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## THE EXTENDED CALCULUS OF INDICATIONS INTERPRETED AS A THREE-VALUED LOGIC

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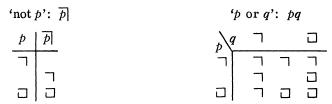
1 Introduction The point of view of indication, as a foundational notion for mathematical thinking, was introduced by G. Spencer Brown in 1969 [1]. Taking as primitive only the intuitive notion of distinction or indication, he presented a simple yet amazingly powerful calculus, the Calculus of Indications (CI), whose full import is slowly being recognized [2]. In its two values, indicated, '7', and not indicated, ',', this calculus embodies the general form of any two-valued situation. Many possible interpretations of CI are thus possible, but a particularly interesting one is for classical propositional logic, where statements can be true or false (cf. [1], Appendix 2). I have taken the Calculus of Indications as a starting point in an attempt to produce adequate tools to deal with self-referential situations.\* Selfreference is, of course, of great historical importance; it was responsible for a major crisis in mathematical thinking at the turn of the century. More recently, with the development of cybernetics and systems theory, other aspects of self-referential situations have become apparent, namely, the fact that many highly relevant systems have a self-referential organization. The key character of self-production in living systems is, perhaps, the most obvious instance; examples from the neurological, cognitive, and social domains also abound [3,4,5,6]. With his motivation I developed an Extended Calculus of Indications (ECI), capable of dealing with the basic forms of self-reference, and thus, providing a foundation to interpret any possible instance of them [7]. The point of view of indication greatly simplifies the discussion of self-referential situations, by simply having an expression indicate itself. Expressions where self-indication is allowed, are called boolean expressions of higher degree by Spencer Brown, and in his [1] he hinted at their possible applications. In [7] I showed that CI is not consistent with self-indicating expressions and derived ECI, where not two but three values exist: indicated, not indicated, and self-referring or

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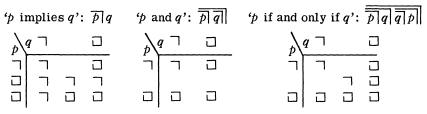
autonomous, ''\_'. It can be showed that, although all self-referring forms are allowed in ECI, their diversity can essentially be reduced to the atomic case of the autonomous value.

These introductory statements do not attempt to be a recapitulation of all these results, but only a background motivation for the rest of this work. I will not discuss here the varieties of self-referential forms that can be accomodated in ECI (i.e. expressions of higher degree) and their interpretation in a propositional logic (cf. Section III.B). My intention here, is only to interpret the values of ECI as truth values and thus to produce a three-valued logic from ECI. The interest of such three-valued logic is twofold. It is on the one hand theoretical, since it introduces into the theory of many-valued logic an approach for calculations of self-referential linguistic forms. On the other hand, it is of interest in some applications of the ECI to deal with self-referential systems, since the present logic, being derived from ECI itself, can serve as a language to provide access to computer modeling and simulation.

2 Interpretation I will define language  $\mathcal{L}(\mathsf{E})$ , to be understood as **ECI** interpreted for a propositional logic. In  $\mathcal{L}(\mathsf{E})$  there are three possible truth-values:  $\neg$ , true; , false; and  $\square$ , autonomous (i.e. self-naming). The following tables define negation and disjunction in  $\mathcal{L}(\mathsf{E})$ :



The rest of the connectives are defined as follows:



The truth value mark  $\neg$  can be seen to act also as a logical operator in  $\mathcal{L}(\mathsf{E})$ . In other words, we chose to view true as 'not false', and false as 'not true'; similarly, autonomous is taken to be 'not itself', since if p is  $\Box$ , then p is identical to  $\overline{p}$ , thence the symbol  $\Box$ .

In order to construct expressions in  $\mathcal{L}(\mathbf{E})$  we adopt the following rules of formation: (i)  $\neg$ ,  $\neg$ , are expressions; (ii) if p is an expression so is  $\overline{p}$ ; (iii) if p, q are expressions, then pq is an expression. Expressions p, q which can be shown to simplify (via the tables) to the same value are taken to be identical in which case we write 'p = q'. For example, from the tables and rules of formation, we have the identities of the following expressions:

We adopt the rules of substitution as usual, that is, identical expressions can be replaced for one another. For detachment, however, we shall not adopt a modus ponens but an equivalence rule, by detaching expressions at their point of equivalence. The reasons for this choice will be apparent below.

The axioms of the  $\mathcal{L}(\mathbf{E})$  are the following:

A1 
$$\overline{pq}p = p$$
  
A2  $\overline{pr}\overline{qr} = \overline{pq}r$   
A3  $\overline{p}p = p$ 

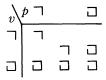
Within the above interpretation, the main theorems of ECI [7] can be rendered thus:

- (i) If in an expression all the variables take a value, then the expression has a value.
- (ii) For any choice of values for the variables of an expression, the value of the expression is unique. In other words,  $\mathcal{L}(\mathbf{E})$  is consistent.
- (iii) The axioms of  $\mathcal{L}(\mathbf{E})$  are complete. That is, if p has the same value as q, then 'p = q' can be derived from the axioms.
- (iv)  $\mathcal{L}(E)$  is strongly complete. If a non-derivable expression is added as an axiom, the  $\mathcal{L}(E)$  becomes inconsistent.
- **3** Discussion In this section I wish to compare  $\mathcal{L}(\mathsf{E})$  with other well-known three-values systems. I will discuss separately those issues pertaining to the purely formal aspects or 'syntax', and those issued pertaining to the intended use and meaning or 'semantics'.
- A. Syntax The truth tables of  $\mathcal{L}(\mathbf{E})$  are identical to those first described by Kleene [8], and redefined as the variant-standard system S<sub>3</sub> in the study of Dienes [9]. This is easily seen by changing '¬' into 'T' (or '1'), ' ' into 'F' (or '0') and '\(\sigma'\) into '\(\mathbf{U}'\) (or '2') in the preceeding tables. Further, any expression of  $\mathcal{L}(E)$  can be immediately transcribed into an expression in Kleene's system (K) simply by rewriting the connectives as defined in section 2. The real difference between these classic systems and  $\mathcal{L}(\mathbb{E})$  lies in the latter's use of the '='-equivalence. In this connection it should be noticed that an expression which is derivable from K's axioms, must have either the value 'true' or 'undefined', since these are the tautologies of K [9]. Let us transcribe ' $\forall p$ ' in K for ' $p = \Box p$ ' in  $\mathcal{L}(E)$ . Since  $\mathcal{L}(E)$  is complete, all identities of the form 'p =  $\Box p$ ' are derivable; therefore all derivable expressions p in **K** are recovered in  $\mathcal{L}(\mathbf{E})$  as exactly those expressions satisfying  $p = \Box p$ . Conversely, if an expression in  $\mathcal{L}(E)$  is such that  $p = \Box p$ , then we must have  $\not \in p$ , since it has a designated value, and thus it must be derivable in K. This justifies the choice of an equivalence

form for  $\mathcal{L}(\mathsf{E})$ : by doing so, not only do we recover all expressions derivable with implication, but have also access to many *other* classes of expressions. By taking an implication rule for detachment, we force ourselves to confuse expressions which are 'T' or 'U' in K. In this sense,  $\mathcal{L}(\mathsf{E})$  is richer than the classical systems with which it has coincidence in the truth-tables.

B. Semantics The expressions of  $\mathcal{L}(\mathbf{E})$  were derived as an interpretation of ECI for the domain of logical discourse. Thus, by necessity they carry an immediate interpretation, and the semantic rendering for the truthtables poses no problem. The real semantic issues stem from the fact that the intended use of  $\mathcal{L}(\mathsf{E})$  is to deal with self-referential linguistic forms. When interpreted for classical logic, self-reference engenders the paradoxes of self-denying sentences. To this problem several authors have addressed themselves, in an attempt to solve it by means of a three-valued logic. Moh Shaw-Kwei [10] used the three-valued system of Łukasiewicz, interpreting the third value as 'paradoxical'. He showed that in this case, as in all the family of lukasiewiczian systems, paradoxes will recurr; however, his results do not apply for systems of the K-type, where 'p implies p' is untrue. Later, Asenjo [11] proposed a calculus of antinomies, where the third value is taken as 'antinomic', with truth-tables of the K-type; his axioms, however, are incomplete. The present work extends these author's attempts by providing a consistant, complete system that can accept self-denying statements as non-paradoxical. We depart from them in various aspects. First,  $\mathcal{L}(\mathsf{E})$  is an interpretation of indicative expressions which stand for more general forms; conversely all logical forms in  $\mathcal{L}(\mathbf{E})$  have access to an underlying simple calculus of indications not available in other logics. Second, this interpretation for a three-valued logic is a first step in an attempt to deal with self-reference in general, not to circumvent paradoxes, which are seen at this light as only a particular case of atomic self-reference or autonomous value. Third, this logic can be a linguistic carrier for description, modeling, and simulation of selfreferential systems other than logic; in this sense  $\mathcal{L}(E)$  is, so to speak, constructive.

Beyond these considerations, let us look more in detail the autonomous value as a paradigm for self-reference. As it now stands in  $\mathcal{L}(\mathbf{E})$  it is only a third value which can deal with self-reference in a very loose way, namely, insofar self-referring statements require a value which is identical to its negation. In [10,11] these paradoxical statements are taken for granted, or borrowed from the calculus of classes in the form of Russell's paradox. To study self-reference in a deeper and more rigorous way, we must stay at the level of the calculus used, propositional logic in the case at hand, and introduce self-referential forms only through the means available in it. A classic treatment is [12] through the use of the 'quotation' of a proposition. In a simpler form, this can be done by defining a two-place connective Vol(v,p), to be taken as 'the value of p is v'. Let 'Vol' be defined by the following table:



A self-referring statement can then be constructed, asserting its own value as in

$$p = Val( ,p)$$
  
 $p = Val( ,p),$ 

both of which are autonomous. This is, of course, not surprising since indeed both statements refer to themselves, and thus are self-indicatory. No "paradoxical" results are produced through the use of this metaliguistic connective. However, it is highly unpalatable to have a statement

$$q = Val(\Box, p)$$

to be autonomous, since if p is not identical to q, q is not asserting anything about itself, but merely the fact that another statement is self-referring. This clearly points to the weakness of approaching the problem of recovering self-referential forms in language from the point of view of truthtables. Several modern logicians, for this very reason, have tried to find other ways of dealing with self-referential statements, notably van Frasser [13] and Skyrms [14]; these new attempts, to be sure, imply a departure from the customary form of valuation and/or substitution. I will not discuss these results here. The purpose of this paper was only to present the basic values of ECI as interpreted for logic, leaving aside higher degree or re-entering expressions. Consequently let me only suggest that the access in  $\mathcal{L}(\mathbf{E})$  to an underlying calculus where these expressions can be accomodated, might be an alternative and powerful way of circumventing the pitfalls of truth-tables and of dealing with self-reference more directly. I will expand on this at length in a forthcoming work.

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