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AN ANALYSIS OF THE CONCEPT OF CONSTRUCTIVE CATEGORICITY

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Introduction* In 1962 Andrzej Grzegorczyk introduced the concept of constructive categoricity. In his paper [9], he defined this new notion, displayed some of its properties and made some claims for the superiority of this notion over those used over the past sixty years. 'Superior,' he claimed, in that his notion captured more closely the intuitive concept of categoricity. Yet it was not made clear what the intuitive concept of categoricity was nor how and why these previous attempts to formalize the intuitive content of categoricity failed. Thus Grzegorczyk's claim to success became as problematic as the failure of previous attempts.

Chapter I of this paper determines what is the intuitive content of the concept of categoricity. It does this by surveying some of the major works on the concept of categoricity and determining what properties these logicians and mathematicians claimed for systems that they denoted as categorical. A common thread of analysis runs through the works of Dedekind, Veblen, Huntington, Tarski, Carnap, Łoś, and Vaught. Once this has been established Chapter I proceeds with a discussion of the formalizations of this notion previous to Grzegorczyk and concludes by indicating that some of the properties of the intuitive concept are mutually exclusive; others problematical. Chapter II is a formal analysis of some of the properties of the notion of constructive categoricity. By building on the results of Grzegorczyk a second characterization of the concept of constructive categoricity is arrived at, and, subsequently, shown to be equivalent to Grzegorczyk's original characterization. The formal relationships between constructive categoricity and another formalization, namely, categoricity in power are proved. Having determined some further formal properties of constructive categoricity, it is possible to assess Grzegorczyk's claims to having more closely formalized the intuitive content of categoricity. This is done in Chapter III which concludes with a summary of the unsolved problems that have been raised by this thesis about the concept of categoricity.

CHAPTER I

In dealing with representation theorems[1] and attempted axiomatizations of specific mathematical theories, it is known beforehand (disregarding its philosophically problematical nature) what specific domains of mathematical objects are to be described and at least some of the properties

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these objects are to have. It does happen that in describing some mathematical domains not only is the minimal set of properties known, but also the maximal set. Thus, in such a case, it is intended to describe completely one and only one set of mathematical objects. For example, this is the intention in axiomatizing the arithmetic of natural numbers. Of course, having various individual mathematical domains, it is natural to ask what objects and properties are common to several of these domains. Here the intention is to describe interesting properties of many mathematical domains and such is the intention of group theory. Thus, we have, for example, the theory of natural numbers, the theory of real numbers, the theory of dense ordering, and the theory of Euclidean 3-space which attempt to describe one and only one mathematical domain whereas the theory of groups, theory of n-dimensional vector spaces, Boolean algebras, and the theory of categories attempt to describe properties common to many. It is natural to ask, then, of any axiomatization of the natural numbers whether the axioms do, indeed, generate only a single domain or whether through lack of specification do they incidentally describe some other domain also? This, although vague now, can be tentatively identified as the question of categoricity. Thus, an axiom system is categorical if and only if it describes a single mathematical domain. How, it may be asked, can it be determined whether one or many domains are being described? In talking about axioms, meaning abstract or uninterpreted sentences, a method of comparison of all possible realizations or interpretations of the axioms would yield the desired result. Thus if it were possible to choose arbitrarily any two realizations, call them A and B, produce an interpretation of A in B and produce an interpretation of B in A it could be said that they are essentially the same realization. When it is shown for an axiom system that the collection of all possible interpretations is reducible by pair-wise mutual interpretability to but a single realization, then obviously the axiom system describes a single domain and is categorical. If such a reduction is not possible then the axioms are non-categorical. The word 'same' used in the phrase 'same interpretation' has in that context several meanings. 'Same interpretation' can mean that: (1) the respective domains contain an identical set of true sentences (under the interpretation); (2) the respective domains talk about an equivalent number of objects; and (3) the respective domains contain vocabulary to talk about an identical set of distinguished individual objects. Those axiom systems whose realizations reduce to a single realization under the relation of same interpretation in the sense of (1) are said to be *categorical in the general sense*, or simply, *categorical*; those under the relation of same interpretation in the sense of (1) and (2) are said to be *categorical in power*; and those under the relation of same interpretation in the sense of (1), (2), and (3) can be said to be *constructively categorical* provided the number of objects in the domain is *at most* a countable infinity.

Although, historically, Veblen was the first to use the word categoricity, the concept had been employed in earlier logical or mathematical works. In Dedekind's [7], published in 1887, there is presented the following discussion of the natural numbers (they are called the number-series N):

> §132. Theorem. All simply infinite systems are similar (read: isomorphic) to the number-series N and consequently by §33 also to one another.[2]
> §133. Theorem. Every system which is similar to a simply infinite system and therefore by §132, §33 to the number-series N is simply infinite.[3]
> §134. Remark. By the two preceding theorems §132, §133, all simply infinite systems form a class in the sense of §34. At the same time, with reference to §71, §73 it is clear that every theorem regarding numbers, i.e., regarding the elements n of the simply infinite system N set in order by the transformation ϕ and indeed every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the arrangement ϕ, *possesses perfectly general validity for every other simply infinite system* Ω set in order by the transformation θ and its elements ν, and that the passage from N to Ω (e.g., also the translation of an arithmetic theorem from one language into another) is effected by the transformation ψ considered in §132, §133, which changes every element n of N into an element ν of Ω, i.e., into $\psi(n)$. . . . By these remarks, as I believe, the *definition of the notion of numbers given in* §73 *is fully justified.*[4]

Here Dedekind implies that since the domain of the natural numbers is but a single mathematical domain and his definition of the natural numbers leads to the result that all systems satisfying his definition are isomorphic, his is indeed a correct definition. This is just what has been described above as the intuitive content of the notion of categoricity. It, moreover, gives some hint as to the part the notion of categoricity can play in the theory of definition. It is perhaps worth remembering that the first use of what came to be known as categoricity was with respect to the theory of definitions.

In 1902, *cf.* [12], E. V. Huntington wrote:

> The object of the work which follows is to show that these six postulates form a complete set; that is, they are (I) *consistent,* (II) *sufficient,* (III) *independent* (or irreducible). By these three terms we mean: (I) there is at least one assemblage in which the chosen rule of combination satisfies all the six requirements; (II) there is essentially *only one* such assemblage possible; (III) none of the six postulates is a consequence of the other five.[5]
>
> Theorem II. Any two assemblages M and M' which satisfy the postulates 1-6 are equivalent; that is, they can be brought into one-to-one correspondence in such a way $a \circ b$ will correspond with $a' \circ b'$ whenever a and b in M correspond with a' and b' in M' respectively.[6]
>
> 36. From Theorems I and II we may say that the postulates 1-6 define essentially a single assemblage. This assemblage we call the system of absolute continuous magnitude, and the rule of combination addition.[7]

Here, for the first time, is a special word for the intuitive concept that categorical axiom systems determine *only one* mathematical domain. Huntington's use of the word 'sufficient' suggests that no other specification is necessary to determine this domain other than the specification given by the six postulates.

In 1904 O. Veblen in [21], introduced the word 'categoricity' and opposed it with the word 'disjunctive. He credits the suggestion of such terminology to John Dewey. Categoricity is there described as follows:

> Inasmuch as the terms *point* and *order* are undefined one has the right, in thinking of propositions, to apply the terms in connection with any class of objects of which the axioms are valid propositions. It is part of our purpose however to show that there is *essentially only one* class of which the twelve axioms are valid. In more exact language, any two classes K and K' of objects that satisfy the twelve axioms are capable of one-to-one correspondence such that if three elements A, B, C of K are in the order ABC, the corresponding elements of K' are also in the order ABC. Consequently any proposition which can be made in terms of points and order either is in contradiction with our axioms or is equally true of all classes that verify our axioms. The validity of any possible statement in these terms is therefore completely determined by the axioms; and so any further axiom would have to be considered redundant. Thus, if our axioms are valid geometrical propositions, they are sufficient for the complete determination of Euclidean geometry.[8]
> A system of axioms such as we have described is called *categorical*, whereas one to which it is possible to add independent axioms is called *disjunctive*.[9]

> Theorem 84. If K and K' are any two classes that verify axioms I-XII, then any proposition stated in terms of points and order that is valid of the class K is valid of the class K'.[10]

These passages are remarkable for several reasons: (I) the theorem purporting to show the categoricity of Euclidean geometry does, in fact, show a slightly different property, which, as it is stated, avoids what came to be recognized as the major difficulty with the concept of categoricity, and (II) these passages anticipate several theorems which are important with respect to the theory of categoricity. However, the discussion of these two points will be deferred for the discussion leads naturally into the next topic, namely categoricity in power.

Dedekind, Huntington, and Veblen agree that the property they are trying to describe is one that is applicable only to systems that, because of correct definition or sufficient determination, are validated by only a single class of objects. Moreover, this property is or should be applicable only to axiom systems describing mathematical domains which are intended to specify realms intuitively felt to be single domains or accepted as being single domains. Thus Huntington writes, "From another point of view, the propositions 1-6 may be accepted as expressing in precise mathematical form the essential characteristics of magnitude in the *popular* sense of the work."[11] Dedekind and Veblen also agree that all domains satisfying categorical axiom systems verify the same propositions. Thus classes of objects satisfying categorical axiom systems are reducible to a single class both by the relation of isomorphism and by the relation of verifying the same propositions. These relations are not equivalent and failure to distinguish them led to the difficulties with the original notion of categoricity. Dedekind and Huntington felt that the property of categoricity reflected the correctness of proposed definition (here the term 'definition' means a combination of sentences which are axioms and sentences considered as definition proper) but Veblen takes them to task for this.

Adopting an attitude which came to be known as the Russellian theory of definition, Veblen writes, "It would probably be better to reserve the word definition for the substitution of one symbol for another, and to say that a system of axioms is categorical if it is sufficient for the complete determination of a class of objects or elements."[12]

The notions of categoricity discussed so far were meant to be applicable to axiomatic theories in general. However, they were applied only to first-order theories because of the many difficulties that arise in using it with respect to second- or higher-order theories. Some of the difficulties are, for example, existence of models for higher-order theories and the two distinct notions of completeness which arise for higher-order theories.

But, strictly speaking, first-order theories can have two distinctive types of individuals; those ordinarily and properly called individuals as distinguished from those sets which can be considered as individuals also. Thus elementary theories are based on individual variables and constants, functional constants and predicate constants as the only syntactical elements and with quantification allowed only over individual variables. It should be noted that for the remainder of the introduction, and for the remainder of this paper, with the exception of the historical section concerning categoricity as considered by writers prior to Łoś, only elementary theories will be considered. Putting aside terminological differences, Dedekind, Huntington, and Veblen intended that the term 'categorical' be applicable to all and only those axiom systems with the following three properties:

1. all classes of objects which verify the axioms are isomorphic;
2. all classes of objects which verify the axioms also verify the same set of propositions;
3. the axiom system describes fully a domain which is intuitively considered to be but a single domain.

Returning to an unanswered question, why was Veblen's definition of categoricity particularly remarkable? First, Veblen's discussions justifying the notion that axioms added to a categorical axiom system are superfluous anticipates a theorem consciously enunciated only in 1953 by Łoś and Vaught which became a celebrated test for completeness, namely, if an axiom system has no finite models and is categorical in some infinite power then it is complete. Secondly, Veblen's Theorem 84 states that Axioms I-XII are verified by classes of objects which verify the same set of postulates (stated in terms of points and order). However, his definition of categoricity requires that they be put into one-to-one correspondence. Classes of objects which verify the same set of propositions are called arithmetically equivalent. This terminology is due to Tarski.[13] Arithmetically equivalent classes are not necessarily isomorphic. This is what was meant above by saying that the relation of isomorphism and the relation of verifying the propositions are not equivalent. This is the defect of the original notion of categoricity. Through the work of Löwenheim and

Skolem it has been shown that if an axiom system was satisfied by an infinite class of objects, say of cardinality $\mathfrak{m}$, then there were classes of objects of cardinality $\mathfrak{n}$ and $\mathfrak{p}$ respectively, satisfying these axioms such that $\mathfrak{n} \leqslant \mathfrak{m} < \mathfrak{p}$. These classes of unequal cardinality can be constructed which verify the same set of propositions and *a fortiori* are not isomorphic. Obviously, properties 1 and 2 of the original concept of categoricity are not compatible as they thus stood. This was rectified in 1953 by Vaught and Łoś, who worked independently. Curiously enough, Łoś approached the correction via isomorphism and Vaught approached it via arithmetical equivalence.[14]

In 1934 Tarski published a paper [19], whose intent it was to investigate among other concepts the concept of categoricity itself and not simply to apply it to this or that axiom system as the previously mentioned papers had. Although he says, "... a set of sentences is called categorical if any two of its interpretations (realizations) of this set are isomorphic," his principal definition of categoricity, formulated in terms of Russell's *Principia Mathematica,* is as follows:

Let us say that the formula

$$R\frac{x', y', z' \ldots}{x'', y'', z'' \ldots}$$

is to have the same meaning as the conjunction

$$R \in 1 \to 1. \quad V \underset{R}{\sim} V. \quad x' \underset{R}{\sim} x''. \quad y' \underset{R}{\sim} y''. \quad z' \underset{R}{\sim} z'' \ldots$$

(In words: 'R is a one-one mapping of the class V of all individuals onto itself, by which x', y', z', ... are mapped onto x'', y'', z'', ..., respectively.') Consider now any finite set Y of sentences; 'a', 'b', 'c', ... are all specific terms which occur in the sentences of Y, and '$\psi(a, b, c, \ldots)$' is the conjunction of all these sentences. The set Y is called categorical if the formula

$$(x', x'', y', y'', z', z'', \ldots) : \psi(x', y', z', \ldots) \;.$$

$$\psi(x'', y'', z'', \ldots) \;.\supset.\; (\exists R) \;.\; R\frac{x', y', z', \ldots}{x'', y'', z'', \ldots}$$

is logically provable.[15]

This idea, as so many of the ideas in Tarski's early papers, is seminal. Simply by changing the emphasis Tarski avoids many of the difficulties that the other definitions fall prey to. The emphasis has switched from the determination of single domains, an intuitive concept, to the determination of the completeness of specific terms, or formal concepts. Tarski's definitions are always with respect to a given concrete formal context. Hence the above definition of categoricity is given in terms of syntax rather than semantics, or intuitions of single domains. Then, also, by requiring that relation R is a map from the domain of individuals onto the domain of individuals Tarski's definition is not subject to the criticism that isomorphism of realizations and arithmetical equivalence are not equivalent concepts. Once the class of all possible realizations is restricted to one domain, in this case the domain of individuals, then these concepts are equivalent. The only drawback to such a procedure is that the definition is rather restrictive. This, however, is perhaps due to the fact that the definition is cast in Russellian terminology and perhaps with an intended interpretation in mind. The same reason can be advanced to explain two other criticisms of the definition, namely, that it is not restricted to elementary theories, but is intended to apply to higher-order theories, with all the attendant difficulties of such a procedure.

Tarski himself realizes a serious difficulty with categoricity as he has defined it and his intended use of it, namely, the determination of the completeness of concepts. By this is meant that given a logical system, have we, with respect to its lowest logical level (for example, with elementary theories the class of individuals) defined all possible concepts or all possible subclasses of this level? The idea is not to have done so actually, but to have within the defined systems enough primitive concepts to be able to do so. That is, it is not possible to add to the system a predicate which is not definable in terms of the primitive concepts. Strictly speaking, what Tarski intends is that at the n'th level we are able to define all possible subclasses of the n-1'st level. As Tarski points out in his paper, it is possible to have categorical systems which can be extended through the introduction of concepts not definable in terms of the original theory. The reason for this is that if the isomorphism of realizations of a particular theory can be established on the basis of transformation other than the identity transformation, then the individuals of the theory are not fully fixed in logical space. Tarski then further refines his definition of categoricity by defining the concept on mono-transformability:

$$(x', x'', y', y'', \ldots, R', R'') : \phi(x', y', \ldots) \,.\, \phi(x'', y'', \ldots) \,.\, R' \frac{x', y', \ldots}{x'', y'', \ldots}$$

$$. \, R'' \frac{x', y', \ldots}{x'', y'', \ldots} \, .\supset . \; R' = R''$$

is logically provable.[16]

Before beginning the topic of categoricity in power let us note that in 1942 Carnap introduced alternative terminology to categoricity which he called monomorphism.

> Ein AS heisst monomorph (oder kategorisch), wenn es widerspruchsfrei ist und alle seine Modelle miteinander isomorph sind. Der Begriff der Isomorphie von Modellen ist umfassender als der fruher definierte Begriff der Isomorphie von Klassen oder Relationen (19). Das Modell $\mathfrak{M}$ B_1, $B_2, \ldots, B_n$ fur die n axiomatischen Grundzeichen bestehe aus den Begriffen (oder Extensionen) des Systems S; ein anderes Modell $\mathfrak{M}'$ bestehe aus $B_1', \ldots, B_n'$. $\mathfrak{M}$ heisst isomorph mit $\mathfrak{M}'$, wenn es einen Korrelator zwischen den Individuen in $\mathfrak{M}$ und denen in $\mathfrak{M}'$ gibt derart, dass jedes B_p (p = 1 bis n) auf Grund dieses Korrelators isomorph im fruheren Sinn mit B_p' ist. Wenn das AS dagegen nicht-isomorphe Modelle besitzt, so heisst es polymorph. Wenn ein AS monomorph ist, so besitzt es eine gewisse Vollstandigkeit in dem Sinn, dass es alle strukturellen Eigenschaften moglicher Modelle festlegt.[17]

In the above passage AS is an abbreviation for 'axiom system.' It is notable that at this later date the emphasis has shifted from the determination of a single mathematical domain to the determination of the structural properties of all possible models. It indicates a higher degree of abstraction, away from systems with definite content toward uninterpreted systems in general. It is also worth noting that this concept suffers the same defect as the original notion of categoricity in that it assumes that monomorphism is a necessary and sufficient condition for the determination of structural properties of all possible models when, in fact, it is only a sufficient condition.

How would it be possible to force monomorphism and arithmetical equivalence to become equivalent notions? Why are they not equivalent? Simply because if the domain of objects which satisfy an axiom system is infinite it is possible to construct a domain of objects containing a significantly larger number of objects which satisfy the same axioms. They are arithmetically equivalent but not isomorphic. Consequently, if consideration were limited to domains with an equal number of objects then monomorphism and arithmetical equivalence become equivalent concepts. This is exactly the step taken by the introduction of the notion of categoricity in power. Łoś has written:

> A deductive system is categorical if it possesses only one model, in other words: if each two models are isomorphic. It is well known that no elementary system which has an infinite model is categorical. Usually to prove this theorem, two models of different powers are constructed. It is evident that such two models may not be isomorphic. The problem arises, whether this theorem can be proved in a different way, e.g., by proving that for each elementary system which has an infinite model, there exist two non-isomorphic denumerable models.
>
> The answer is: no. There are such elementary systems which have only one denumerable model. Such a system is called *categorical in power* $\aleph_0$. In general, we say that a system is *categorical in power* $\mathfrak{m}$, if it possesses only one model of the power $\mathfrak{m}$.[18]
>
> A system is *categorical in power* $\mathfrak{m}$, if it possesses a model of the power $\mathfrak{m}$ and all its models of this power are isomorphic in pairs.[19]

Obviously, since this notion resolves the conflict between isomorphism and arithmetical equivalence it is more successful in capturing the intuition behind the concept of categoricity. However, as will be shown later, it does have some undesirable consequences. As was said above, Łoś approached categoricity in power by way of isomorphism but Vaught by way of arithmetical equivalence.

> It follows easily from the generalized Skolem-Löwenheim theorem: If K is a non-empty arithmetically closed class of algebras (with at most a denumerable many relations and operations) such that for some infinite power, all members of K with that power are isomorphic, then all members of K are arithmetically equivalent. A metamathematical consequence of this result is: If T is a consistent theory formalized within first-order logic such that every model of T is infinite, and for some infinite power, all models of that power are isomorphic, then T is complete.[20]

Although this paragraph does not address itself to the problem of categoricity or explicitly define the notion of categoricity, the key words are "all members (models) of that power are isomorphic." This paragraph contains the theorem, also announced by Łoś and at least intuitively anticipated by Veblen, namely:

If theory T is categorical in power $\mathfrak{m}$ *then T is complete.*

This theorem, however, indicates at what point categoricity in power is unfaithful to the intuitive concept to be formalized. It indicates that completeness is a necessary condition for categoricity in power. But through the pioneering work of Gödel it is known that the arithmetic of natural numbers is incomplete. So is the arithmetic of real numbers. Obviously, they cannot be categorical in any power. Thus, the concept of categoricity in power is incapable of distinguishing between those theories which are intended to describe but a single domain and those theories which are intended to describe several mathematical domains. The notion of categoricity in power is capable of reconciling the opposition of isomorphism and arithmetic equivalence which has proven to be a defect in the original definition of categoricity but is unable to reconcile the opposition of isomorphism and distinguishability of axiom systems intended to describe a single domain from those intended to describe several. This, of course, was also a defect with the original definition of categoricity but historically it was not the major one. It is worthwhile noting that in the context of categoricity in power the equivalence of isomorphism and arithmetical equivalence indicates that also 'distinguishability' and arithmetical equivalence are incompatible. In general, incomplete theories will have classes of objects which satisfy their axioms but these classes are not arithmetically equivalent. Consequently, neither isomorphism nor arithmetical equivalence will help in distinguishing axiom systems meant to determine single or several theories.

All these, however, are rather obvious considerations which point our thoughts to a more interesting question: in what sense is it said that incomplete theories do determine mathematical domains? For example, to the theory of the arithmetic of natural numbers add a sentence which is independent of the axioms to form theory A and add the negation of this sentence to form theory B. After this point of ramification in what sense do they determine a single domain? Yet it could be maintained there is a definite and distinct difference between the theory (as exemplified by an axiom system, finite or infinite) and the mathematical domain (given by intuition). Then it could be concluded (which seems to be a philosophical conclusion of the Gödel incompleteness theorem) that present axiomatic devices and perhaps mathematical language in general is incapable of describing an intuitive domain. Ultimately, this path of reasoning leads to the necessity of some decision as to the role of intuition in mathematics or the relation between intuition and mathematics or even to the relation between thought and language.

After such an analysis how much of the original intention of categoricity can be preserved? From what has been said above, a workable concept of categoricity must be limited to: (1) complete theories, and (2) some one particular power or cardinality for the domain of objects. Yet to keep both of these forces a change in the notion of single domain. Instead of calling a single domain that 'given' by intuition or those which verify the same set of propositions only, call a single domain those classes which verify the same set of propositions and whose 'points' or individual objects 'act' or 'behave' identically with respect to the determination forced by the axiom system. For example, to the theory of dense ordering without beginning or end (restricting consideration to domains of cardinality $\aleph_0$ and theories which are complete), it is possible to add a sentence, formulated entirely within the grammar of this theory, concerning the behavior of a triple of individuals, i.e., $a_1 < a_2 < a_3$ which is independent of the axiom system.[21] However, within the same restrictions, to a modified version of Tarski's axiom system for the arithmetic of real numbers, it is not possible, within the grammar of the theory, to find a sentence concerning the individuals of the domain which is independent of the axioms of the theory.[22] In the former theory, the domain is not absolutely determined with respect to the individuals of the domain, there is some degree of freedom, whereas, by contrast, in the latter theory all individual numbers of the domain are absolutely determined. To use Carnap's terminology, in the first theory, "the structural properties of all possible models" are determined; in the second theory, the elemental as well as the structural properties of all possible models are determined. Theories whose axioms determine classes which verify the same propositions and elemental sentences will be said to determine a single domain. Thus the idea behind constructive categoricity introduced by Grzegorczyk in [9], can be preliminarily stated as follows: if the set of constructible atomic formulas of T behave the same in any two of its models, then T is constructively categorical.

In order to determine the elemental sentences it must be possible to enumerate effectively all the individuals of the theory in question. This can be done by replacing all existential quantifiers occurring in the axioms of the theory by suitable parameterized functions and then letting the resultant quantifier-free theory determine its own model by the familiar process of allowing the elements of the syntax to name themselves as semantic objects and the predicates of the syntax determine the relations of the semantics. The term 'constructive' is used because (1) the process of effective enumeration and (2) the self-determination of models is employed.

Unfortunately, such a goal is excellent in theory but practical considerations force severe limitations. Anyone who understands the general mathematical meaning of the word 'constructive' (deliberately used in the term 'constructive categoricity') can anticipate the restriction that must be placed on the notion of constructive categoricity. To be able to ascertain whether two domains verify the same individual propositions there need be an effective procedure for 'running through' the individuals and individual propositions. Effective procedures must be recursive. Automatically, either of the following restrictions are imposed: (1) the cardinality of the domains under consideration must be less than or equal to $\aleph_0$ as adopted by Grzegorczyk or (2) the domains under consideration must be well-ordered or of a given ordinal type, as I now suggest, to modify constructive categoricity in order that it might be applicable to a larger class of theories.[23] Both restrictions are severe, yet acceptable, because the concept of constructive categoricity with either restriction yields useful or interesting information. No systematic research has been done concerning the concept of categoricity based on well-ordered domains.

Constructive categoricity does meet generally the original requirements of the notion of categoricity. Within the class of *complete* theories, those axiom systems which are constructively categorical (1) verify the same set of elemental and general sentences, and (2) have up to isomorphism but a single model, i.e., every model is mutually interpretable in every other model. The property of constructive categoricity is capable of distinguishing between those axiom systems which determine a single mathematical domain and those which do not. However, the differences between the properties of constructive categoricity and the original intention are notable. Since the formal requirements for isomorphism or mutual interpretability are more comprehensive, the intuitive notion of single domain also changes since they are directly related. Not only does the relation of mutual interpretability require that any pair of domains has an equivalent number of individuals, and that the set of statements formalizable in the grammar of one domain be interpretable in the grammar of any other, but also that there be no implicit functions on the domain of individuals into the domain of individuals. Also the implicit functions of one domain must be interpretable in any other domain.

CHAPTER II

Introduction The main topic of this paper is Grzegorczyk's notion of constructive categoricity. In the first chapter the history of the notion of categoricity was traced up to and including constructive categoricity. Each variant of the original notion was accompanied by an informal and terse analysis of its points of application and its limitations with the intention of describing its success in formalizing the original intuition. This chapter gives a formal analysis of the variant of categoricity called constructive categoricity. As was noted in Chapter I, the key to using constructive procedures is the ability to enumerate the elements of the theory in question. This enumeration is accomplished through the use of Skolem functions (to be defined later). Consequently, the simplest system to which the notion of constructive categoricity can be applied is an elementary theory with functions. Moreover, it is only applicable to elementary theories.

The contents of this chapter are: (1) a description of elementary theories with functions and the relation of equality (since the majority of the theorems are based on an elementary theory with equality) and some simple theorems concerning their properties; (2) a description of Skolem theories (which is not given by Grzegorczyk) and theorems concerning their properties (as far as is known at this moment, Skolem theories have not been studied for themselves but always with respect to some other notion. Although a thorough study of the properties of Skolem theories is not included in this thesis, several theorems are included relating to these properties); (3) a description of the work that Grzegorczyk did with respect to constructive categoricity; (4) a new characterization of a subclass of constructively categorical theories; (5) and, finally, the relation between constructive categoricity and categoricity in power. In the third chapter, building on the formal results of this chapter, the information conveyed by the predicate 'constructively categorical' will be clearly delineated.

Before proceeding, let it be noted that in this chapter several theorems are stated without proof. This means that these theorems are well known in the literature of logic and are only included for ease and completeness of presentation. These theorems and necessary definitions will be herein printed in lower cast. Also, several well known theorems are given with proof. This will occur when the context demands variations from the classic proof or where there is a new proof possible based on results of this thesis. Where this occurs it will be duly noted. All other theorems given with proof are those of the author of this paper.

1 Elementary Theories with Functions and the Relation of Equality This section will be primarily terminological. A description of the syntax and semantics of elementary theories will be given along with definitions of model, isomorphism, cardinality, inferential equivalence, arithmetic equivalence, extensionality, and completeness.

Definition 2.1 The alphabet of formal language $\mathcal{L}$ is the following:

1. a set of subscripted letters $a_0, a_1, a_2, \ldots b_0, b_1, b_2, \ldots c_0, c_1, c_2, \ldots$ in general a_i, b_i, c_i where i is any ordinal number less than some given ordinal I (usually $\mathrm{I} = \omega$). This set of subscripted letters is called the set of *individual constants*, denoted IC.
2. a set of subscripted letters $x_0, x_1, x_2, \ldots y_0, y_1, y_2, \ldots z_0, z_1, z_2, \ldots$ in general x_i, y_i, z_i where i is any ordinal number less than some given ordinal I (usually $\mathrm{I} = \omega$). This set of subscripted letters is called the set of *individual variables*, denoted IV.
3. a set of letters each with a subscript and superscript $f_0^1, f_1^1, f_2^1, \ldots f_0^2, f_1^2, f_2^2, \ldots$ in general f_j^i where i is a finite ordinal less than some given finite ordinal I and j is an ordinal number less than some given ordinal J (usually $\mathrm{J} = \omega$). This set of letters with subscript and superscript is called the set of *function constants*, denoted FC.
4. a set of letters each with a subscript and superscript $P_0^1, P_1^1, \ldots P_0^2, P_1^2, \ldots Q_0^1, Q_1^1, \ldots Q_0^2, Q_1^2, \ldots R_0^1, R_1^1, \ldots R_0^2, R_1^2, \ldots$ in general $P_i^j, Q_i^j, R_i^j, \ldots$ where i and j are any ordinal numbers less than some given ordinals I and J respectively (usually $\mathrm{I} = \omega$ and $\mathrm{J} < n \leqslant \omega$). This set of letters with subscript and superscript is called the set of *predicate constants*, denoted PC.
5. the following set of symbols called *logical signs*, denoted LS, $\neg, \rightarrow, \vee, \&, \leftrightarrow, (\ldots), (\exists \ldots)$ and a set of symbols (,) called *punctuation signs*, denoted PS.

The following two conditions hold:

1. sets IC, IV, FC, PC, LS, PS are mutually pairwise disjoint;
2. $\mathrm{IC} \cup \mathrm{IV} \cup \mathrm{FC} \cup \mathrm{LS} \cup \mathrm{PS}$ = alphabet of $\mathcal{L}$.

Definition 2.2 An *expression* is any finite contiguous concatenation of symbols of the alphabet of language $\mathcal{L}$. A *term* of language $\mathcal{L}$ is an expression determined by the following rules:

1. an individual constant or individual variable is a term;
2. if $t_1, \ldots, t_i$ is a sequence of i terms, not necessarily distinct, and f_j^i is a function constant then $f_j^i(t_1, \ldots, t_i)$ is a term;
3. only expressions determined by the above rules are terms.

Definition 2.3 An atomic formula of $\mathcal{L}$ is an expression determined by the following rules:

1. if t_1 and t_2 are terms, not necessarily distinct, then $t_1 = t_2$ is an atomic formula;
2. if $t_1, \ldots, t_i$ is a sequence of i terms, not necessarily distinct, and P_j^i is a predicate constant then $P_j^i(t_1, \ldots, t_i)$ is an atomic formula;
3. all and only those expressions determined by the above rules are called atomic formulas.

Definition 2.4 A well-formed formula of $\mathcal{L}$ is an expression determined by the following rules:

1. an atomic formula is a well-formed formula;
2. if A is a well-formed formula (denoted wff) of $\mathcal{L}$ then $\neg A$ is a wff of $\mathcal{L}$, called the *negation* of A;
3. if A and B are the wff of $\mathcal{L}$ then

a. $(A \mathbin{\&} B)$ is a wff of $\mathcal{L}$, called the *conjunction* of A and B;
b. $(A \vee B)$ is a wff of $\mathcal{L}$, called the *disjunction* of A and B;
c. $(A \rightarrow B)$ is a wff of $\mathcal{L}$, called the *implication* of B from A;
d. $(A \leftrightarrow B)$ is a wff of $\mathcal{L}$, called the *equivalence* of A and B;

4. if A is a wff of $\mathcal{L}$ and x_i is an individual variable occurring as part of A then

a. $(x_i)\, A$ is a wff of $\mathcal{L}$;
b. $(\exists x_i)\, A$ is a wff of $\mathcal{L}$.

Two things should be noted in these definitions:

1. the term "$\mathcal{L}$" and "language $\mathcal{L}$" are used interchangeably and will continue to be so;
2. if the occurrence of an individual constant a_i or individual variable x_i as a part of a wff A is of particular interest, such a case will be noted by $A(x_i)$ or $A(a_i)$.

A part of a well-formed formula A, whether or not it is a proper part, if it is itself a wff, is called a *subformula* of A. A variable x_i occurring in wff A, i.e., $A(x_i)$ is called a *bound occurrence of variable* x_i provided it occurs in a subformula of A of the form $(x_i)\, B(x_i)$ or $(\exists x_i)\, B(x_i)$; otherwise it is called a *free occurrence of variable* x_i. A wff A in which no variable occurs free is called a *sentence* of $\mathcal{L}$.

Definition 2.5 An *elementary theory* based on language $\mathcal{L}$ with equality, denoted T_e, is a set of sentences of $\mathcal{L}$. The sentences of set T_e are either called *axioms*, sentences accepted as elements of T_e without proof (*cf.*, Def. 2.6) or *theorems*, sentences accepted on the basis of a proof.

A proof is a sequence of sentences or well-formed formulas constructed from the rules of procedure. For elementary theories there are two *rules of procedure*:

1. if $A \rightarrow B$ is an accepted sentence and A is an accepted sentence then B is an accepted sentence;
2. if A is an accepted sentence, then $(x_i)B$ is an accepted sentence where $B = A(x_i)$.

Definition 2.6 A *proof of sentence* B is a finite sequence of sentences or wff $A_1, \ldots, A_n$ such that $A_n = B$ and each A_i is either (1) an axiom, or (2) follows from the preceding members of the sequence by an application of one of the rules of procedure.

T_e then might more easily be defined as a set of sentences including the set of axioms and closed with respect to the rules of procedure. The set of axioms for elementary theories is the union of two disjoint sets: the *logical* axioms and the non-logical or *proper* axioms. The proper axioms could be called the characteristic or determining axioms of T_e.

The logical axioms of T_e are any sentences of the following form:

T1. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
T2. $A \rightarrow (\neg A \rightarrow B)$
T3. $(\neg A \rightarrow A) \rightarrow A$
T4. $(A \rightarrow B(x_i)) \rightarrow (A \rightarrow (x_i)B(x_i))$ *provided* x_i *does occur free in* A
T5. $(x_i)A(x_i) \rightarrow A(t)$ *where* t *is a term of* $\mathcal{L}$ *and* t *contains no variable* x_j *such that* x_i *in* A *occurs in a subformula* B *of* A, $B = (x_j)C(x_i)$ *or* $B = (\exists x_j)C(x_i)$
E1. $(x)\, x = x$
E2. $(x)(y)(x = y) \rightarrow (y = x)$
E3. $(x)(y)(z)(x = y) \rightarrow ((y = z) \rightarrow (x = z))$
E4. $(x)(y)(x = y) \rightarrow (A(x) \rightarrow A(y))$

The proper axioms cannot be given generally but are given in the construction of a particular elementary theory. For example, the elementary theory of groups is the set of sentences of $\mathcal{L}$ which are closed with respect to the rules of procedure and include the following three sets of sentences as axioms: I. T1-T5, II. E1-E4, and

III.
G1. $(x)(y)(\exists z)\, x \circ y = z$
G2. $(x)(y)(z)(x \circ (y \circ z)) = ((x \circ y) \circ z)$
G3. $(x)\, x \circ a_0 = x$
G4. $(x)(\exists y)\, x \circ y = a_0$

For the discussion of Skolem theories and constructive categoricity it is not necessary to have the relation of equality as an element of the language. Subsequently, some theorems and an unsolved problem will be given with respect to such theories without equality. Elementary theories with functions only will be denoted by T_f.

Theory T_f is based upon language $\mathcal{L}'$, identical to language $\mathcal{L}$, with the following two exceptions:

1. the relation of equality, symbolized by '=', does not occur in $\mathcal{L}'$;
2. condition (1) of Def. 2.3 concerning the generation of wff does not occur in $\mathcal{L}'$.

Otherwise $\mathcal{L}$ and $\mathcal{L}'$ are identical.

The logical axioms for T_f consist of the set T1-T5 only. For example, the elementary theory of partial ordering is a set of sentences of $\mathcal{L}'$ which are closed with respect to the rules of procedure and include the following two sets of sentences as axioms: I. T1-T5 and

II. 01. $(x) \neg(x < x)$
02. $(x)(y)(z)(x < y \ \& \ y < z) \rightarrow (x < z)$

A realization or interpretation of a language $\mathcal{L}$ with functions and equality is a domain of objects (or a set of objects) upon which some functions and relations are defined. A relation is a subset of the countably infinite sequences of elements of the domain of objects. Simply, let A represent the domain of objects, then relation R is a subset of A^{ω} i.e., $R \subset A^{\omega}$. It is easiest to consider functions simply as many-one relations. Ordinarily, in defining a realization reference must be made to the rank of the functions and relations. Rank of a function or relation is the number of arguments of the function or relation. However, because the functions and relations are herein defined on countably infinite sequences, comparisons can be made between several realizations without cumbersome notation relating rank of the corresponding functions and relations.

A set A upon which is defined functions $F_i^1, F_i^2, F_i^3, \ldots$ and relations $R_1, R_2, R_3, \ldots$ is said to be an *interpretation* of language $\mathcal{L}$ if:

1. to every individual constant of language $\mathcal{L}$ there corresponds an individual $a \in A$;
2. to every function constant f_j^i of language $\mathcal{L}$ there corresponds a function F_j^i;
3. to every predicate constant P_j^i of language $\mathcal{L}$ there corresponds a relation R_j.

Having defined a realization it is necessary to be able to define 'satisfaction of a wff of $\mathcal{L}$ by sequence $s \in A^{\omega}$.' Once this is done then a model for theory T_e can be defined. The following preliminary definition is required.

Definition 2.6.a A term t_i of language $\mathcal{L}$ is assigned an element of set A by sequence s according to the following rules: Let $s = (a_1, \ldots, a_i, \ldots)$ then

1. if t_i is an individual constant then s assigns to t_i the element a of A assigned to it by the interpretation;
2. if t_i is an individual variable x_i then s assigns to t_i element a_i of s;
3. let a_i^t be the element of A assigned by s to terms t_i and F_i^j the function assigned to the function constant f_i^j by interpretation A then s assigns to $f_i^j(t_1, \ldots, t_j)$ the individual $a_{fj}\,(t_1, \ldots, t_j) = F_i^j(a_1^{t_1}, a_2^{t_2}, \ldots a_j^{t_j})$.

Definition 2.7 An atomic formula of language $\mathcal{L}$ is said to be *satisfied* by sequence $s \in A^{\omega}$ in interpretation A:

1. if $a_1^{t_1}$ and $a_2^{t_2}$ are individuals of A assigned to terms t_1 and t_2 by sequence s and $t_1 = t_2$ is any atomic formula then it is satisfied by sequence s if and only if $a_1^{t_1} = a_2^{t_2}$;
2. if $a_1^{t_1}, \ldots, a_k^{t_k}$ is a sequence of k individuals of A assigned to terms $t_1, \ldots, t_k$ by sequence s and $P_i^k(t_1, \ldots, t_k)$ is an atomic formula then it is satisfied by s if and only if

$$(a_1^{t_1}, \ldots, a_k^{t_k}, a_{k+1}, \ldots) \in R_i \subset A^{\omega}$$

where R_i is the relation corresponding to predicate P_i^k.

Normally Def. 2.7 (1) can be stated in terms of any binary congruence relation R^2 on domain A and $t_1 = t_2$ is satisfied if $R(a_1^{t_1}, a_2^{t_2})$. However, models in which the symbol of equality is interpreted as equality and not simply a congruence relation are called *normal models*. In this thesis all models for T_e are normal models.

Definition 2.8 · A well-formed formula of language $\mathcal{L}$ is said to be *satisfied by sequence* $s \in A^{\omega}$ in interpretation A:

1. wff $\neg A$ is satisfied by sequence s if and only if A is not satisfied by sequence s;
2. if A and B are the wff of T, then

a. wff $(A \;\&\; B)$ is satisfied by sequence s if and only if A is satisfied by sequence s and B is satisfied by sequence s;
b. wff $(A \vee B)$ is satisfied by sequence s if and only if A is satisfied by sequence s or B is satisfied by sequence s;
c. wff $(A \rightarrow B)$ is satisfied by sequence s if and only if A is not satisfied by sequence s or B is satisfied by sequence s;
d. wff $(A \leftrightarrow B)$ is satisfied by sequence s if and only if A is satisfied by sequence s if and only if B is satisfied by sequence s;

3. A is a wff of $\mathcal{L}$ and x_i is an individual variable occurring as part of A then:

a. wff $(x_i)A(x_i)$ is satisfied by sequence s if and only if *every* sequence of A^{ω} which differs from sequence s in at most the i'th place satisfies A;
b. iff $(\exists x_i)A(x_i)$ is satisfied by sequence s if and only if some sequence of A which differs from sequence s is at most the i'th place satisfies A.

A wff or a sentence of language $\mathcal{L}$ is said to be *true in interpretation* A if it is satisfied by every sequence s of A. An interpretation A is said to be a *model* for elementary theory T_e iff the axioms of T_e are true in A and the rules of procedure of T_e preserve truth in A. A rule of procedure is said to preserve truth in A provided an application of that rule to a wff of T_e which is true in A produces another wff of T_e which is true in A. A wff or sentence of T_e is said to be logically *valid* if and only if it is true in every model.

The notation used for a model will be the following sequence

$$\mathfrak{M} = \langle M, F_1^0, \ldots F_1^1, \ldots F_j^i, \ldots R_1, R_2, \ldots \rangle$$

where M is a set of objects and F_j^i and R_i are functions and relations defined on M. This sequence will be called model $\mathfrak{M}$. The phrase 'the cardinality of model $\mathfrak{M}$' or '*the cardinality of* $\mathfrak{M}$' is an ellipsis for the phrase 'the cardinality of the set M in the model $\mathfrak{M}$.'

Above, theory T_e was defined in terms of axioms and rules of procedures, i.e., it was defined syntactically. It can be defined with an equal degree of satisfaction as that system or set of sentences of $\mathcal{L}$ which are true under a particular given interpretation or true under a particular set of interpretations, i.e., T_e can be defined semantically. Since constructive categoricity which is being investigated is a property of axiom systems the syntactic definition of T_e is sufficient.

Definition 2.9 Models $\mathfrak{M}$ and $\mathfrak{N}$ are said to be *similar* provided the same number of relations of equivalent rank (see p. 526) are defined on each model. A function φ is a *homomorphism* between similar models $\mathfrak{M}$ and $\mathfrak{N}$ provided

1. φ maps M into N;
2. $\varphi(F_i^j(a_1, \ldots, a_j, \ldots)) \subset \varphi(F_i^j)(\varphi(a_1), \ldots \varphi(a_j) \ldots)$;

and

3. $\varphi(R_i(a_1, \ldots, a_j, \ldots)) \subset \varphi(R_i)(\varphi(a_1), \ldots \varphi(a_j) \ldots)$.

The above definition contains many abbreviations. Strictly speaking

$$\varphi\colon\ M \to N.$$

$\varphi(F_i^j)$ is an abbreviation for the F_i^j of $\mathfrak{N}$ which is associated with the corresponding F_i^j of $\mathfrak{M}$. Similarly, for $\varphi(R_i)$. Condition 3 says that for $s \in M^\omega$ φ generates a corresponding $s' \in N^\omega$ such that $(s)\ s \subset R_i \to s' \subset \varphi(R_i)$.

Function φ is an *isomorphism* between similar models $\mathfrak{M}$ and $\mathfrak{N}$ provided

1. φ maps M one-one and onto N;
2. $\varphi(F_i^j(a_1, \ldots, a_j, \ldots)) = \varphi(F_i^j)(\varphi(a_1), \ldots, \varphi(a_j), \ldots)$;

and

3. $\varphi(R_i(a_1, \ldots, a_j, \ldots)) = \varphi(R_i)(\varphi(a_1), \ldots, \varphi(a_j), \ldots)$.

Definition 2.10 Elementary theories S_e and T_e are said to be *inferentially equivalent* provided:

1. the axioms of T_e are theorems of S_e and vice versa;
2. the rules of procedure of T_e are provable in S_e as metarules or theorems of S_e and vice versa;
3. the definitions in T_e are theorems or definitions of S_e and vice versa.

This is, of course, a syntactic type of equivalence.

Definition 2.11 Models $\mathfrak{M}$ and $\mathfrak{N}$ are said to be *arithmetically equivalent* provided sentence S is true in $\mathfrak{M}$ if and only if sentence S is true in $\mathfrak{N}$.

This is a semantic type of equivalence.

Definition 2.12a Property P is said to be *syntactically extensional* if for some elementary theory T_e, T_e possesses property P and for any elementary theory S_e such that S_e and T_e are inferentially equivalent then S_e also possesses property P.

Definition 2.12b Let $\mathfrak{M}$ and $\mathfrak{N}$ be models of an elementary theory. Property P is said to be *semantically extensional* if $\mathfrak{M}$ possesses property P and for $\mathfrak{N}$ such that $\mathfrak{M}$ and $\mathfrak{N}$ are arithmetically equivalent then $\mathfrak{N}$ also possesses property P.

Definition 2.13a Elementary theory T_e is said to be *syntactically consistent* if there is at least one well-formed formula of language $\mathcal{L}$ which is not a theorem of T_e.

Definition 2.13b Elementary theory T_e is said to be *semantically consistent* if it has a model.

Definition 2.14a Elementary theory T_e is said to be *syntactically complete* if the addition of a wff of language $\mathcal{L}$ which is not a theorem of T_e produces a syntactically inconsistent system.

If the theory possesses a negation sign and is based on classical propositional calculus then this definition is equivalent to the sentence, "*Theory T_e is syntactically complete if for every sentence A of $\mathcal{L}$ either A is a theorem of T_e or $\neg A$ is a theorem of T_e.*"

Definition 2.14b Elementary theory T is said to be *semantically complete*, or simply, *complete*, if and only if every true sentence of T is a theorem of T.

2 Some Properties of Elementary Theories

Theorem 2.18a *Let T_f be an elementary theory based only on logical axioms. Let T_1 be formed from T_f by the addition of a set of non-logical axioms $S_1, \ldots, S_n$ whose alphabet is identical with the alphabet of T_f. If A is a well-formed formula of T_f, then A is a theorem of T_1 if and only if there is a theorem of T_f of the form $S_1 \to \ldots \to S_n \to A$.*

Theorem 2.18b *Let T_e be an elementary theory (with equality) based only on logical axioms. Let T_1 be formed from T_e by the addition of a set of non-logical axioms $S_1, \ldots, S_n$ whose alphabet is identical with the alphabet of T_e. If A is a well-formed formula of T_e then A is a theorem of T_1 if and only if there is a theorem of T_e of the form $S_1 \to \ldots \to S_n \to A$.*

Theorem 2.19 *If $A \to B$ is a theorem of propositional calculus there is a formula C whose propositional variables occur in both A and B such that $A \to C$ and $C \to B$ are theorems of propositional calculus.*

3 Elementary Skolem Theories Skolem theories are subsets of elementary theories with functions or elementary theories with functions and equality. What distinguishes them from other elementary theories is that all existential quantifiers are replaced by a special or distinguished set of functional constants called Skolem functions (some existential quantifiers are replaced by unique individual constants but if individual constants are considered as zero-placed functions the above statement needs no modification). Skolem theories, therefore, are free variable calculi with the rule of detachment as the only rule of procedure (the rule of substitution being subsumed by the use of axiom schemata) and all logical axioms concerning quantifiers, which occur in elementary theories with functions are removed except the law of particularization. In summary, a Skolem theory is either a free-variable elementary theory with functions (to be denoted T_f') or free variable elementary theory with equality (to be denoted T_e') based on a full propositional calculus with a logical axiom for particularizing and the rule of detachment as the sole rule of procedure.

T_f' or T_e' are syntactically identical to T_f and T_e except for the following points:

1. the logical axioms of T_f' or T_e' are simply those of any full propositional calculus and the axiom scheme $P(x) \to P(t)$;
2. the only rule of procedure is the rule of detachment and thus *a fortiori* the definition of proof changes;
3. there are no quantifiers in T_f' or T_e'.

The semantics of T_f or T_e and T_f' and T_e' are the same and this is the reason for their importance. Elementary Skolem theories are not so much studied for themselves but for what they reveal concerning the properties of the elementary theories to which they are correlated. This correlation is accomplished through a constructive procedure. The language of the Skolem theory is simply the language of the original elementary theory, be it T_f or T_e, to which has been added Skolem function constants which are constants not occurring in the original theory. The following definition gives the rules for generating a Skolem theory from an elementary theory. The resultant theory is called the equivalent Skolem theory.

Definition 2.20 A wff A of the language upon which T_f' or T_e' is based is said to be the *Skolem equivalent* of a sentence S of elementary theory T_f or T_e if and only if sentence S is in prenex normal form and A is the last member of the finite sequence $S_1, \ldots, S_n$ where

1. $S = S_1$;
2. $S_n = A$;
3. if S_1 is of the form $(\exists x_1)(\exists x_2) \ldots (\exists x_i)\, P(x_1, x_2, \ldots, x_i)$ then S_2 is of the form $P(a_1, a_2, \ldots, a_i)$ where each a_i is a Skolem individual constant, distinct from each other and any other constant occurring in T_f or T_e;
4. if S_1 is of the form $(x_1) \ldots (x_i)(\exists x_j)\, A(x_1, \ldots, x_i, x_j, x_{j+1})$ then S_2 is of the form $(x_1) \ldots (x_i)\, A(x_1, \ldots, x_i, f_j^i(x_1, \ldots, x_i), x_{j+1})$ where f_j^i is a Skolem functional constant different from all others used in the transition;
5. if S_i is any member of the series $S_1, \ldots, S_n$ of the form $(x_1) \ldots (x_i)(\exists x_j)\, A(x_1, \ldots x_i, x_j, x_{j+1})$ then S_{i+1} is of the form $(x_1) \ldots (x_i)\, A(x_1, \ldots x_i, f_n^i(x_1, \ldots x_i), x_{j+1})$ where f_n^i is a Skolem functional constant different from all Skolem functions in transition from $S_j \rightarrow S_{j+1}$ and different from all functional constants in any S_j, $j < i$;
6. S_{n-1} will be a sentence in prenex normal form whose prefix contains only universal quantifiers. S_n is obtained from S_{n-1} by deleting the prefix.[24]

Let $T_e = T_1, \ldots, T_n$ be an axiom system for elementary theory T_e, excluding the logical axioms. $\mathrm{Skl}(T_e)$ will be used to denote the Skolem equivalent of axiom system T_e, i.e., $\mathrm{Skl}(T_e)$ is the set consisting of the Skolem equivalents of axioms T_1 through T_n inclusive. Let $\mathrm{SFC}(T_i)$ represent the set of Skolem functional constants and Skolem individual constants used in the formation of the Skolem equivalent of sentence T_i. It is important to remember that in the formation of $\mathrm{Skl}(T_e)$ the following condition holds:

sets $\mathrm{SFC}(T_1)$, $\mathrm{SFC}(T_2), \ldots, \mathrm{SFC}(T_n)$ *are pairwise disjoint.*

An example of a Skolem form of an elementary theory is a set of sentences of $\mathcal{L}$ which are closed with respect to the rules of detachment and they include the following four sets of sentences: I. T1-T3, II. S1. $A(x_i) \rightarrow A(t)$, III. E1-E4, and

IV. SG1. $x \circ y = f_1^2(x, y)$
SG2. $((x \circ y) \circ z) = (x \circ (y \circ z))$
SG3. $x \circ a_o = x$
SG4. $x \circ f_1^1(x) = a_o$

This theory is obviously the Skolem form of the theory of groups.

Theorem 2.21a *Let $T_f^!$ be a Skolem theory based only on logical axioms. Let $T_1^!$ be formed from $T_f^!$ by the addition of a set of non-logical axioms $S_1, \ldots, S_n$ whose alphabet is identical with the alphabet $T_f^!$. If A is a well-formed formula of $T_f^!$ then A is a theorem of $T_1^!$ if and only if there is a theorem of $T_f^!$ of the form $S_1 \rightarrow \ldots S_n \rightarrow A$.*

Theorem 2.21b *Let $T_e^!$ be a Skolem theory (with equality) based only on logical axioms. Let $T_1^!$ be formed from $T_e^!$ by the addition of a set of non-logical axioms $S_1, \ldots, S_n$ whose alphabet is identical with the alphabet of $T_e^!$. If A is a well-formed formula of $T_e^!$ then A is a theorem of $T_1^!$ if and only if there is a theorem of $T_e^!$ of the form $S_1 \rightarrow \ldots S_n \rightarrow A$.*

Theorem 2.22a *Let S be a prenex normal form formula of elementary* *theory T_f. Then there exists a free variable formula S' whose language does not differ from S except for the addition of a finite number of Skolem constants and Skolem function constants such that*

(1) *models for S can be transformed into models for S' and models for S' are models for S;*
(2) *if S' is a theorem of* $\mathrm{Skl}(T_f)$ *then S is a theorem of T_f.*[25]

Proof: Let $\mathfrak{M}$ be a model for S. Prenex normal form of formula S can only be of three forms:

(1′) $(x_i)\,S(x_i)$ a formula whose prefix contains only universal quantifiers;
(2′) $(\exists x_i)\,S(x_i)$ a formula whose prefix begins with an existential quantifier;
(3′) $(x_i)\ldots(x_j)(\exists x_k)S(x_i,\ldots,x_j,x_k)$ a formula whose prefix begins with universal quantifiers followed by an existential quantifier.

Then

(1′) if S has the form $(x_i)S(x_i)$ then S' has the form $S(x_i)$ and any model for S will also be a model for S' since S and S' are semantically identical;
(2′) if S has the form $(\exists x_i)S(x_i)$ then $\mathfrak{M}$ will be its model iff for any sequence $S \in M^{\omega}$ there is a sequence $s' \in M^{\omega}$ which differs from s in at most the i'th place and s' satisfies $S(x_i)$. S' will have the form $S(a_i)$. Now to a_i let $\mathfrak{M}' \in \mathfrak{M}(S')$ assign $s_i' \in s'$ satisfying $S(x_i)$. Then any model $\mathfrak{M}$ which is a model of $(\exists x_i)S(x_i)$ will also be a model for $S(a_i)$;

Note: Even though the language of S may contain individual constants s_1' can always be assigned to a_i by $\mathfrak{M}$ because in the transformation from $S \rightarrow S'$, a_i is chosen from a set of constants which have not occurred in any previous wff, i.e., the Skolem constants are distinct from the individual constants and from each other.

(3′) if S has the form $(x_i)\ldots(x_j)(\exists x_k)S(x_i,\ldots,x_j,x_k)$ then $\mathfrak{M}$ will be its model iff for any sequence $(s_1,\ldots,s_k,\ldots) = s \in M^{\omega}$ there is a sequence $s' \in M^{\omega}$ which differs from s in at most the i'th place and s' satisfies $S(x_i,\ldots,x_j,x_k)$. S' will have the form $S(x_i,\ldots x_j, f_i^{j-i}(x_i,x_j))$. If $f_i^{j-1}(s_1,\ldots,s_k,\ldots)$ is defined for every sequence $s \in M^{\omega}$ having the value s_k' then $\mathfrak{M}$ will also be a model for S'.

Note: Again, as in (2′) the language of S may contain function constants but since the Skolem functions are distinct from the functional constants and from each other the above assignment can always be made.

Consequently, in virtue of (1′), (2′), and (3′), point (1) is proven. In virtue of Theorems 2.18a and 2.21a point (2) can be restated as follows:

If $\mathrm{Skl}(T_1) \rightarrow \ldots \rightarrow \mathrm{Skl}(T_n) \rightarrow S'$ *is a theorem of* $\mathrm{Skl}(T_f)$ *then* $T_1 \rightarrow \ldots \rightarrow T_n \rightarrow S$ *is a theorem of* T_f, *where* $\mathrm{Skl}(T_1),\ldots,\mathrm{Skl}(T_n)$ *are the non-logical axioms of theory* T_f, *respectively.*

Now, assume $\mathrm{Skl}(T_1) \rightarrow \ldots \rightarrow \mathrm{Skl}(T_n) \rightarrow S'$ is a theorem of $\mathrm{Skl}(T_f)$. Let $\mathfrak{M}$ be a model for $\mathrm{Skl}(T_f)$. By point (1) it is also a model for theory T_f and satisfies $T_1 \rightarrow \ldots \rightarrow T_n \rightarrow S$. But by the well known completeness theorem for elementary theories T_f, $T_1 \rightarrow \ldots \rightarrow T_n \rightarrow S$ is a theorem (in virtue of the logical axioms solely). Thus S is a theorem of theory T_f.

N.B. In virtue of the given transformation and Def. 2.24, the model resulting from the transformation is a constructive model for S'.

Theorem 2.22b *Let S be a prenex normal form formula of elementary theory with equality T_e. Then there exists a free variable formula S' whose language does not differ from S except for the addition of a finite number of Skolem constants and Skolem functional constants such that*

(1) *models for S can be transformed into constructive models for S' and models for S' are models for S;*
(2) *if S' is a theorem of* $\mathsf{Skl}(T_e)$ *then S is a theorem of T_e.*

Proof: Similar to 2.22a.

Theorem 2.23a *Let A be a conjunction of the axioms of* $\mathsf{Skl}(T_f)$ *and* $\mathsf{L}(S)$ *be the list of atomic formulas contained in wff S. If $(A \& B) \to C$ is a theorem of* $\mathsf{Skl}(T_f)$ *then there is a formula D of* $\mathsf{Skl}(T_f)$ *such that* $\mathsf{L}(D) = \mathsf{L}(A \& B) \cap \mathsf{L}(C)$ *and $(A \& B) \to D$ and $D \to C$ are theorems of* $\mathsf{Skl}(T_f)$.

Proof: Since $(A \& B) \to C$ is a quantifier free formula of $\mathsf{Skl}(T_f)$ it is a theorem of the propositional calculus on the atoms of $(A \& B) \to C$. Thus by Theorem 2.19 there is a formula D whose propositional variables occur both in $(A \& B)$ and C such that $(A \& B) \to D$ and $D \to C$ are theorems of the propositional calculus and consequently of $\mathsf{Skl}(T_f)$.

Theorem 2.23b *Let A be a conjunction of the axioms of* $\mathsf{Skl}(T_e)$. *If $(A \& B) \to C$ is a theorem of* $\mathsf{Skl}(T_e)$ *then there is a formula D of* $\mathsf{Skl}(T_e)$ *such that* $\mathsf{L}(D) = \mathsf{L}(A \& B) \cap \mathsf{L}(C)$ *and $(A \& B) \to D$ and $D \to C$ are theorems of* $\mathsf{Skl}(T_e)$.

Proof: Similar to 2.23a.

Definition 2.24 If theory $T_f(T_e)$ is an elementary theory (with equality) based on axioms $T_1, \ldots T_n$ then model

$$\mathfrak{M} = \langle M, a_1, \ldots, a_n, F_1^1, F_2^1, \ldots R_1, R_2 \ldots \rangle$$

is called *constructive* if $\mathfrak{M}$ is a model for $\mathsf{Skl}(T_f)$ ($\mathsf{Skl}(T_e)$) where $\mathsf{SC}(T_1) \ldots \mathsf{SC}(T_n)$ are assigned the distinguished constants $a_1, \ldots, a_n$ and $\mathsf{SFC}(T_1) \ldots \mathsf{SFC}(T_n)$ are assigned functions F_j^i and set M is identical with the least set containing $a_1, \ldots a_n$ and closed with respect to F_j^i's.[26]

Theorem 2.25a *Every syntactically consistent elementary theory has a constructive model.*[27]

Proof: The axioms for an elementary theory either all begin with universal quantifiers or have at least one sentence which begins with an existential quantifier. If it begins with all universal quantifiers on the basis of the following theorem (which is provable from the logical axioms only):

$$(x)R(x) \to (\exists x)R(x)$$

it can be transformed into an axiom-system which has at least one sentence beginning with an existential quantifier and is inferentially equivalent to the original. Namely, if theory T_f or T_e is based on axioms $(x_i)A_1, \ldots, (x_k)A_n$ then the transformation of T_f or T_e is based on axioms $(x_i)A_1, \ldots, (x_k)A_n$, $(\exists x_k)A_n$. Therefore the theorem will be proved if the following is proved:

(1) *if A is any consistent set of sentences with predicate constants $P_i^j \ldots$ and if at least one sentence begins with an existential quantifier, then A has a constructive model.*

The model to be constructed is a model on the terms of $\mathrm{Skl}(A)$. To the sentence beginning with an existential quantifier add a Skolem constant a_1 and continue transforming it to form its Skolem equivalent. Let $\{a_1, \ldots, a_n\}$ be the added Skolem constants and $\{f_i^j\}$ be the added Skolem function constants to form $\mathrm{Skl}(A)$. Let T be the set of terms constructed according to Definition 2.2 from $\{a_1, \ldots, a_n\}$ and $\{f_i^j\}$. If there are constants and functions occurring in the original theory, they must be included in the generation of T.

Extend $\mathrm{Skl}(A)$ to a set Z by adding to it atomic sentences formed by the application of the predicate constants of $\mathrm{L}(A)$ to the terms of the set T. Well-order the set of generated atomic sentences. Define Z recursively:

a. $Z_0 = \mathrm{Skl}(A)$;

b. $Z_{n+1} = \begin{cases} Z_n \cup P_i^j(t_1, \ldots, t_n) \text{ if } Z_n \cup \neg P_i^j(t_t, \ldots, t_n) \\ \text{is syntactically inconsistent} \\ Z_n \cup \neg P_i^j(t_1, \ldots, t_n), \text{ otherwise;} \end{cases}$

c. $Z = \bigcup_{n \in N} Z_n$ where N is the set of natural numbers.

Define model $\mathfrak{M}$ for set Z as follows:

a. let $m_1, \ldots, m_n \in M$ be assigned to the added Skolem constants $a_1, \ldots, a_n$ respectively;
b. let $F_i^j, F_i^k, \ldots$ be assigned to the added Skolem function constants $\{f_i^j\}$ so that $F_i^j(m_1, \ldots, m_j)$ is assigned to $f_i^j(a_1, \ldots, a_j)$;
c. let $R_1, R_2, \ldots$ be assigned to the predicate constants P_i^j and define $(m_1, \ldots, m_j, \ldots) \in R_i$ if and only if $P_i^j(a_1, \ldots, a_i) \in Z$.

Show that $\mathfrak{M}$ is a model for $\mathrm{Skl}(A)$. First, if A_i is an atomic sentence then by point c. A_i is satisfied by $\mathfrak{M}$. Second, if A_i is of the form $\neg A_k$ where A_k is an atomic sentence, $\mathfrak{M}$ satisfies A_i by point c. and the construction of Z. If A_i is of the form $A_j \rightarrow A_k$, A_i will be satisfied since it verifies the atomic sentences of $\mathrm{Skl}(A)$. (This can be shown by a truth-table evaluation and the definition of $\rightarrow$ in $\mathfrak{M}$.) $\mathfrak{M}$ is a model of $\mathrm{Skl}(A)$ since the Skolem equivalent sentences are quantifier free.

If T_f or T_e is an elementary theory and $\mathfrak{M}$ is a model for $\mathrm{Skl}(T_f)$ or $\mathrm{Skl}(T_e)$ it is also a model for T_f or T_e in virtue of Theorem 2.22a or 2.22b, point (1). From Definition 2.24 it is seen that $\mathfrak{M}$ is, moreover, a constructive model.

Theorem 2.25b *Every syntactically consistent elementary theory with equality has a constructive normal model.*

Proof: It follows directly from Theorem 2.25a and well-known methods for "reducing" models to normal models through the introduction of equivalence classes.

4 Some Relations between Skolem Theories and Definability In the transition from a theory T_e to the Skolem form of the theory $\mathrm{Skl}(T_e)$ there are added to the original theory new elements, the Skolem functions. Are these totally new elements or can they be defined in the original theory? If the Skolem functions cannot be defined in the original theory, what does this tell us about this theory? Answers to some of these questions will be given in this chapter. Ultimately, these questions and their answers will be important in determining the necessary and sufficient conditions for the applicability of the predicate 'constructively categorical' to complete theories.

Definition 2.25 A function constant f_i^j is said to be definable in theory T_e from predicate constants $P_1^0, \ldots, P_j^i$ and function constants $f_1^0, \ldots, f_j^k$ iff the following two conditions hold:

(1) $(x_1) \ldots (x_i)(x_{i+1})(f_i^j(x_1, \ldots, x_i) = x_{i+1} \leftrightarrow F(P_1^0, \ldots P_j^i, f_1^0, \ldots f_j^k))$ is a provable sentence of T where F is a wff containing predicate constants $P_1^0, \ldots, P_j^i$, and function constants $f_1^0, \ldots, f_j^k$ and moreover $x_1, \ldots, x_i, x_{i+1}$ are the only free variables occurring in F;
(2) some condition for *totality* of the function holds, e.g.,

$$(x_1)(x_2) \ldots (x_n) \exists!(x_{n+1}) f^n(x_1, \ldots, x_n) = x_{n+1}$$

where '$\exists!$' means 'there exists one and only one . . . such that.' Obviously, it can be eliminated in terms of '$\exists x$' and a sentence about the uniqueness of x.

A function f_j^i defined on set A is *total* if the domain of f_j^i is the set of all i-tuples of A^i. Let the condition of totality, sentence 2.25 (2) be denoted by the letter $\mathbf{U}$.

Definition 2.25 is formulable in T_e but not in $\mathrm{Skl}(T_e)$. Sentence 2.25 (2) must be modified since it contains an existential quantifier. However, there are many ways an equivalent condition can be given. For example, if in $\mathrm{Skl}(T_e)$ there occurs the nowhere-defined function or the zero-function, let us denote it $\bar{f}$, then sentence 2.25 (2) can be formulated as follows:

$$(2') \neg(f_i^n(x_1, \ldots, x_n) = \bar{f}).$$

Definition 2.26 If sentence 2.25 (1) and sentence 2.25 (2) are not provable in T_e then function constant f_i^j is said to be *undefinable* from predicate constants $P_1^0, \ldots, P_j^i$ and function constants $f_1^0, \ldots, f_j^i$ in theory T_e.

Theorem 2.27 *If function constant f_i^j is not definable in theory T_e in terms of predicate constants $P_1^0, \ldots, P_1^k$ and function constants $f_1^0, \ldots, f_k^j$ of theory T_e, then function f_i^j is not definable in $\mathrm{Skl}(T_e)$ in terms of predicate constants $P_1^0, \ldots, P_1^k$ and function constants $f_1^0, \ldots f_k^j$ and added Skolem functional constants $f_{k+1}^0, \ldots, f_i^j$ of $\mathrm{Skl}(T_e)$.*

Proof: Suppose function f_i^j is not definable in theory T_e but is definable in $\mathsf{Skl}(T_e)$ in terms of $P_1^0, \ldots, P_l^k$ function constants $f_1^0, \ldots, f_k^j$ and added Skolem function constants $f_{k+1}^0, \ldots, f_l^j$. So, if f_i^j is definable in $\mathsf{Skl}(T_e)$ then both

$$f_i^j(x_1, \ldots, x_j) = x_{j+1} \leftrightarrow F(P_1^0, \ldots, P_l^k, f_1^0, \ldots, f_k^j, f_{k+1}^0, \ldots, f_l^j)$$

and sentence U concerning totality of the functions are theorems of $\mathsf{Skl}(T_e)$. From Theorem 2.22b it follows that

$$f_i^j(x_i, \ldots, x_j) = x_{j+1} \leftrightarrow F(P_1^0, \ldots, P_l^k, f_1^0, \ldots, f_k^j)$$

is a theorem of T_e and U' which is the analogate of U in T_e is a theorem of T_e. This constitutes a definition of f_i^j in terms of $P_1^0, \ldots, P_l^k$ and $f_k^0, \ldots, f_k^j$ contrary to assumption. Thus f_i^j is not definable in $\mathsf{Skl}(T_e)$.

Theorem 2.28 *Let theory* $\mathsf{Skl}(T_e)$ *with equality be based on predicates* $P_1^0, \ldots, P_n^j$ *and function constants* $f_1^0, \ldots, f_i^j$ *and Skolem function constants* $f_{i+1}^0, \ldots, f_k^j$. *If function constant* f^{P} *is not definable in* $\mathsf{Skl}(T_e)$ *in terms of* $P_1^0, \ldots, P_n^j, f_1^0, \ldots, f_i^j, f_{i+1}^0, \ldots, f_k^j$ *then there are two normal constructive models of* $\mathsf{Skl}(T_e)$

$$\mathfrak{M} = \langle M, P_1^0, \ldots, P_n^j, f_1^0, \ldots, f_i^j, f_{i+1}^0, \ldots, f_k^j, f^{\mathsf{P}} \rangle$$

and

$$\mathfrak{M}' = \langle M', P_1^{0\prime}, \ldots, P_n^{j\prime}, f_1^{0\prime}, \ldots, f_i^{j\prime}, f_{i+1}^{0}{}', \ldots, f_k^{j\prime}, f^{\mathsf{P}\prime} \rangle$$

such that $M = M'$, $P_1^0 = P_1^{0\prime}, \ldots, P_n^j = P_n^{j\prime}, f_1^0 = f_1^{0\prime}, \ldots, f_k^j = f_k^{j\prime}$ *and* $\neg(f^{\mathsf{P}} = f^{\mathsf{P}\prime})$.[28]

Proof: Let S be the following sentence:

$$f^{\mathsf{P}}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \leftrightarrow f^{\mathsf{P}\prime}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \mathbin{\&} \mathsf{U} \leftrightarrow \mathsf{U}'.$$

The two normal models of the theorem can be merged into a single structure $\mathfrak{M}^*$ where $M = M' = M^*$ (Note: the new model is normal) $P_1^0 = P_1^{0\prime} = P_1^{0*}, \ldots, P_n^j = P_n^{j\prime} = P_n^{j*}, f_1^0 = f_1^{0\prime} = f_1^{0*}, \ldots, f_k^j = f_k^{j\prime} = f_k^{j*}$ and containing f^{P} and $f^{\mathsf{P}\prime}$ (note again $f^{\mathsf{P}} \neq f^{\mathsf{P}\prime}$). That is

$$\mathfrak{M}^* = \langle M^*, P_1^{0*}, \ldots, P_n^{j*}, f_1^{0*}, \ldots, f_k^{j*}, f^{\mathsf{P}}, f^{\mathsf{P}\prime} \rangle.$$

Let A be a conjunction of the axioms of $\mathsf{Skl}(T_e)$, (in which, incidentally, f^{P} may occur and A' be a conjunction of the axioms of $\mathsf{Skl}(T_e)$ where every occurrence of f^{P} is replaced by $f^{\mathsf{P}\prime}$. Assume that f^{P} is not definable in $\mathsf{Skl}(T_e)$ and, moreover, that it is not possible to find the two models mentioned in the theorem; consequently it is not possible to find a model $\mathfrak{M}^*$.

$\mathfrak{M}^*$ is characterized by being a model for $\{A\} \cup \{A'\} \cup \{\neg S\}$. But since no such model exists, the set $\{A\} \cup \{A'\} \cup \{\neg S\}$ is inconsistent. Thus S must be derivable from the set $\{A\} \cup \{A'\}$. Thus

$$A \mathbin{\&} A' \rightarrow ((f^{\mathsf{P}}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \mathbin{\&} \mathsf{U}) \rightarrow (f^{\mathsf{P}\prime}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \mathbin{\&} \mathsf{U}'))$$

is a theorem. Also the following is consequently a theorem

$$(A \,\&\, f^{\mathsf{P}}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \,\&\, \mathsf{U}) \rightarrow (A' \rightarrow f^{\mathsf{P}'}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \,\&\, \mathsf{U}').$$

By Theorem 2.23b there exists a formula D of $\mathsf{Skl}(T_e)$ whose propositional variables occur both in $(A \,\&\, f^{\mathsf{P}}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \,\&\, \mathsf{U})$ and $(A' \rightarrow f^{\mathsf{P}'}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \,\&\, \mathsf{U}')$ such that $(A \rightarrow f^{\mathsf{P}}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \,\&\, \mathsf{U}) \rightarrow D$ and $D \rightarrow (A' \rightarrow f^{\mathsf{P}'}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \,\&\, \mathsf{U}')$. Since in the second formula $f^{\mathsf{P}'}$ may be replaced by f^{P} throughout, on the basis of S being a theorem (if $f^{\mathsf{P}'}$ occurs in a non-equational context the replacement follows directly from S; if it occurs in an equational context the replacement follows from S and the axiom of extensionality for equality)

$$(A \,\&\, f^{\mathsf{P}}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \,\&\, \mathsf{U}) \rightarrow D$$

and

$$D \rightarrow (A \,\&\, f^{\mathsf{P}}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \,\&\, \mathsf{U}).$$

Thus

$$A \rightarrow (D \leftrightarrow f^{\mathsf{P}}(x_1, \ldots, x_{\mathsf{P}}) = x_{\mathsf{P}+1} \,\&\, \mathsf{U})$$

is a thesis. Hence f^{P} is definable in $\mathsf{Skl}(T_e)$ by means of formula D, contrary to assumption. Thus $\mathfrak{M}^*$ must exist, and consequently $\mathfrak{M}$ and $\mathfrak{M}'$ with the required properties.

5 Constructive Categoricity For fullness of treatment it will be worthwhile to review the important definitions and theorems proven by Grzegorczyk in [9] in which the concept of constructive categoricity was introduced. The following definitions were given there.

Definition 2.29 The axiom system A is *constructively categorical* if any two of its constructive models are isomorphic.[29]

The predicate R_i is called *decidable* in a set of sentences if and only if for any terms $t_1, \ldots, t_k$ taken from the language of the sentences A the following condition holds:

$$R_i(t_1, \ldots, t_k) \in \mathsf{Cn}(A) \vee \neg R_i(t_1, \ldots, t_k) \in \mathsf{Cn}(A).$$[30]

Besides a number of examples of theories which are or are not constructively categorical (cited at various places in this thesis) Grzegorczyk proved the following important central theorems.

Theorem 2.31 *If A is any consistent set of sentences with extra-logical constants $R_1, \ldots, R_m$ and if at least one sentence belonging to A begins with an existential quantifier, then A has a constructive model, namely a model on terms of* $\mathsf{Skl}(A)$.[31]

Theorem 2.32 *The axiom system A is constructively categorical if and only if all its primitive predicates are decidable in the set* $\mathsf{Skl}(A)$.[32]

Theorem 2.32a *R. M. Robinson's arithmetic* **Q** *of the natural numbers has a constructively categorical axiom system.*

Theorem 2.32b *The theory of dense ordering with at least two elements is essentially non-categorical.*

The remainder of this chapter is concerned with a second characterization of the notion of constructive categoricity, the equivalence of this notion with that given by Grzegorczyk in Theorem 2.32, and the relation of constructive categoricity to other notions of categoricity.

6 Completeness and Constructive Categoricity

Theorem 2.33 *If theory T_e is syntactically complete and has finite models, then all its models are finite and have the same cardinality.*

Proof: If T_e has finite models and since T_e is based on the full predicate calculus with equality if there is not already in T_e an explicit statement as to exactly how many elements are in its domain, then a statement of the form

$$(x)(\exists x_1)(\exists x_2)\ \dots\ (\exists x_n)((x_1 \neq x_2)\ \&\ (x_1 \neq x_3)\ \&\ \dots\ \&\ (x_{n-1} \neq x_n)\\ \&\ (x = x_1 . \vee . x = x_2 . \vee . \ . \ . \vee . x = x_n))$$

can be formulated in the language of T_e. Such a sentence for some one n must be provable in T_e because it is syntactically complete and, moreover, for only one n since T_e is consistent. Consequently all finite models have the same cardinality n.

The above theorem does not work, however, if syntactic completeness is changed to semantic completeness. Even though there will be such a sentence provable in T_e its exact form will depend on the model with respect to which T_e is complete. A quite obvious remark which should however be made at this point is that for syntactically complete theories with equality all constructive models have but one cardinality, either one given finite cardinality or cardinality $\aleph_0$.

Theorem 2.34 *If theory T_e is syntactically complete and the Skolem function constants added to T_e to form* $\mathrm{Skl}(T_e)$ *are definable in T_e then T_e is constructively categorical.*

Proof: To show that T_e is constructively categorical it must be shown that for any two constructive models of T_e, $\mathfrak{M}$ and $\mathfrak{N}$, such that

$$\mathfrak{M} = \langle M, f_1^0, \dots, f_j^i, f_{j+1}^{i+1}, \dots, f_k^j, R_1^0, \dots, R_j^i \rangle$$

and

$$\mathfrak{N} = \langle N, f_1^{0\prime}, \dots, f_j^{i\prime}, f_{j+1}^{i+1\prime}, \dots, f_k^{j\prime}, R_1^{0\prime}, \dots R_j^{i\prime} \rangle$$

where f_{j+1}^{i+1} (resp. $f_{j+1}^{i+1\prime}$) . . . f_k^j (resp. $f_k^{j\prime}$) are interpretations of the Skolem function constants there is a function φ such that

a. φ maps M 1-1 and onto N;
b. $\varphi(f_m^n(a_1, \dots, a_n, \dots)) = \varphi(f_m^n)(\varphi(a_1), \dots, \varphi(a_n), \dots)$ for $1 \leq j$ and $m \leq k$;

and

c. $\varphi(R_m^n(a_1, \dots, a_n, \dots)) = \varphi(R_m^n)(\varphi(a_1), \dots, \varphi(a_n), \dots)$.

Since the added Skolem function constants are definable in T_e condition

b. above reduces to the following:

b$'$. $\varphi(f_m^i(a_1, \ldots, a_i, \ldots)) = \varphi(f_m^i)(\varphi(a_1), \ldots, \varphi(a_i), \ldots)$ for $1 \leqslant i,\ m \leqslant j$.

Suppose, however, φ does not exist. Then either conditions a., b$'$., or c. or all are violated. Concerning condition a., by Theorem 2.33 and the supposition of syntactical completeness, $\mathfrak{M}$ and $\mathfrak{N}$ are of some one definite finite cardinality or since $\mathfrak{M}$ and $\mathfrak{N}$ are constructive, they are of power $\aleph_0$. In either case, a 1-1 map can always be found.

On the other hand, if condition b$'$. or c. are violated, then there must be some sentence S of T_e such that S is true in one model and not in the other. Consequently, S is independent of T_e and T_e is not syntactically complete, against our assumption.

Theorem 2.35 *If theory T_e having no finite models is complete and the Skolem function constants added to T_e to form* $\mathsf{Skl}(T_e)$ *are definable in T_e then T_e is constructively categorical.*

Proof: Similar to Theorem 2.34 with two exceptions:
(1) considerations of cardinality other than $\aleph_0$ are not necessary;
(2) models $\mathfrak{M}$ and $\mathfrak{N}$ do not verify the same atomic sentences and thus T_e cannot be complete since it is consistent.

The above theorem is mentioned by Grzegorczyk but not proved.

Theorem 2.36 *If theory T_e is constructively categorical then the Skolem function constants added to T_e to form* $\mathsf{Skl}(T_e)$ *are definable in T_e in terms of the predicates $R_1^0, \ldots R_j^i$ and function constants $f_1^0, \ldots, f_k^i$ of theory T_e.*

Proof: Suppose theory T_e is constructively categorical and some Skolem function constant, say $f_j^{i\prime}$ is not definable in terms of the predicate $R_1^0, \ldots, R_j^i$ and function constants $f_1^0, \ldots, f_k^i$. Then by Theorem 2.28 there exist two constructive models

$$\mathfrak{M} = \langle M, R_1^0, \ldots, R_j^i, f_1^0, \ldots, f_k^i, f_j^{i\prime}\rangle$$
$$\mathfrak{N} = \langle N, R_1^0, \ldots, R_j^i, f_1^0, \ldots, f_k^i, f_j^{i\prime\prime}\rangle$$

such that

$$\neg(f_j^{i\prime} = f_j^{i\prime\prime}).$$

Thus, $\mathfrak{M}$ and $\mathfrak{N}$ cannot be isomorphic and theory T_e, contrary to assumption, cannot be constructively categorical.

The above theorems are summarized in the following characterization of complete categorical theories. Theorem 2.38 was conjectured by Grzegorczyk but not proved.

Theorem 2.37 *If theory T_e is syntactically complete then T_e is constructively categorical if and only if the Skolem function constants added to T_e to form* $\mathsf{Skl}(T_e)$ *are definable in T_e.*

Proof: Follows directly from Theorem 2.34 and Theorem 2.36.

Theorem 2.38 *If theory T_e having no finite models is complete then T_e is constructively categorical if and only if the Skolem function constants added to T_e to form* $\mathrm{Skl}(T_e)$ *are definable in T_e.*

Proof: Follows directly from Theorem 2.35 and Theorem 2.36.

If it could be proved that constructive categoricity implies semantic completeness, then on the strength of Theorem 2.37 and 2.38 completeness and definability would be necessary and sufficient conditions for constructive categoricity. However, this implication is in general not true. It is not a necessary condition. This follows from two facts: (1) Grzegorczyk's proof of the constructive categoricity of R. M. Robinson's arithmetic of natural numbers, Theorem 2.32a; and (2) incompleteness of R. M. Robinson's arithmetic of natural numbers. Also, as can be seen from Grzegorczyk's theorem, Theorem 2.32b, completeness is not a sufficient condition for constructive categoricity. Thus, unlike categoricity in power which is directly related to completeness,[33] completeness is neither a necessary nor a sufficient condition for constructive categoricity.

It appears also that the same conditions obtain with respect to syntactic completeness. If one were to attempt to prove the sentence, "if theory T_e is constructively categorical then it is syntactically complete," the most obvious way is to assume the hypothesis is true but the conclusion is false. Suppose T_e is syntactically incomplete, then there is a sentence S which is independent of T_e. However, to arrive at a contradiction one must be able to produce two constructive models $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{M}$ is a constructive model for the axioms of T_e and S while $\mathfrak{N}$ is a constructive model for the axioms of T_e and $\neg S$. In general, this probably is not possible.

For the sake of the completeness of the treatment of the notion of constructive categoricity on the field of complete theories with equality, the following theorems prove the equivalence of Grzegorczyk's characterization of constructive categoricity (*cf.*, Theorem 2.31) and the one given in this paper.

Theorem 2.39 *If all the primitive predicates of the language of theory T_e are decidable in* $\mathrm{Skl}(T_e)$ *then the added Skolem functions used to transform T_e into* $\mathrm{Skl}(T_e)$ *are definable in T_e.*

Proof: Suppose at least one of the Skolem functions f_i^0 added to T_e to form $\mathrm{Skl}(T_e)$ is not definable in T_e. Then, by using the argument of Theorem 2.28 the existence of two models for $\mathrm{Skl}(T_e)$ $\mathfrak{M}$ and $\mathfrak{N}$ can be shown such that $f_i^0 \in \mathfrak{M}$ and $f_i^{0\prime} \in \mathfrak{N}$ and $\neg(f_i^0 = f_i^{0\prime})$. Since we may suppose for the purposes of the theorem that the other Skolem functions of T_e are definable in T_e then it is obvious that $\mathfrak{M}$ and $\mathfrak{N}$ are two normal models for $\mathrm{Skl}(T_e)$. Since, however, they are non-isomorphic T_e cannot be constructively categorical. Looking at the construction of models $\mathfrak{M}$ and $\mathfrak{N}$, for some primitive predicate which has f_i^0 as an argument (there must be one or f_i^0 would not appear in the constructive model) that primitive predicate cannot be decidable, at some point in the range of f_i^0 where $f_i^0 \neq f_i^{0\prime}$, contrary to assumption. Therefore, the theorem is proved.

Theorem 2.40 *If theory T_e is syntactically complete and the functions added to T_e to form* $\mathsf{Skl}(T_e)$ *are definable in T_e then the primitive predicates of T_e are decidable in* $\mathsf{Skl}(T_e)$.

Proof: Suppose the hypothesis true but the conclusion false. If the primitive predicates of T_e are not decidable, then for some primitive predicate P and terms $t_1, \ldots, t_n$ of $\mathsf{Skl}(T_e)$ neither $P(t_1, \ldots, t_n) \in \mathsf{Cn}(\mathsf{Skl}(T_e))$ nor $\neg P(t_1, \ldots, t_n) \in \mathsf{Cn}(\mathsf{Skl}(T_e))$. Consequently the sets $\{\mathsf{Skl}(T_e) \cup P(t_1, \ldots, t_n)\}$ and $\{\mathsf{Skl}(T_e) \cup \neg P(t_1, \ldots, t_n)\}$ are consistent and possess models $\mathfrak{N}$ and $\mathfrak{N}'$, respectively. But $\mathfrak{N}$ and $\mathfrak{N}'$ are models for theory T_e by Theorem 2.22a. However, since the added Skolem functions are definable in T_e sentences $P(t_1, \ldots, t_n)$ and $\neg P(t_1, \ldots, t_n)$ are expressible in T_e. Thus, on the basis of Padoa's method and models $\mathfrak{N}$ and $\mathfrak{N}'$ of T_e, theory T_e cannot be syntactically complete which contradicts our assumption of syntactic completeness of T_e. This proves the theorem.

Theorem 2.41 *If theory T_e possessing no finite models is complete and the functions added to T_e to form* $\mathsf{Skl}(T_e)$ *are definable in T_e then the primitive predicates of T_e are decidable in* $\mathsf{Skl}(T_e)$.

Proof: Similar to 2.40.

Theorem 2.42 *If theory T_e is syntactically complete, then the primitive predicates of T_e are decidable in* $\mathsf{Skl}(T_e)$ *if and only if the Skolem functions added to T_e to form* $\mathsf{Skl}(T_e)$ *are definable in T_e.*

Proof: Theorem 2.39 and Theorem 2.40.

Theorem 2.43 *If theory T_e possessing no finite models is complete, then the primitive predicates of T_e are decidable in* $\mathsf{Skl}(T_e)$ *if and only if the Skolem functions added to T_e to form* $\mathsf{Skl}(T_e)$ *are definable in T_e.*

Proof: Theorem 2.39 and Theorem 2.41.

7 Categoricity in Power and Constructive Categoricity This section will briefly state the relationship between the notions of constructive categoricity and categoricity in power $\aleph_0$. It is easily stated on the basis of previous theorems.

Theorem 2.44 *If theory T_e having no finite models is categorical in power $\aleph_0$ then theory T_e is constructively categorical if and only if the Skolem functions added to T_e to form* $\mathsf{Skl}(T_e)$ *are definable in T_e.*

Proof: Łoś' theorem on completeness[33] and Theorem 2.38.

The notions of categoricity in power and of constructive categoricity though related are not subsets of each other. This is easily shown by means of examples. First, categoricity in power $\aleph_0$ is not a subset of constructive categoricity. Grzegorczyk shows[34] that the theory of dense ordering without beginning or end is categorical in power but is not constructively categorical. Secondly, constructive categoricity is not a subset of categoricity in power $\aleph_0$. Robinson's arithmetic of the natural numbers is as proved by Grzegorczyk[35] constructively categorical but not categorical in power $\aleph_0$. It is well known that Robinson's arithmetic is essentially incomplete. Consequently on the basis of Łoś' theorem it cannot be categorical in power, *a fortiori*, it cannot be categorical in power $\aleph_0$.

CHAPTER III

There are, implicit in the preceding chapter, consequences concerning the structures which are constructively categorical. They are nevertheless of interest in themselves and will be explicated in this chapter. Specifically, this chapter will contain a description of syntactic and semantic features, peculiar to constructively categorical theories, and the formal consequences of these properties. Then the initial purpose of Grzegorczyk's introduction of the concept of constructive categoricity will be reviewed with some accompanying criticism. Finally a list of some unsolved problems yet remaining with respect to this logical notion will be listed in addition to some observations concerning them.

1 Some Syntactic and Semantic Features of Constructively Categorical Theories Referring to Definition 2.25 it should be observed that what is there defined is a total function, which is not the usual definition of function, but which can be seen as necessary by reviewing the definition of constructive categoricity, Definition 2.29 and Definition 2.9. Obviously, if the Skolem functions are not total then they may be interpreted in one model as having certain but arbitrary values at the points of non-totality and in the other model as having arbitrary but different values so that the models cannot be isomorphic. If the Skolem functions are not total, the theory in general will not be categorical. What does this tell us about the syntactic properties of constructive categorical theories? Simply, that it must be meaningful to apply every predicate of such a theory to every individual contained in the universe discussed by that theory. This point is perhaps seen a little more easily by reviewing Theorem 2.31 and Definition 2.29 where Grzegorczyk proves that theory T is constructively categorical if and only if all the primitive predicates of T are decidable in $\mathsf{Skl}(T)$.

Thus for any predicate P and any individual i of a constructively categorical theory it must be possible to express syntactically and derive from the axioms whether P applies at i or not. This property of "universality" of the predicates is a strong condition. It should not be thought, however, that in constructively categorical theories it is not possible to distinguish subclasses of elements and talk about properties of such a subclass but it must also be possible to say syntactically just which elements do not belong to that subclass and, moreover, that the properties of the subclass elements do not belong to them. As restrictive as this might seem, it is certainly consonant with some of the original intuitions behind the notion of categoricity.

Turning now to the semantic peculiarities of constructively categorical theories there are two of particular note: (1) every syntactic existential element has a *unique* semantic correlate, and (2) the cardinality of any constructive model is at most a countable infinity.

Factor 1 is very important because it specifically differentiates Skolem functions as used in conjunction with constructive theories from Hilbert's epsilon.[36] It is this same factor which makes these models constructive. In conjunction with these two points Skolem functions herein used are definitely weaker than the axiom of choice in contrast to Hilbert's epsilon which, under certain conditions, is equivalent to the axiom of choice. More on this point later.

Returning now to the statement that every syntactic element has a unique semantic correlate, what does the term "unique" mean and how is this statement proved? With respect to constructive models "unique" means one and only one, i.e., if $\mathfrak{M}$ is a constructive model for a theory T then $\mathfrak{M}$ contains one and only one element t for each existential statement formulable in T. Contrast this with non-constructive models when an existential statement can often be correlated with more than one element of the model. With respect to constructively categorical theories, then, "unique" means identical up to isomorphism.

In order to prove Factor 1 a recap of how existential sentences are resolved using Skolem functions will be helpful:

1. any sentence with an initially placed existential quantifier has that quantified variable replaced by a Skolem constant—a zero-placed Skolem function and with the important proviso that this Skolem function be a "new" function, i.e., not a function used in resolving any previous sentence of T with initially placed existential quantifiers. Thus any axiom or sentence of T with an initially placed existential quantifier has a unique semantic correlate;

2. replace all non-initial occurrences of existential quantifiers with Skolem functions whose arguments are all the universally quantified variables preceding the existential variables, e.g.,

$$(x)(\exists y) \ldots y \ldots$$

by

$$(x) \ldots f(x) \ldots$$

again with the proviso that this Skolem function be a "new" function. However, at first sight $f(x)$ appears to be at most a parameterized name and not unique. But x ranges over the unique names of model $\mathfrak{M}$ for theory T and generates a unique name at every point in $\mathfrak{M}$. Obviously this is because the Skolem functions are single valued.

This is a very important point and the reason why constructive models are so different from other models. Let this point be reemphasized by stating it in another way. Assume that

$$S = (x)(\exists y) P(xy)$$

is a sentence of theory T. Let $\mathfrak{M}$ be a model of T and $\mathfrak{N}$ be a constructive model for T. For S to be satisfied in $\mathfrak{M}$ all that is required is that for any pair $\langle m_1, m_2\rangle$, $m_1, m_2 \epsilon M$ of $\mathfrak{M}$ is that there is a pair p of elements of M which differ from $\langle m_1, m_2\rangle$ in at most the second place such that $p \epsilon P'$ of $\mathfrak{M}$ where P' is the interpretation of predicate P in model $\mathfrak{M}$. However, note, that there may be another pair p' of elements of M such that $p = \langle m_1, m_2\rangle$ and $p' = \langle m_1, m_3\rangle$ and such that $p \epsilon P'$ and $p' \epsilon P'$ nevertheless $m_2 \neq m_3$. In contradistinction to this in constructive model $\mathfrak{N}$ the conditions for satisfaction remain the same as for $\mathfrak{M}$ yet, let $n_1, n_2, n_3 \epsilon N$ of $\mathfrak{N}$ and $p = \langle n_1, n_2\rangle$ and $p' = \langle n_1, n_3\rangle$ and if $p \epsilon P'$ and $p' \epsilon P'$ then $n_2 = n_3$ where P' is the semantic correlate of predicate P in $\mathfrak{N}$.

The proof that Factor 2 concerning cardinality is in fact the case is based on the following assertions:

1. the number of sentences of the formal language $\mathcal{L}$ is at most a denumerable infinity;
2. the substitution in such sentences of at most a denumerable infinity of Skolem functions yields at most a denumerable infinity of resolved sentences;
3. the Skolem functions in each sentence range over at most a denumerable infinity of previously generated points of the constructive model.

The total number of sentences resulting from the process yields at most $\aleph_0 \times \aleph_0$ sentences, i.e., a set of sentences of at most $\aleph_0$.

2 Skolem Functions and Hilbert's Epsilon In [11] Hilbert and Bernays introduce a logical operator, ϵ, which is referred to as Hilbert's epsilon. The epsilon operator was introduced in order to aid in the reduction of a predicate calculus to a free variable calculus. The rules for use of the epsilon to accomplish such a reduction are exactly analogous to the use of Skolem functions, as given in Definition 2.20, except where Skolem constants appear the symbol $\epsilon_x S(x)$ would occur, ($S(x)$ is a formula in prenex normal form with initially placed existential quantifier $\exists x$), and $f(y, z, \ldots)$ would occur as an abbreviation for $\epsilon_x S(x, y, z, \ldots)$. What is the difference then between Hilbert's epsilon and Skolem functions?

Quite obviously, the intent of both operators is the symbolic resolution of sentences of predicate calculus in order to eliminate existential quantifiers. However, the levels at which this intention was carried out were completely different. Skolem functions appear to be used where semantics is of prime importance. They were introduced in conjunction with the determination of the conditions under which existential sentences could be satisfied in a model.[37] Hilbert's epsilon is used in cases where syntax is of prime concern. The famous Hilbert epsilon theorems[38] are of a purely syntactic nature.

The first and most important difference is the method of introduction of the operators into the theory. Hilbert's epsilon is introduced into a theory through the use of a special axiom:

$$F(y) \rightarrow F(\epsilon_x F(x))$$

where y is a free variable. Skolem functions require no special axiom and are introduced with the express intent of model construction by the now familiar process of letting the theorems of the theory define the properties of the model (i.e., letting syntactic objects name themselves as semantic objects).

Secondly, the Hilbert epsilon can be considered as a formalization of the indefinite description, which is a phase of the form 'a so-and-so' (contrasted to Russell's famous definition description—'the so-and-so'). The Skolem functions are not designed to be such a formalization and it would be stretching the point quite a bit to make it so. What makes this so, and this brings up the third point, is that Hilbert's epsilon functions chiefly as a selection or choice operator. The axiom given by Hilbert is interpreted as follows: if some constant satisfies F then $\epsilon_x F(x)$ denotes some object, not specified, which satisfies F. Modern practitioners of the epsilon, Asser and Hermes, continue in this tradition. Until recently, the epsilon had not been investigated in itself. Now predicate calculi have been built upon the epsilon and the properties of these calculi determined. Asser[39] interprets the epsilon:

> "Finally, the sign ϵ is a variable for the choice-function of the domain of individuals J. Thus a choice-function of J is a representation ϕ which assigns to each non-empty subset of J a uniquely determined element of this subclass . . ." Hermes in his paper gives the following semantics: "(3.10) ν is a choice operator over ω. For every non-empty subset ρ of ω, $\nu(\rho)$ is an element of ρ."[40]

As was pointed out previously, Skolem functions do not have to be total functions. Moreover, the semantics of these functions is such that they may be defined either over the entire set of individuals I (as is required for constructive categoricity) or over some subset of this domain but it is only required that their range of values be in the set of individuals. They may but do not have to function as a choice operator: the intersection of the range of values of a Skolem function and its domain of definition can be null. This is the most fundamental difference between Hilbert's epsilon and Skolem functions. As a consequence of this ability of Hilbert's epsilon to function as a choice function, we are led back to our original point that under certain conditions Hilbert's epsilon is equivalent to the axiom of choice. Why is it said "under certain conditions"? In their work on set theory Fraenkel and Bar-Hillel remark:

> There clearly exists a close connection between the ϵ-formula and the axiom of choice. This connection should not be overstated, as it is occasionally done, in the form that the ϵ-formula is a kind of logical (or generalized) axiom of choice. Indeed, the ϵ-formula allows for a singleton selection only, while the axiom of choice allows for a simultaneous selection from each member of an (infinite) set of sets, and guarantees the existence of the set comprising the selected entities. There is no reason to suppose that in a set theory

> constructed on the basis of an ϵ-calculus the principle of choice would become generally derivable, unless the specific axioms of set theory contain ϵ-terms themselves.[41]

However, for just such a construction of set theory see Bourbaki.[42]

Skolem functions as used herein are limited to elementary theories and cannot be used to construct a logical equivalent of the axiom of choice. If not restricted to elementary theories then they still suffer the same defect as noted above, the intersection of their ranges and domains can be null, and would not in general act as a choice operator.

It might be of some interest to contrast the development of a concept of constructive categoricity based on Hilbert's epsilon limited to elementary theories versus the one based on Skolem functions. The most obvious benefit would be that those "functions" introduced in the resolution of existential sentences would not have to be total because if one used that which Asser calls "ϵ-calculus of the second kind" the places of non-totality are arbitrarily defined to have the null object as a value. To gain this advantage which is considerable due to its naturalness means however that the following must hold. First, the calculus must be able to talk about the null object. Second, the axioms must be strong enough so that the predicates can be interpreted as only applying to some one definite cardinality of objects (contrasted to constructive categoricity where the primitive predicates must be "universal").

On the other hand, a constructive categoricity based on Hilbert's epsilon has the distinct disadvantage of having to be artificially limited to models of cardinality of $\aleph_0$ or less if some semblance of constructivity is to be maintained. (Contrast this to constructive models based on Skolem functions which are limited to at most $\aleph_0$ by the process of construction.) However, in the future it might be worth pursuing a study of constructivity based on Hilbert's epsilon.

3 Critique of the Notion of Constructive Categoricity The major thrust of the critique revolves around the fact proven in Chapter II, section **4**, that completeness is neither a necessary nor a sufficient condition for constructive categoricity. First, it was questioned near the end of Chapter I as to the appropriateness of applying the notion of categoricity, no matter which type, to theories which are incomplete. That question remains. Secondly, there is the following rather obvious theorem:

Theorem 3.1 *If theory T_e is constructively categorical then* $\mathsf{Skl}(T_e)$ *is complete with respect to its constructive models.*

Proof: If $\mathsf{Skl}(T_e)$ is not complete with respect to constructive model $\mathfrak{M}$ then there is a sentence S of $\mathsf{Skl}(T_e)$ such that S is satisfied in $\mathfrak{M}$ but such that $S \notin \mathsf{Cn}(\mathsf{Skl}(T_e))$ and $\neg S \notin \mathsf{Cn}(\mathsf{Skl}(T_e))$. The symbol '$\mathsf{Cn}$' means 'the consequences of,' i.e., $\mathsf{Cn}(\mathsf{Skl}(T_e))$ is the least set of sentences S containing $\mathsf{Skl}(T_e)$ and closed under the rule of detachment. Since $\mathsf{Skl}(T_e)$ is a propositional calculus with some added extralogical axioms, this can only happen if some atomic part Y of S is such that $Y \notin \mathsf{Cn}(\mathsf{Skl}(T_e))$ and

$\neg Y \notin \mathrm{Cn}(\mathrm{Skl}(T_e))$. Thus for some primitive predicate P of T_e, P is not decidable in $\mathrm{Skl}(T_e)$. Consequently, by Theorem 2.31, T_e is not constructively categorical contrary to assumption.

Consider now that Grzegorczyk proves in his paper that R. M. Robinson's arithmetic of the natural numbers is constructively categorical. He annotates his proof as follows:

> It is worth recalling that this arithmetic is undecidable, and hence also incomplete. It is even known that very simple general theorems, e.g.
>
> $$(x)(y)(x + y = y + x)$$
>
> are independent of these axioms. Yet this axiom determines exactly one constructive model. It is, of course, *the classical model consisting of the natural numbers.*[43]

In what sense is this constructive model the classical model consisting of the natural numbers? First, it is known that this axiom system is essentially incomplete. To say that it is incomplete means that it is incomplete with respect to some model. It happens that it is incomplete with respect to this classical model. Thus the constructive model and the classical model are not identical because the axioms are not incomplete with respect to the constructive model which they determine.

Just suppose for a moment that the constructive and the classical models are identical. It is known that Robinson's arithmetic is ω-incomplete. Are we then justified in saying that Robinson's arithmetic determines the classical model of the natural numbers? The statement cited by Grzegorczyk, $(x)(y)(x + y = y + x)$, is a valid statement about the natural numbers. In Robinson's arithmetic statements such as $2 + 5 = 5 + 2$ are provable but are these of scientific interest other than perhaps computability? Yet commutativity of addition for the natural numbers is an important and interesting property!

Theorems 2.40-2.43 and Theorem 3.1 suggest that the notion of constructive categoricity only yields interesting information when applied to complete theories. It should, perhaps, be considered as an extension of the notion of categoricity in power $\aleph_0$. However, the study of constructive categoricity should be extended to models of arbitrary power whose domain of individuals is ordered according to some ordinal type. Then the notion of constructive categoricity could be considered *in general* to be a refinement of the notion of categoricity in power.

4 Unsolved Problems There are some unsolved problems or further directions of research that remain from Grzegorczyk's paper or have arisen in the course of this thesis. They will be listed here along with some hints to their solution. Problem 1 was formulated by Grzegorczyk; all others are formulated by the author of this thesis, as are all hints.

Problem 1. ". . . has a constructive model of a constructively categorical axiom system no automorphisms (other than identity), if we disregard the

Skolem functions, for together with them it obviously cannot have any automorphisms. . ."[44]

It seems that one way to approach this problem would be to investigate the Lindenbaum-Tarski algebras generated by the Skolem form of the axiom system and then use the results concerning automorphisms of Boolean algebras[45] to attack the problem. Since the Skolem form of the theory is a 'syntactic copy' of the constructive models the results concerning the algebras then carry over to the models. One difficulty, however, will be a guarantee of the existence of Boolean meets and joins over a countable infinity of objects.

Problem 2. This is a direction for further research which has been mentioned several times already. It would be interesting to develop a theory of constructive categoricity based on models of some ordinal type $\alpha, \omega < \alpha$. Since the syntax and models are correlated in the process of generating the models the syntax for such theories would have to be capable of dealing with infinitely long expressions (*viz.* K. Karp, *Theories with Expressions of Infinite Length,* North Holland Publishing Co., 1964).

Problem 3. This follows upon Problem 2. If such a theory can be developed, then if theory T_e is constructively categorical for models of ordinal type α, $\omega < \alpha$ is T_e constructively categorical for ordinal type β, $\beta < \alpha$? Is T constructively categorical for ordinal type γ, $\alpha < \gamma$?

Problem 4. The results given in Theorems 2.40-2.43 are based on theories which have equational definitions as the mode or method of definition. For complete theories, would definability remain a necessary and sufficient condition for constructive categoricity if the method of definition were biconditional implication instead of equational? Seeing how very dependent constructive categoricity was upon the behavior of the individuals of the Skolem form of the theory, it would appear that if biconditional implication were the mode of definition, definability would not be a necessary and sufficient condition. This, however, remains to be proven.

APPENDIX

Is the concept of constructive categoricity an extensional concept or not? On page 49 of his paper, *cf.* [9], Grzegorczyk says, "As a possible drawback of this concept of categoricity, we ought to mention that constructive categoricity is not an extensional property." Yet no examples are forthcoming to illustrate this. Theorem 2.32a indicates that Robinson's arithmetic **Q** has a constructively categorical axiom system. This is produced by Grzegorczyk, *cf.* [9], p. 53, but this is not equivalent to the originally given axiom system on p. 52. In the former, one can prove $P(0) = 0$ whereas in the latter one cannot. Are these two axiom systems equivalent? The word 'equivalent' is nowhere made precise enough.

To put it in a more precise context, use the term syntactically equivalent and syntactically extensional as given in Definition 2.10 and Definition 2.12a. Then on the basis of Theorem 2.37 and Definition 2.10 the following theorem results:

Theorem. *For syntactically complete theories the concept of constructive categoricity is syntactically extensional.*

NOTES

1. The most immediate example of representation theorems is Stone's representation theorem for Boolean algebras. See [17], pp. 23, 97, 115, 117. Examples of representation theorems as applied specifically to logic, see pp. 194 ff, 198 ff.
2. See [7], p. 92.
3. [7], p. 93.
4. [7], pp. 95-96. Italics are mine.
5. See [12], p. 264. It is interesting to note how much the terminology Huntington uses to describe postulates has developed in the fifteen years since Dedekind. It is probably due to Peano's influence, whom Huntington cites in his bibliography.
6. [12], p. 277.
7. [12], p. 278.
8. See [21], p. 346.
9. [21], p. 346. Why Dewey suggested the terms categorical and disjunctive is not immediately apparent from his logical works and remains at present a mildly interesting open problem.
10. [21], p. 383.
11. See [12], p. 264. Italics are mine.
12. See [21], pp. 346-347.
13. See [18], p. 712. The idea of arithmetical equivalence is closer to the intuitive idea of 'same mathematical domain' than is categoricity.
14. See [14], p. 59.
15. See [19], p. 310.
16. [19], p. 313.
17. See [5], p. 149. This terminology is very similar to Tarski's. *Cf.* notes 14 and 15. The following is a free translation of this text by the author:

    An AS (axiom system) is called monomorphic (or categorical) if it is consistent and all its models are pairwise isomorphic. The concept of the isomorphism of models is related to the previously defined concept of isomorphism of classes or relations (19). Model $\mathfrak{M}$ depends on the relations (or extensions) $B_1, B_2, \ldots, B_n$ of the finitely axiomatizable foundations of system S; another model $\mathfrak{M}'$ depends on relations $B_1', \ldots, B_n'$. $\mathfrak{M}$ is said to be isomorphic to $\mathfrak{M}'$ if there is a mapping between the individuals of $\mathfrak{M}$ and of $\mathfrak{M}'$ such that each B_p ($p = 1$ to n) is isomorphic (in the previously defined sense) to B'_p on the basis of this mapping. If these models of AS are not isomorphic then AS is said to be polymorphic. If an AS is monomorphic, consequently complete, then it determines all structural properties of any possible model.

18. See [14], p. 58.

19. [14], p. 60.

20. See [20], pp. 396-397. Working independently Łoś and Vaught came up with the same result. Vaught started from Tarski's notion of arithmetical class (see note 13). But when you add the notion of cardinality to the notion of arithmetical equivalence the resulting notion is categoricity in power.

21. See [3], p. 69.

22. See [9], p. 55.

23. See Chapter III, section **4**.

24. See [9], pp. 47-48, Definition 5.1.

25. Variants of condition (2) of this theorem are known in the literature of logic, e.g., [16], p. 55. However, by proving condition (1), condition (2) becomes an easy matter.

26. The essence of this definition is due to Grzegorczyk. See [9], p. 48, definition 5.3. However, it has to be slightly modified to use the terminology adopted in this thesis.

27. This is a modification and an improvement of a theorem of Grzegorczyk. See [9], p. 49, Theorem 5.5. The model $\mathfrak{M}$ is more general than that constructed by Grzegorczyk.

28. Essence of this proof is due to E. W. Beth. See [2]. It has been modified for Skolem theories with their different rules of procedure and the definition of functions as being total, which is necessary for constructive categoricity.

29. See [9], p. 48, Definition 5.4.

30. See [9], p. 51. The symbol 'Cn' stands for 'consequence of' e.g., $\mathsf{Cn}(A)$ is the least set containing A and closed under the rules of procedure.

31. See [9], p. 49, Theorem 5.5.

32. See [9], p. 51, Theorem 5.7.

33. This fact is proved by Łoś [14], p. 60.

34. See [9], p. 51.

35. See [9], pp. 52-54, Theorem 5.8.

36. See [11], p. 18ff.

37. See pp. 224-233 of [6] for a detailed explanation of Skolem normal form for satisfiability.

38. See [11], p. 18 and p. 79.

39. See [1]. Translation of German text is mine.

40. See [10]. Translation of German text is mine.

41. See [8], pp. 183-185.

42. In [4], the τ operator functions as Hilbert's epsilon does.

43. See [9], pp. 53-54.

44. See [9], p. 56.

45. See [17].

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