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Provable Forms of Martin's Axiom

GARY P. SHANNON*

Abstract It is shown that if Martin's Axiom (MA) is restricted to well-orderable sets, and if the countable antichain condition is replaced by either a finite antichain condition or a finite "Q-strong antichain" condition, then the resulting statements are forms of MA provable in ZF. Variations of these weak forms of MA are shown to be equivalent to the Axiom of Choice and some of its weak forms.

Introduction The purpose of this paper is to set forth the proof in ZF of some weak forms of Martin's Axiom (MA). This may be of some philosophical interest with respect to the work of Maddy ([7],[8]). To obtain such forms: (i) MA is restricted to quasi-orders on well-orderable sets, and (ii) the countable antichain condition is replaced by a finite antichain condition. The conclusion of MA is also strengthened, for under these conditions there exist filters which intersect all dense subsets. These theorems are applied to prove MA-type equivalents of forms of the Axiom of Choice (AC).

One known result related to this paper is the statement that, in ZF, $AC^{\aleph_0}$ implies MA ($\aleph_0$) (Kunen [6], Lemma 2.6(c), pp. 54–55). Another related result (due to Goldblatt [3]) is that, "In ZF, the principle of dependent choices is equivalent to the statement that if $(P,\leq)$ is a nonempty partial order and $\mathfrak{D}$ is a countable collection of dense subsets of P, then there exists a $\mathfrak{D}$-generic filter".

Definitions Let $(P,\leq)$ be a quasi-order, i.e., $\leq$ is reflexive and transitive (if in addition $\leq$ is antisymmetric, then $(P,\leq)$ is a partial order).

Two elements x,y of P are said to be *incompatible* if there does not exist $z \in P$ such that $z \leq x$ and $z \leq y$ (otherwise x and y are said to be *compatible*).

A subset I of P is said to be an *antichain* if any two elements of I are incompatible. $c(x)$ denotes the set of elements of P that are compatible with x.

A subset D of P is said to be *dense* if for all $x \in P$ there exists $y \in D$ such that $y \leq x$.

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A subset F of P is said to be a *filter* if for all $x, y \in F$ there exists $z \in F$ such that $z \leq x$ and $z \leq y$, and if $x \in F$ and $y \geq x$ imply $y \in F$.

If $\mathfrak{D}$ is a collection of subsets of P, then a filter F is called $\mathfrak{D}$-*generic* iff $F \cap D \neq \emptyset$ for all $D \in \mathfrak{D}$. A filter F is called *P-generic* iff $F \cap D \neq \emptyset$ for all dense subsets D of P.

$A \approx B$ denotes that there exists a 1-1 correspondence between A and B, and $A \lesssim B$ denotes that there exists a 1-1 map from A into B.

$\downarrow x = \{y \in P : y \leq x\}$, and $\uparrow x = \{y \in P : Y \geq x\}$.

$\downarrow L = \{y \in P : y \leq x \text{ for some } x \in L\}$.

For each cardinal κ and λ, let $\mathrm{AC}^{\kappa}_{\lambda}$ ($\mathrm{AC}^{\kappa}_{\mathrm{fin}}$) denote the following statement:

If $\mathcal{C}$ is a collection of nonempty, pairwise disjoint sets of cardinality λ (finite cardinality), and if $\mathcal{C} \approx \kappa$, then there exists a choice function on $\mathcal{C}$.

For each cardinal κ and n, let $\mathrm{ACS}^{\kappa}_{n}$ ($\mathrm{ACS}^{\kappa}_{\mathrm{fin}}$) denote the following statement:

If $\mathcal{C}$ is a collection of nonempty, pairwise disjoint sets of cardinality n (finite cardinality), and if $\mathcal{C} \approx \kappa$, then there exists a subcollection $\mathfrak{D}$ of $\mathcal{C}$ with $\mathfrak{D} \approx \kappa$ such that there exists a choice function on $\mathfrak{D}$.

The finite antichain condition

Lemma 1 *In* ZF, *if* $(P, \leq)$ *is a quasi-order such that* P *is well-orderable and contains antichains of arbitrarily large finite cardinality, then* P *contains a denumerable antichain.*

Proof: Assume that $(P, \leq)$ is a quasi-order such that P is well-orderable and contains antichains of arbitrarily large finite cardinality. P is well-orderable, thus there exists an ordinal α such that $P \approx \alpha$. Let $\mathcal{C}$ be the collection of finite antichains, I, of P with $|I| \geq 2$ and such that there exists $y \in I$ such that $\downarrow y$ contains antichains of arbitrarily large finite cardinality.

If $\mathcal{C} = \emptyset$ then let $\mathcal{R}$ be the collection of all finite antichains of P. Then $\mathcal{R} \approx \alpha$ — say $\mathcal{R} = \{D_\delta : \delta \in \alpha\}$.

Let τ_1 be the least ordinal such that D_1 is a proper subset of D_{τ_1} (τ_1 exists, since $P \neq \bigcup\{\mathrm{c}(y) : y \in D_1\}$. Suppose $P = \bigcup\{\mathrm{c}(y) : y \in D_1\}$. D_1 is finite and for each $d \in D_1$ there exists $\mathrm{n}(d) \in \omega$ such that each antichain in $\downarrow d$ has at most $\mathrm{n}(d)$ elements. If $I \subseteq P$ is an antichain with at least $(\sum_{d \in D_1} n(d)) + 1$ elements, then by the pigeon-hole principle one of the elements of $\downarrow d$ contains an antichain with more than $\mathrm{n}(d)$ elements — a contradiction. Let μ be the least ordinal such that $p_\mu \in P - \bigcup\{c(y) : y \in D_1\}$; then $D_1 \cup \{p_\mu\} \in \mathcal{R}$).

For each $n > 1$ define, by induction, τ_{n+1} to be the least ordinal such that D_{τ_n} is a proper subset of $D_{\tau_{n+1}}$.

Then $\bigcup\{D_{\tau_n} : n \in \omega\}$ is a denumerable antichain.

Assume that $\mathcal{C} \neq \emptyset$ and $\mathcal{C} \approx \beta > \omega$ — say $\mathcal{C} = \{I_\tau : \tau < \beta\}$. For each $n \in \omega$, define (since P is well-orderable) $p_n \in I_n$ such that $\downarrow p_n$ contains antichains of arbitrarily large finite cardinality; and for each $n \in \omega$, let $\mathcal{C}_n = \{I \in \mathcal{C} : I \subseteq \downarrow p_n\}$. If $\mathcal{C}_n = \emptyset$ for any n then (by the case $\mathcal{C} = \emptyset$ relativized to $\downarrow p_n$) P contains a denumerable antichain. Therefore assume that $\mathcal{C}_n \neq \emptyset$ for all $n \in \omega$. Set

$\mu(0) = 1$, and define $r_1 \in I_{\mu(0)} - \{p_{\mu(0)}\}$. Assume that r_n is defined ($n \geq 1$), $r_n \in I_{\mu(n-1)} - \{p_{\mu(n-1)}\}$. Let $\mu(n)$ be the least natural number such that $I_{\mu(n)} \in \mathcal{C}_{\mu(n-1)}$. Then define $r_{n+1} \in I_{\mu(n)} - \{p_{\mu(n)}\}$. Then $\{r_n : n \in \omega\}$ is a denumerable antichain.

If $\mathcal{C} \neq \emptyset$ is finite, then for some n, $\mathcal{C}'_n = \{I \in \mathcal{C} : I \subseteq \downarrow p_n - \uparrow p_n\}$ is empty (for $I_{n+1} \in \mathcal{C}_n$, $p_{n+1} \in I_{n+1}$ is inductively defined such that $\downarrow p_n$ has arbitrarily large finite antichains and arrives after at most $|\bigcup\mathcal{C}|$ steps at the conclusion). By the case $\mathcal{C} = \emptyset$ relativized to $\downarrow p_n - \uparrow p_n$, there is a denumerable antichain.

Lemma 1 is a special case, in ZF, of a result of Erdös and Tarski [2].

Theorem 1 *In* ZF, *if* $(P,\leq)$ *is a quasi-order such that* P *is nonempty, well-orderable and such that every antichain is finite, then there exists a* P*-generic filter.*

Proof: Assume that $(P,\leq)$ is a quasi-order such that P is nonempty, well-orderable and such that every antichain is finite. P is well-orderable, thus there exists an ordinal α such that $P \approx \alpha$. By Lemma 1 there is an upper bound, say m, on the cardinality of any antichain of P. Let I be an antichain of P such that $|I| = m$. Let $x \in I$ be such that $\mathrm{c}(x) \approx \alpha$ (such an x must exist since $P = \bigcup\{\mathrm{c}(y) : y \in I\}$ and $|I| = m$), and let $F = \mathrm{c}(x)$.

Claim *F is a filter.*

Proof: If $y, z \in F$ then y and z are each compatible with x, and thus there exist $u, v \in P$ such that $u \leq x$, $u \leq y$, $v \leq x$, $v \leq z$. Then there exists $w \in P$ such that $w \leq u$ and $w \leq v$ (since if w does not exist, u and v are incompatible, and then $(I - \{x\}) \cup \{u, v\}$ is an antichain of cardinality $m + 1$—a contradiction). Then $w \in F$ (since w is compatible with x) and $w \leq y$ and $w \leq z$.

If $y \in F$ and $z \in P$ such that $z \geq y$, then there exists $u \in P$ such that $u \leq y$ and $u \leq x$. Then $u \leq z$ and $u \leq x$, and thus z is compatible with x, and thus $z \in F$.

Therefore F is a filter.

If D is dense in P then $F \cap D \neq \emptyset$, since there exists $y \in D$ such that $y \leq x$, thus y is compatible with x, and thus $y \in F$. Thus F is P-generic.

Corollary 1 *In* ZF *the following statements are equivalent*:

(i) AC

(ii) *If* $(P,\leq)$ *is a quasi-order such that* P *is nonempty and every antichain is finite, then there exists a* P*-generic filter which is well-orderable as a set.*

Proof: (i) ⇒ (ii). By the same argument used for the proof of Theorem 1 (using the fact that AC implies P is well-orderable).

(ii) ⇒ (i). Assume (ii) and let $\mathcal{C}$ be a collection of pairwise disjoint nonempty sets. Define $\leq$ on $P = \bigcup\mathcal{C}$ by $x \leq y$ for all $x, y \in P$ (then every antichain of $(P,\leq)$ is clearly finite). Then $\mathcal{C}$ is a collection of dense subsets of P, and thus there exists a P-generic filter F and an ordinal β with $F \approx \beta$. Let $F = \{f_\delta : \delta < \beta\}$. Define g on $\mathcal{C}$ by $g(C_\delta) = f_\tau$, where τ is the least ordinal such that $f_\tau \in F \cap C_\delta$. Then g is a choice function on $\mathcal{C}$.

Corollary 2 *In* ZF *the following statements are equivalent for well-orderable cardinals* κ *and* $\lambda \geq \kappa$:

(i) AC^{κ}_{fin}

(ii) *If $(P,\leq)$ is a quasi-order such that P is nonempty and is a union of κ finite sets, and such that every antichain is finite, then there exists a P-generic filter which is well-orderable as a set*

(iii) *If $(P,\leq)$ is a quasi-order such that P is nonempty and is a union of κ finite sets, and such that every antichain is finite, then for each family $\mathfrak{D}$ of λ dense sets there exists a $\mathfrak{D}$-generic filter which is well-orderable as a set.*

Proof: (i) $\Rightarrow$ (ii). By the same argument used for the proof of Theorem 1 (using the fact that AC^{κ}_{fin} implies $P \approx \kappa$).

(ii) $\Rightarrow$ (iii) is clear. (iii) $\Rightarrow$ (i). By the same argument used in the proof of Corollary 1.

Strong antichains

Definitions Let $(P,\leq)$ be a quasi-order.

A chain C of $(P,\leq)$ is called a *covering chain* iff for each $x \in C$ that is not maximal in C, there exists $y \in C$ such that y covers x in P (y covers x in P iff $y > x$ and there does not exist $z \in P$ such that $y > z$ and $z > x$).

Let Q be any subset of P, and let $x \in P$. The *depth*of x (relative to Q) is the order type of the smallest well-ordered (under $\leq$) covering chain C which begins at x and ends at an element of Q (if no such covering chain exists then x does not have any depth relative to Q). The depth of x is denoted $\text{dp}(x)$ (the reference to Q will be omitted, and will be understood).

Two elements x,y of P are said to be *Q-strongly incompatible* if x,y are incompatible and $\text{dp}(x) \neq \text{dp}(y)$ (assuming that $\text{dp}(x)$ and $\text{dp}(y)$ exist). A subset I of P is said to be a *Q-strong antichain* if any two elements of I are Q-strongly incompatible.

Theorem 2 *In* ZF, *the following statements are equivalent*:

(i) AC^{ω}_{fin}

(ii) *Assume that $(P,\leq)$ is a partial order such that P is a countable union of finite sets and contains a nonempty set Q relative to which depths are defined for all elements of P. If all Q-strong antichains of P are finite, then for each countable family $\mathfrak{D}$ of dense sets there exists a $\mathfrak{D}$-generic filter.*

Proof: Assume (ii). Let $\mathcal{C}$ be a sequence of finite nonempty sets, say $\mathcal{C} = \langle C_n : n \in \omega\rangle$, and let $\mathcal{S}$ be the set of choice functions on the finite initial segments of $\mathcal{C}$.

Define $\leq$ on $\mathcal{S}$ by $f \leq g$ iff $g \subseteq f$. Then $(\mathcal{S},\leq)$ is a partial order, and $\mathcal{S}$ is a countable union of finite sets. Define depth in $(\mathcal{S},\leq)$ relative to the set of choice functions on $\{C_0\}$.

If every Q-strong antichain of $(\mathcal{S},\leq)$ is finite, then let $D_i = \{f \in \mathcal{S} : C_i \in \text{dom}(f)\}$. Then D_i is dense in $\mathcal{S}$ (if $f \in \mathcal{S}$ and $C_i \notin \text{dom}(f)$ then let m be the largest natural number such that $C_m \in \text{dom}(f)$ – and let $g = f \cup \bigcup_{j=m+1}^{i}\{(C_j,x_j)\}$, where x_j is any element of C_j – then $g \in D_i$ and $g \leq f$). Thus there exists a $\{D_i : i \in \omega\}$-generic filter, F.

If $f,g \in F$ then there exists $h \in F$ such that $h \leq f$ and $h \leq g$. Therefore $f \subseteq h$ and $g \subseteq h$, and thus $f \cup g$ is a function. It follows that $\bigcup F$ is a function, and since $F \cap D_i \neq \emptyset$ for each $i \in \omega$, $\bigcup F$ is a choice function on $\mathcal{C}$.

If $(\mathcal{S},\leq)$ contains an infinite Q-strong antichain then call it I. I contains at most one element of each depth, and every depth is finite, thus I must be denumerable—say $I = \{f_n : n \in \omega\}$. For each n, let $g(n)$ denote the largest natural number such that $C_{g(n)} \in \mathrm{dom}(f_n)$ (if no such $g(n)$ exists then f_n is a choice function on $\mathcal{C}$ and so the proof is done). Since I is a Q-strong antichain, if $n \neq m$ then $g(n) \neq g(m)$, and thus $C_{g(n)} \neq C_{g(m)}$. Then $\bigcup\{f_n | C_{g(n)}\}$ is a choice function on a denumerable subfamily of $\mathcal{C}$. Thus (ii) implies $\mathrm{ACS}^{\omega}_{\mathrm{fin}}$ and thus (since $\mathrm{ACS}^{\omega}_{\mathrm{fin}}$ is equivalent to $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ (see Brunner [1])) (ii) implies $\mathrm{AC}^{\omega}_{\mathrm{fin}}$.

The proof that (i) implies (ii) uses the fact that $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ is equivalent to the statement that a denumerable collection of finite nonempty pairwise disjoint sets is denumerable, and follows the proof of Lemma 2.6(c) in Jech [4] (the proof does not need the assumption that every Q-strong antichain of P is finite).

Theorem 3 *Assume that $(P,\leq)$ is a quasi-order such that P is well-orderable and contains a nonempty set Q relative to which depths are defined for all elements of P such that $x < y$ implies* $\mathrm{dp}(x) > \mathrm{dp}(y)$. *In* ZF, *if all Q-strong antichains of P are finite, then there exists a P-generic filter.*

Proof: If every antichain in P is finite then the existence of a P-generic filter can be shown as in the proof of Theorem 1.

P is well-orderable, thus there exists an ordinal α such that $P = \{p_\delta : \delta < \alpha\}$. Assume that P contains an infinite antichain, J. Then there exists an infinite antichain I such that all of the elements of I have the same depth, δ_1. Otherwise there are infinitely many sets $J_\delta = \{x \in J : \mathrm{dp}(x) = \delta\} \neq \emptyset$, and thus $K = \{p_{\tau_\delta} \in J_\delta : \tau_\delta$ is the least ordinal such that $p_{\tau_\delta} \in J_\delta\}$ is an infinite Q-strong antichain.

There exists $x \in I$ and $y \in \downarrow x$ such that $\downarrow y - \uparrow y = \emptyset$. Otherwise let β_1 be the least ordinal such that $p_{\beta_1} \in I$ (thus $\mathrm{dp}(p_{\beta_1}) = \delta_1$), and assume that p_{β_n} and δ_n are defined. Let δ_{n+1} be the least ordinal such that $\delta_{n+1} > \delta_n$ and $\delta_{n+1} = \mathrm{dp}(w)$ for some $w \in \downarrow I - \downarrow\{p_{\beta_1}, \ldots, p_{\beta_n}\}$, and define β_{n+1} to be the least ordinal such that $p_{\beta_{n+1}} \in \downarrow I - \downarrow\{p_{\beta_1}, \ldots, p_{\beta_n}\}$ and $\mathrm{dp}(p_{\beta_{n+1}}) = \delta_{n+1}$ (δ_{n+1} and $p_{\beta_{n+1}}$ exist by the assumption that for all $x \in I$ there does not exist $y \in \downarrow x$ such that $\downarrow y - \uparrow y = \emptyset$, and by the hypotheses that $u > v$ implies $\mathrm{dp}(v) > \mathrm{dp}(u)$ and that every element of P has a depth relative to Q). Then $\{p_{\beta_n} : n \in \omega\}$ is an infinite Q-strong antichain.

Let $G = \{u \in P : y \geq u \text{ and } u \geq y\}$, and let $F = \{z \in P : z \geq y\}$. Then F is a filter, and if D is dense in P then $D \cap G \neq \emptyset$, thus $F \cap D \neq \emptyset$, and therefore F is a P-generic filter.

Remark If ZF is consistent then it is not possible to prove the following statement in ZF + $\forall n \geq 1$, AC_n:

> *If $(P,\leq)$ is a partial order such that P is a countable union of finite sets, and such that all antichains are finite, then for each countable family $\mathfrak{D}$ of dense sets there is a $\mathfrak{D}$-generic filter.*

Proof: The permutation model that follows is from Theorem 7.11 of [4], and it satisfies AC_n for all n:

Let $A = \bigcup\{T_n : n \in \omega\}$, where $T_n = \{a_{n1}, a_{n2}, \ldots, a_{np_n}\}$ and p_n is the nth prime number. $\pi_n : a_{n1} \to a_{n2} \to \cdots \to a_{np_n} \to a_{n1}$ and $\pi_n(x) = x$ for all $x \notin T_n$.

Let $\mathcal{S}$ be the set of choice functions on sets of the form $\{T_0, T_1, \ldots, T_n\}$ ($\mathcal{S} \in V$ since if $E = \{a_{01}\}$ then $\operatorname{fix}(E) \subseteq \operatorname{sym}(\mathcal{S})$).

Define $\leq$ on $\mathcal{S}$ by $f \leq g$ iff $g \subseteq f$ ($\leq \in V$).

Claim $(\mathcal{S}, \leq)$ *does not contain an infinite antichain.*

Proof: The idea behind the proof is that an infinite antichain cannot have a finite support.

Suppose that I is an infinite antichain of $\mathcal{S}$. Let E be a finite subset of A such that $\operatorname{fix}(E) \subseteq \operatorname{sym}(I)$. Let n be the largest natural number such that $E \cap T_n \neq \varnothing$. Then for all $m > n$, $\pi_m \in \operatorname{fix}(E)$.

Assume that $m > n$, $f \in \mathcal{S}$, and $T_m \in \operatorname{dom}(f)$. Then $\pi_m(f) \in I$, and thus $(f - \{(T_m, f(T_m))\}) \cup \{(T_m, a_{mj})\} \in I$ for all j, $1 \leq j \leq p_m$. Thus for all $m > n$, and for each j, $1 \leq j \leq p_m$, $(f - \bigcup_{m>n}\{(T_m, f(T_m))\}) \cup \bigcup_{m>n}\{(T_m, a_{mj})\}$ is an element of I. (For each j, $1 \leq j \leq p_m$, call $(f - \bigcup_{m>n}\{(T_m, f(T_m))\}) \cup \bigcup_{m>n} ((T_m, a_{mj})\}$ a *generalization* of f.)

Let $g, h \in I$ be such that $g|_{\{T_0, \ldots, T_n\}} = h|_{\{T_0, \ldots, T_n\}}$ and $\operatorname{dom}(g) \subsetneq \operatorname{dom}(h)$ (g and h must exist since I is infinite and there are only finitely many possibilities for choice functions on $\{T_0, \ldots, T_n\}$). Then (by the previous paragraph) there must exist $\hat{h}$, a "generalization" of h, and $\hat{g}$, a "generalization" of g, such that $\hat{g} \subsetneq \hat{h}$. Then $\hat{h} \leq \hat{g}$, and thus I is not an antichain.

Therefore, every antichain of $\mathcal{S}$ is finite. Let $D_i = \{f \in \mathcal{S} \mid T_i \in \operatorname{dom}(f)\}$ (then D_i is dense for each $i \in \omega$, as shown in the proof of Theorem 2). If there exists a $\{D_i : i \in \omega\}$-generic filter, F, then $\bigcup F$ is a choice function on $\{T_n \mid n \in \omega\}$. But no such choice function exists in this model, and thus such a filter cannot exist.

This result transfers to ZF ([5], pp. 208–212).

Note that it follows from Theorem 2 that AC^{ω}_{fin} implies the statement: *If* $(P, \leq)$ *is a partial order such that P is a countable union of finite sets, and such that all antichains are finite, then for each countable family* $\mathfrak{D}$ *of dense sets there is a* $\mathfrak{D}$*-generic filter.* I do not know if this statement implies AC^{ω}_{fin}, nor do I know if it implies AC^{ω}_{n} for any $n \geq 2$. It may turn out (as a result of the above remark) that this statement is a weak form of the Axiom of Choice that is strictly between $\forall n \geq 1$: AC^{ω}_{n} and AC^{ω}_{fin}.

Let $\operatorname{MAS}(\kappa)$ denote the statement obtained from $\operatorname{MA}(\kappa)$ (Kunen [6], p. 54) by replacing 'antichain' with 'Q-strong antichain', and adding the condition that P contains a nonempty set Q relative to which depths are defined for all elements of P. Then, in ZF, $\operatorname{MAS}(\kappa)$ implies $AC^{\aleph_0}$ for $\kappa \geq \aleph_0$, and $\operatorname{MAS}(\omega_1)$ implies ACS^{ω_1} (the proofs of each of these are similar to the proof of Theorem 2).

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Department of Mathematics and Statistics
California State University
Sacramento, California