M^u_μ be the uncentered maximal operator. Then for every $\lambda > 0$,*

$$\lambda\mu\{M^u_\mu f > \lambda\} \leq 2\|f\|_1.$$

Theorem 3.6. *Let μ a radial measure on $\mathbb{R}^d$. Suppose there exists a $C > 0$ such that for all $x \in \mathbb{R}^d$ and all r with $0 < r \leq 1$, we have*

$$\mu(B(x\sqrt{1-r^2}, r\|x\|_2)) \leq C\mu(B(x, r\|x\|_2)). \tag{3.20}$$

Then, for every radial function $f \in L^1(\mathbb{R}^d, d\mu)$ and every $\lambda > 0$,

$$\lambda\mu\{M_\mu f > \lambda\} \leq 2(C+1)\|f\|_{L^1(\mathbb{R}^d, d\mu)}.$$

Remark 3.7. Obviously, all radial non-decreasing measures (including the Lebesgue d-dimensional measure) satisfy condition (3.20) with $C = 1$, since the size of balls does not increase when they are shifted towards the origin. Note also that when $r = 1$, condition (3.20) simply says that $\mu(B(0, \|x\|_2)) \leq C\mu(B(x, \|x\|_2))$.

Proof of Theorem 3.6. Since μ is radial, the local integrability of $\mu_0(t)t^{d-1}$ on $(0, \infty)$ together with condition (3.20) entail that all balls have finite measure, so we can assume that balls $B(y, s)$ are closed. Let $r > 0$. The idea is to show that for every $x \in \mathbb{R}^d \setminus \{0\}$ and every ball $B = B(x, r\|x\|_2)$, the averages $\frac{1}{\mu(B)} \int_B f d\mu$ are pointwise bounded by the one-dimensional uncentered maximal function evaluated at $\|x\|_2$, times a certain constant (since the set $\{0\}$ has measure zero, we can just forget about it; alternatively, we note that the set D defined below equals B when $x = 0$, and then the result is immediate).

We prove the pointwise bound by passing to spherical coordinates. Let v be a unit vector such that the ray $\{t(v) : t \geq 0\}$ intersects B; in what follows, rays will be denoted just by $t(v)$. If the segment I resulting from this intersection contains $\|x\|_2 v$, then we can use the uncentered operator evaluated at $\|x\|_2 v$, and there is no need to do anything. However, it may happen that I does not contain $\|x\|_2 v$. If so, we enlarge I up to $\|x\|_2 v$, and define D to be the union with B of all these enlarged segments. Now if $r > 1$, then $D = B(0, \|x\|_2)) \cup B(x, r\|x\|_2)$, whence

$$\mu(D) \leq \mu B(0, \|x\|_2) + \mu B(x, r\|x\|_2)$$
$$\leq C\mu B(x, \|x\|_2) + \mu B(x, r\|x\|_2) \leq (C+1)\mu B(x, r\|x\|_2).$$

We show next that if $r \leq 1$, then $D \subset B(x\sqrt{1-r^2}, r\|x\|_2) \cup B(x, r\|x\|_2)$, so

$$\mu(D) \leq \mu B(x\sqrt{1-r^2}, r\|x\|_2)) + \mu B(x, r\|x\|_2) \leq (C+1)\mu B(x, r\|x\|_2)$$

(thus, in both cases the measure of D is comparable to the measure of B).

For each unit vector v such that the ray $t(v)$ intersects D, let the segment $[a(v), b(v)]$ denote this intersection. That is, $a(v)$ is the point of entry (of first intersection) of the ray $t(v)$ in B (or equivalently, in D), and $b(v)$, the point of exit of D, i.e., either $b(v)$ is the point of exit of the ball, or $b(v) = \|x\|_2 v$, whichever is larger.

Suppose next that the angle between two given unit vectors u, w, is acute ($\leq \pi/2$), and let $s > 0$. Let R be the length of the segment joining su with its perpendicular projection over the segment $[0, sw]$. Then R is also the length of the segment joining sw, with its perpendicular projection over $[0, su]$. This observation proves that $D \subset B(x\sqrt{1-r^2}, r\|x\|_2) \cup B(x, r\|x\|_2)$, as follows. Consider the vector x, and let v be any unit vector such that the ray $t(v)$ is tangent to $B = B(x, r\|x\|_2)$. Call this point of tangency $t_0(v)$, and note that the segment from x to $t_0(v)$ is perpendicular to the ray $t(v)$. We use T to denote the set of all unit vectors with rays tangent to B, and S the set of all unit vectors with rays intersecting B (in particular, $T \subset S$).

The observation above, with $x = su$, $\|x\|_2 v = sw$, and $R = r\|x\|_2$, shows that the points in $D \setminus B$ farthest away from the ray tx, i.e., the points of the form $\|x\|_2 v$, are at distance $r\|x\|_2$ from their perpendicular projections over $[0, x]$. These perpendicular projections equal $x\sqrt{1-r^2}$ by the Pythagorean Theorem, so all the points $\|x\|_2 v$, $v \in T$, belong to $B(x\sqrt{1-r^2}, r\|x\|_2)$. The points $t_0(v)$ are in B, so they are also in $B(x\sqrt{1-r^2}, r\|x\|_2)$, since the latter ball is just B displaced towards the origin. By convexity, the segments $[t_0(v), \|x\|_2 v]$ are fully contained in $B(x\sqrt{1-r^2}, r\|x\|_2)$. This proves that $D \setminus B \subset B(x\sqrt{1-r^2}, r\|x\|_2)$, as desired.

Now, in order to obtain the pointwise bound

$$M_\mu f(x) = M_\mu f_0(\|x\|_2) \le (C+1) M^u_{\gamma_0} f_0(\|x\|_2), \tag{3.21}$$

all we have to do is to average f over D instead of B. Writing the integral in polar (spherical) coordinates, the averages of a function over any segment are always controlled by the uncentered one-dimensional maximal operator, evaluated at *any* point of the segment. Since all segments in D contain a point of the form $\|x\|_2 v$ (where $\|v\|_2 = 1$) and since both the measure and the function are radial, by evaluating the one-dimensional maximal operator always at the same point $\|x\|_2$, we are actually averaging a constant function, so we get the same value back. We present the details next.

Recalling the notation from (3.1) and (3.2), let us define the measure γ_0 on $(0, \infty)$ via $d\gamma_0(t) := \mu_0(t) t^{d-1} dt$, so given any subinterval $I \subset (0, \infty)$,

$$\gamma_0(I) = \int_I \mu_0(t) t^{d-1} dt.$$

Writing σ for area on the unit sphere, and integrating in spherical coordinates, we get

$$\frac{1}{\mu(B(x, r\|x\|_2))} \int_{B(x, r\|x\|_2)} |f(y)| d\mu(y) = \frac{\mu(D)}{\mu(B)} \frac{1}{\mu(D)} \int_B |f(y)| d\mu(y) \tag{3.22}$$

$$\le \frac{C+1}{\mu(D)} \int_D |f(y)| d\mu(y) = \frac{C+1}{\mu(D)} \int_D |f_0(\|y\|_2)| \mu_0(\|y\|_2) dy$$

$$= \frac{C+1}{\mu(D)} \int_S \left(\int_{a(v)}^{b(v)} |f_0(t)| \mu_0(t) t^{d-1} dt \right) d\sigma(v)$$

$$\le \frac{C+1}{\mu(D)} \int_S \gamma_0((a(v), b(v))) M^u_{\gamma_0} f_0(\|x\|_2) d\sigma(v) = (C+1) M^u_{\gamma_0} f_0(\|x\|_2).$$

Taking the supremum over $r > 0$ in (3.22), we obtain (3.21). Finally, we express the level sets of $M_\mu f$ in spherical coordinates, and apply Theorem 3.5:

$$\mu\{x \in \mathbb{R}^d : M_\mu f(x) > \lambda\} \le \mu \left\{ x \in \mathbb{R}^d : M^u_{\gamma_0} f_0(\|x\|_2) > \frac{\lambda}{C+1} \right\}$$

$$= \int_{\mathbb{S}^{d-1}} \left(\int_{\left\{ M^u_{\gamma_0} f_0 > \frac{\lambda}{C+1} \right\}} \mu_0(t) t^{d-1} dt \right) d\sigma(\omega) = \int_{\mathbb{S}^{d-1}} \gamma_0 \left\{ M^u_{\gamma_0} f_0 > \frac{\lambda}{C+1} \right\} d\sigma(\omega)$$

$$\leq \frac{2(C+1)}{\lambda} \int_{\mathbb{S}^{d-1}} \left(\int_{(0,\infty)} |f_0(t)| d\gamma_0(t) \right) d\sigma(\omega) = \frac{2(C+1)}{\lambda} \int_{\mathbb{R}^d} |f| d\mu.$$

□

To bound $\mu(B(x, r\|x\|_2))$ from below in the next result, in expressions (3.24) and (3.25) below, it is enough to replace the density by its lowest value on $B(x, r\|x\|_2)$, that is, by $(\|x\|_2(1+r))^{-\alpha}$. We use $\left(\|x\|_2\sqrt{1+r^2}\right)^{-\alpha}$ instead, noting that the density is larger than this constant on at least half the ball. The estimates are not very different, but the second choice gives better constants for high values of α.

Corollary 3.8. *Fix $\alpha > 0$ and set $d\mu(y) = \frac{dy}{\|y\|_2^\alpha}$ on $\mathbb{R}^d$, for $d \geq 1$. If $d \geq 2\alpha$ and $f \in L^1(\mathbb{R}^d, d\mu)$ is radial, then for every $\lambda > 0$,*

$$\mu\{M_\mu f > \lambda\} \leq \frac{2(4 \cdot 6^{\alpha/2} + 1)}{\lambda} \|f\|_{L^1(\mathbb{R}^d, d\mu)}.$$

Proof. We show that (3.20) holds with $C = 4 \cdot 6^{\alpha/2}$. Because μ is radial decreasing, it is clear that the measure of balls increases when they are shifted towards the origin, since the density is always larger on all points of the shifted ball that are not contained in the intersection (of the two balls) than on the points of the original ball not in the intersection. Thus

$$\mu(B(x\sqrt{1-r^2}, r\|x\|_2)) \leq \mu(B(0, r\|x\|_2)) = \frac{d}{d-\alpha}(\|x\|_2 r)^{d-\alpha} v_d \leq 2(\|x\|_2 r)^{d-\alpha} v_d, \tag{3.23}$$

where v_d denotes the Lebesgue d-dimensional measure of the unit ball.

On the other hand,

$$\mu(B(x, r\|x\|_2)) = \int_{B(x,r\|x\|_2)} \frac{dy}{\|y\|_2^\alpha} \geq \int_{B(x,r\|x\|_2) \cap \{y: \|y\|_2 \leq \|x\|_2\sqrt{1+r^2}\}} \frac{dy}{\|y\|_2^\alpha} \geq \tag{3.24}$$

$$\frac{1}{\|x\|_2^\alpha (1+r^2)^{\alpha/2}} \lambda^d (B(x, r\|x\|_2) \cap \{y : \|y\|_2 \leq \|x\|_2\sqrt{1+r^2}\}) \tag{3.25}$$

$$\geq \frac{\lambda^d(B(x, r\|x\|_2))}{2\|x\|_2^\alpha (1+r^2)^{\alpha/2}} = \frac{(r\|x\|_2)^d \ v_d}{2\|x\|_2^\alpha (1+r^2)^{\alpha/2}} \tag{3.26}$$

If $1/\sqrt{5} \leq r \leq 1$, it follows from (3.24-3.26) and (3.23) that in order to obtain (3.20), it is enough to find a $C' > 0$ such that

$$2(\|x\|_2 r)^{d-\alpha} \ v_d \leq C' \frac{(r\|x\|_2)^d \ v_d}{2\|x\|_2^\alpha (1+r^2)^{\alpha/2}}.$$

Simplifying, we see that $C' = 4 \cdot 6^{\alpha/2}$ suffices.

Suppose next that $0 < r \le 1/\sqrt{5}$. Then

$$\begin{aligned}\mu(B(x\sqrt{1-r^2}, r\|x\|_2)) &= \int_{B(x\sqrt{1-r^2}, r\|x\|_2)} \frac{dy}{\|y\|_2^\alpha} \\ &\le \frac{1}{\|x\|_2^\alpha(\sqrt{1-r^2}-r)^\alpha}\int_{B(x\sqrt{1-r^2}, r\|x\|_2)} dy \\ &= \frac{(r\|x\|_2)^d\ v_d}{\|x\|_2^\alpha(\sqrt{1-r^2}-r)^\alpha}\end{aligned}$$

Arguing as in the previous case, we see that it is enough to find a $C'' > 0$ such that

$$\frac{(r\|x\|_2)^d\ v_d}{\|x\|_2^\alpha(\sqrt{1-r^2}-r)^\alpha} \le C'' \frac{(r\|x\|_2)^d\ v_d}{2\|x\|_2^\alpha(1+r^2)^{\alpha/2}}.$$

Simplifying, we see that we can take $C'' = 2 \cdot 6^{\alpha/2}$. Since $C' \ge C''$, (3.20) follows with $C = C'$. □

And with the proof of Corollary 3.8, the proof or Theorem 3.1, Part 3, is also finished.

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