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A SHORTENED PROOF OF SOBOCIŃSKI'S THEOREM CONCERNING A RESTRICTED RULE OF SUBSTITUTION IN THE FIELD OF PROPOSITIONAL CALCULI

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Sobociński [1] proves that in certain circumstances axiomatized systems of the propositional calculus having the rule of simultaneous substitution are not weakened in their deductive power by restricting the application of the substitution rule to the axioms alone. In this paper, a shortened proof of the same result will be presented employing induction on the length of proof sequences. As in Sobociński's proof, it is shown how a proof sequence employing the unrestricted rule may be uniquely and constructively replaced by a proof to the same effect employing only the restricted rule. The proof here draws upon Sobociński's notation and on his proof for certain key steps.

Theorem *If* $\mathbf{T}$ *is an axiom system in the propositional calculus which contains*

(1) *a binary connective* C *among its primitive signs*
(2) *the rule of detachment in regard to* C, R_2
(3) *the rule of simultaneous substitution*, R_1
(4) *an axiom set* A,

and if $\{a_1, \ldots, a_m\}$ *is a finite sequence of axioms and* $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ *constitutes a proof sequence in* $\mathbf{T}$ *of* b_n *employing only* R_1 *and* R_2, *then that proof sequence may be replaced by a proof sequence in* $\mathbf{T}$ *of* b_n *which restricts the applications of* R_1 *to* $\{a_1, \ldots, a_m\}$.

Proof: By induction on the length of proof sequences. Call n the "length" of the proof sequence $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$. Also, where $\{a_1, \ldots, a_m, \ldots, b\}$ is a proof sequence of b in $\mathbf{T}$, $\{a_1, \ldots, a_m\}$ will at times be represented by "α", the rest of the proof sequence by "β_b", and the entire proof sequence by "$\alpha; \beta_b$".

Base Step: $n = 1$. Then the theorem holds directly.

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Induction Step: Suppose the theorem holds for all proof sequences of length $n \leq p$. It will be shown that it then holds for all proof sequences of length $p + 1$. Consider an arbitrary proof sequence of length $p + 1$, $\{a_1, \ldots, a_m, b_1, \ldots, b_p, b_{p+1}\}$, which we represent as $\alpha; \beta_{b_{p+1}}$.

Case 1: In $\alpha; \beta b_{p+1}$, b_{p+1} follows from two earlier lines μ and $C\mu b_{p+1}$ by R_2. Then there are proof sequences (sub-sequences of $\alpha; \beta b_{p+1}$)

$$\alpha; \beta\mu, \text{ and } \alpha; \beta C\mu b_{p+1}$$

such that they both are sequences of length no greater than p. But then by the induction hypothesis $\beta\mu$ and $\beta C\mu b_{p+1}$ may be replaced by sequences to the same effect but which restrict the application of R_1 to α. Call these replacements "β^*_μ" and "$\beta^*_{C\mu b_{p+1}}$". Now form the sequence

$$\alpha; \beta^*_\mu; \beta^*_{C\mu b_{p+1}}$$

Such a sequence contains both μ and $C\mu b_{p+1}$, and restricts R_1 to α. Then annex $\{b_{p+1}\}$ to form

$$\alpha; \beta^*_\mu; \beta^*_{C\mu b_{p+1}}; \{b_{p+1}\} \ ,$$

and since $\{\mu, C_{\mu b_{p+1}}\} \vdash_{\overline{R_2}} b_{p+1}$, the theorem holds.

Case 2: In $\alpha; \beta_{b_{p+1}}$, b_{p+1} follows from some earlier line μ by R_1. There are three sub-cases.

2a. μ appears in α. Then replace $\alpha; \beta b_{p+1}$ by $\alpha; \{b_{p+1}\}$ and the theorem holds.

2b. μ appears in β_p and μ follows from some earlier line ν by R_1. Then there must be some substitution in ν which yields b_{p+1} directly. So drop μ from β_p and add b_{p+1}. Such a sequence is of length no greater than p, and so by the induction hypothesis, the theorem holds.

2c. μ appears in β_p and μ follows from two earlier lines ρ, $C\rho\mu$ by R_2. Then there is a proof sequence for ρ,—call it "$\beta\rho$"—and also one for $C\rho\mu$—call it "$\beta C\rho\mu$"—such that neither is of length greater than $p - 1$.

Now since $\{\rho, C\rho\mu\} \vdash_{\overline{R_2}} \mu$ and $\{\mu\} \vdash_{\overline{R_1}} b_{p+1}$, then there must be some σ and $C\sigma b_{p+1}$ such that $\{\rho\} \vdash_{\overline{R_1}} \sigma$ (or else ρ is already σ), and such that $\{C\rho\mu\} \vdash_{\overline{R_1}} C\sigma b_{p+1}$ (or else $C\rho\mu$ is already $C\sigma b_{p+1}$). Form the sequences

$$\alpha; \beta_\rho; \{\sigma\} \text{ and } \alpha; \beta C\rho\mu; \{C\sigma b_{p+1}\}\,.$$

Neither of these are of length greater than p. Represent $\beta\rho; \{\sigma\}$ by "$\beta\sigma$", and $\beta C\rho\mu; \{C\sigma b_{p+1}\}$ by "$\beta C\sigma b_{p+1}$". By the induction hypothesis, $\alpha; \beta\sigma$ may be replaced by $\alpha; \beta^*_\sigma$, and $\alpha; \beta C\sigma b_{p+1}$ by $\alpha; \beta^*_{C\sigma b_{p+1}}$, which are proof sequences of σ and $C\sigma b_{p+1}$ respectively, and which employ R_1 only in the restricted way. Now form the sequence

$$\alpha; \beta^*_\sigma; \beta^*_{C\sigma b_{p+1}}; \{b_{p+1}\} \ ,$$

and the theorem follows.

REFERENCE

[1] Sobociński, B., "A theorem concerning a restricted rule of substitution in the field of propositional calculi. I and II," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 465-476 and 589-597.

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