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MASS MINIMIZERS AND CONCENTRATION FOR NONLINEAR CHOQUARD EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this paper, we study the existence of minimizers to the following functional related to the nonlinear Choquard equation:

$$E(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|u|^2-\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p$$

on $\widetilde{S}(c)=\{u\in H^1(\mathbb{R}^N)\mid \int_{\mathbb{R}^N}V(x)|u|^2<+\infty,\ |u|_2=c,\ c>0\}$, where $N\geq 1$, $\alpha\in(0,N)$, $(N+\alpha)/N\leq p<(N+\alpha)/(N-2)_+$ and $I_\alpha\colon\mathbb{R}^N\to\mathbb{R}$ is the Riesz potential. We present sharp existence results for $E(u)$ constrained on $\widetilde{S}(c)$ when $V(x)\equiv 0$ for all $(N+\alpha)/N\leq p<(N+\alpha)/(N-2)_+$. For the mass critical case $p=(N+\alpha+2)/N$, we show that if $0\leq V\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $\lim\limits_{|x|\to+\infty}V(x)=+\infty$, then mass minimizers exist only if $0<c<c_*=|Q|_2$ and concentrate at the flattest minimum of V as c approaches c_* from below, where Q is a groundstate solution of $-\Delta u+u=(I_\alpha*|u|^{(N+\alpha+2)/N})|u|^{(N+\alpha+2)/N-2}u$ in $\mathbb{R}^N$.

1. Introduction

In this paper, we consider the following semilinear Choquard problem:

$$-\Delta u-\mu u=(I_\alpha*|u|^p)|u|^{p-2}u,\quad x\in\mathbb{R}^N,\ \mu\in\mathbb{R},\tag{1.1}$$

where $N\geq 1$, $\alpha\in(0,N)$, $(N+\alpha)/N\leq p<(N+\alpha)/(N-2)_+$, here $(N+\alpha)/(N-2)_+=(N+\alpha)/(N-2)$ if $N\geq 3$ and $(N+\alpha)/(N-2)_+=+\infty$ if $N=1,2$.

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The Riesz potential $I_\alpha\colon \mathbb{R}^N \to \mathbb{R}$ is defined as (see [26])

$$I_\alpha(x) = \frac{\Gamma\left(\dfrac{N-\alpha}{2}\right)}{\Gamma\left(\dfrac{\alpha}{2}\right)\pi^{N/2}2^\alpha}\frac{1}{|x|^{N-\alpha}}, \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.$$

Problem (1.1) is a nonlocal one due to the existence of nonlocal nonlinearity. It arises in various fields of mathematical physics, such as quantum mechanics, physics of laser beams, physics of multiple-particle systems, etc. When $N = 3$, $\mu = -1$ and $\alpha = p = 2$, (1.1) turns to be the well-known Choquard–Pekar equation

$$-\Delta u + u = (I_2 * |u|^2)u, \quad x \in \mathbb{R}^3, \tag{1.2}$$

which was proposed as early as in 1954 by Pekar [25], and by a work of Choquard 1976 in a certain approximation to Hartree–Fock theory for one-component plasma, see [14], [16]. Equation (1.1) is also known as the nonlinear stationary Hartree equation since if u solves (1.1) then $\psi(t,x) = e^{it}u(x)$ is a solitary wave of the following time-dependent Hartree equation:

$$i\psi_t = -\Delta\psi - (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N,$$

see [7], [21].

In the past few years, there are several approaches to construct nontrivial solutions of (1.1), see e.g. [5], [14], [17], [18], [20], [21], [27] for $p = 2$ and [22], [23]. One of them is to look for a constrained critical point of the functional

$$I_p(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \tag{1.3}$$

on the constrained L^2-spheres in $H^1(\mathbb{R}^N)$:

$$S(c) = \{u \in H^1(\mathbb{R}^N) \mid |u|_2 = c, c > 0\}.$$

In this way, the parameter $\mu \in \mathbb{R}$ will appear as a Lagrange multiplier and such solution is called a normalized solution. By the following well-known Hardy–Littlewood–Sobolev inequality: For $1 < r, s < +\infty$, if $f \in L^r(\mathbb{R}^N)$, $g \in L^s(\mathbb{R}^N)$, $\lambda \in (0, N)$ and $1/r + 1/s + \lambda/N = 2$, then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} \le C_{r,\lambda,N}|f|_r|g|_s, \tag{1.4}$$

we see that $I_p(u)$ is well-defined and a C^1 functional. Set

$$I_p(c^2) = \inf_{u \in S(c)} I_p(u), \tag{1.5}$$

then minimizers of $I_p(c^2)$ are exactly critical points of $I_p(u)$ constrained on $S(c)$.

Normalized solutions for equation (1.2) have been studied in [14], [17]. In this paper, one of our purposes is to get a general and sharp result for the existence of minimizers for minimization problem (1.5).

To state our main result, we rely on the following interpolation inequality with the best constant: For $(N+\alpha)/N < p < (N+\alpha)/(N-2)_+$,

$$\begin{aligned}(1.6)\quad &\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \\ &\leq \frac{p}{|Q_p|_2^{2p-2}} \bigg(\int_{\mathbb{R}^N} |\nabla u|^2\bigg)^{(Np-(N+\alpha))/2} \bigg(\int_{\mathbb{R}^N} |u|^2\bigg)^{(N+\alpha-(N-2)p)/2},\end{aligned}$$

where equality holds for $u = Q_p$, where Q_p is a nontrivial solution of

$$(1.7)\quad -\frac{Np-(N+\alpha)}{2}\Delta Q_p + \frac{N+\alpha-(N-2)p}{2} Q_p = (I_\alpha * |Q_p|^p)|Q_p|^{p-2}Q_p,$$

for $x \in \mathbb{R}^N$. In particular, $Q_{(N+\alpha+2)/N}$ is a groundstate solution, i.e. the least energy solution among all nontrivial solutions of (1.7). Moreover, when $p = (N+\alpha+2)/N$, all groundstate solutions of (1.7) have the same L^2-norm (see Lemma 3.2 below).

For $p = (N+\alpha)/N$, recall from [15] the Hardy–Littlewood–Sobolev inequality with the best constant:

$$\begin{aligned}(1.8)\quad &\int_{\mathbb{R}^N} (I_\alpha * |u|^{(N+\alpha)/N})|u|^{(N+\alpha)/N} \\ &\qquad\leq \frac{1}{|Q_{(N+\alpha)/N}|_2^{(2(N+\alpha))/N}} \bigg(\int_{\mathbb{R}^N} |u|^2\bigg)^{(N+\alpha)/N}\end{aligned}$$

with equality if and only if

$$u = Q_{(N+\alpha)/N}, \quad \text{where } Q_{(N+\alpha)/N} = C(\eta/(\eta^2 + |x-a|^2))^{N/2},$$

$C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $\eta \in (0, +\infty)$ are parameters. Then our first result is as follows:

THEOREM 1.1. *Assume that $N \geq 1$, $\alpha \in (0, N)$ and $(N+\alpha)/N \leq p < (N+\alpha)/(N-2)_+$.*

(a) *If $p = (N+\alpha)/N$, for any $c > 0$,*

$$I_{(N+\alpha)/N}(c^2) = -\frac{N}{2(N+\alpha)} \bigg(\frac{c}{|Q_{(N+\alpha)/N}|_2}\bigg)^{2(N+\alpha)/N}$$

and $I_{(N+\alpha)/N}(c^2)$ has no minimizer.

(b) *If $(N+\alpha)/N < p < (N+\alpha+2)/N$, then $I_p(c^2) < 0$ for all $c > 0$, moreover, $I_p(c^2)$ has at least one minimizer for each $c > 0$.*

(c) *If $p = (N+\alpha+2)/N$, then there exists $c_* := |Q_{(N+\alpha+2)/N}|_2$ such that*

(i) $I_{(N+\alpha+2)N}(c^2) = \begin{cases} 0 & \text{if } 0 < c \le c_*, \\ -\infty & \text{if } c > c_*, \end{cases}$

(ii) *$I_{(N+\alpha+2)/N}(c^2)$ has no minimizer if $c \neq c_*$,*

(iii) *each groundstate of (1.7) is a minimizer of $I_{(N+\alpha+2)/N}(c_*^2)$,*

(iv) *there is no critical point for $I_{(N+\alpha+2)/N}(u)$ constrained on $S(c)$ for each $0 < c < c_*$.*

(d) *If $(N+\alpha+2)/N < p < (N+\alpha)/(N-2)_+$, then $I_p(c^2)$ has no minimizer for each $c > 0$ and $I_p(c^2) = -\infty$.*

Remark 1.2. Theorem 1.1 can be viewed as a consequence of [14, Theorem 9] for $p = 2$ and [22, Theorem 1]. However, we still state and prove Theorem 1.1 here, using an alternative method, since our result is more delicate and it provides a framework to our subsequent considerations.

Remark 1.3. (a) Since until now the uniqueness (up to translations) of the positive solution to (1.7) is only proved for the case $\alpha = p = 2$, see e.g. [3], [11] and [13], it follows that if $N = 4$ and $\alpha = 2$, then up to translations, *the minimizer of $I_{(N+\alpha+2)/N}(c_*^2)$ is unique and there exists no critical point for $I_{(N+\alpha+2)/N}(u)$ constrained on $S(c)$ for each $c \neq c_*$.*

(b) For $N \ge 3$ and $(N+\alpha+2)/N < p < (N+\alpha)/(N-2)$, it has been proved in [12] that for each $c > 0$, $I_p(u)$ has a mountain pass geometry on $S(c)$ and there exists a solution $(u_c, \mu_c) \in S(c) \times \mathbb{R}^-$ of (1.1) with $I_p(u_c) = \gamma(c)$, where $\gamma(c)$ denotes the mountain pass level on $S(c)$.

By Theorem 1.1, $p = (N+\alpha+2)/N$ is called the L^2-critical exponent for (1.5). In order to get critical points under the mass constraint for such L^2-critical case, we add a nonnegative perturbation term to the right-hand side of (1.3), i.e. consider the following functional:

$$
\begin{aligned}
(1.9)\quad E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 \\
- \frac{N}{2(N+\alpha+2)}\int_{\mathbb{R}^N} (I_\alpha * |u|^{(N+\alpha+2)/N})|u|^{(N+\alpha+2)/N},
\end{aligned}
$$

where

$$
(\mathrm{V}_0)\qquad V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N), \quad \inf_{x\in\mathbb{R}^N} V(x) = 0 \quad \text{and} \quad \lim_{|x|\to+\infty} V(x) = +\infty.
$$

Based on (V_0), we introduce a Sobolev space

$$
\mathcal{H} = \left\{ u \in H^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} V(x)|u|^2 < +\infty \right\}
$$

with its associated norm

$$
\|u\|_{\mathcal{H}} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2 + V(x)|u|^2) \right)^{1/2}.
$$

THEOREM 1.4. *Assume that $N \geq 1$, $\alpha \in (0,N)$ and (V_0) holds. Set*

$$e_c = \inf_{u\in\widetilde{S}(c)} E(u), \tag{1.10}$$

where $\widetilde{S}(c) = \{u \in \mathcal{H} \mid |u|_2 = c\}$. Let c_* be given by Theorem 1.1.

(a) *If $0 < c < c_*$, then e_c has at least one minimizer and $e_c > 0$.*
(b) *Let $N-2 \leq \alpha < N$ if $N \geq 3$ and $0 < \alpha < N$ if $N = 1,2$, then for each $c \geq c_*$, e_c has no minimizer. Moreover,*

$$e_c = \begin{cases} 0 & \text{if } c = c_* \\ -\infty & \text{if } c > c_*. \end{cases}$$

(c) $\lim\limits_{c\to(c_*)^-} e_c = e_{c_*}$.

We also study concentration phenomena of minimizers of e_c as c converges to c_* from below. Let u_c be a minimizer of e_c for each $0 < c < c_*$, then, by (1.6) and Theorem 1.4, we see that $\int_{\mathbb{R}^N} V(x)|u_c|^2 \to 0$ as $c \to (c_*)^-$, i.e. u_c can be expected to concentrate at the minimum of V. To show this fact, besides condition (V_0), we assume that there exist $m \geq 1$ distinct points $x_i \in \mathbb{R}^N$ and $q_i > 0$ $(1 \leq i \leq m)$ such that

$$(\mathrm{V}_1) \qquad V > 0 \quad \text{in } \mathbb{R}^N \setminus \{x_1,\ldots,x_m\} \quad \text{and} \quad \mu_i := \lim_{x\to x_i} \frac{V(x)}{|x-x_i|^{q_i}} \in (0,+\infty).$$

Set $q := \max\{q_1,\ldots,q_m\}$. Let $\{c_k\} \subset (0,c_*)$ be a sequence such that $c_k \to c_*$ as $k \to +\infty$.

THEOREM 1.5. *Suppose that $N \geq 1$, $\alpha \in [N-2,N)$ if $N \geq 3$ and $\alpha \in (0,N)$ if $N = 1,2$, and (V_0) and (V_1) hold. Then there exist a sequence $\{y_k\} \subset \mathbb{R}^N$ and a groundstate solution W_0 of the following equation:*

$$-\Delta W_0 + W_0 = (I_\alpha * |W_0|^{(N+\alpha+2)/N})|W_0|^{(N+\alpha+2)/N-2}W_0, \quad x \in \mathbb{R}^N, \tag{1.11}$$

$$\lambda := \min_{1\leq i\leq m}\left\{\lambda_i \;\middle|\; \lambda_i = \left(\frac{q_i\mu_i}{2c_*^2}\int_{\mathbb{R}^N}|x|^{q_i}|W_0(x)|^2\right)^{1/(q_i+2)}\right\},$$

such that up to a subsequence

$$\lim_{k\to+\infty}\frac{e_{c_k}}{[1-(c_k/c_*)^{2(\alpha+2)/N}]^{q/(q+2)}} = \frac{\lambda^2c_*^2(q+2)}{2q}\left(\frac{N}{\alpha+2}\right)^{q/q+2} \tag{1.12}$$

and

$$\begin{aligned} &\left[1-\left(\frac{c_k}{c_*}\right)^{2(\alpha+2)/N}\right]^{N/(2(q+2))} u_{c_k} \\ &\qquad\cdot\left(\left[1-\left(\frac{c_k}{c_*}\right)^{(2(\alpha+2))/N}\right]^{1/(q+2)} x + y_k\right) \\ &\qquad\qquad\to \left[\left(\frac{\alpha+2}{N}\right)^{1/(q+2)}\lambda\right]^{N/2} W_0\left(\left(\frac{\alpha+2}{N}\right)^{1/(q+2)}\lambda x\right) \end{aligned} \tag{1.13}$$

in $L^{2Ns/(N+\alpha)}(\mathbb{R}^N)$ *for* $(N+\alpha)/N \le s < (N+\alpha)/(N-2)+$ *as* $k \to +\infty$*. Here* W_0 *is, up to translations, radially symmetric about the origin. Moreover, there exists* $x_{j_0} \in \{x_i \mid \lambda_i = \lambda,\ 1 \le i \le m\}$*, such that* $y_k \to x_{j_0}$ *as* $k \to +\infty$*.*

REMARK 1.6. (a) It has been proved in [22] that for $\alpha \in [N-2, N)$ if $N \ge 3$ and $\alpha \in (0,N)$ if $N = 1,2$, each groundstate solution u of (1.11) satisfies

$$\lim_{|x|\to+\infty} |u(x)||x|^{(N-1)/2} e^{|x|} \in (0,+\infty).$$

Hence $\lambda_i \in (0,+\infty)$.

(b) By Remark 1.3 (a), $W_0 = Q_{(N+\alpha+2)/N}$ if $N = 4$, $\alpha = 2$ (see Remark 3.3 below).

The result of Theorem 1.5 is different from that in [19] studying $E(u)$ by replacing $p = (N+\alpha+2)/N$ with $p < (N+\alpha+2)/N$, where one considered the concentration behavior of minimizers as $c \to +\infty$. Concentration phenomena have also been studied in [24] and [4] by considering semiclassical limit of the Choquard equation

$$-\varepsilon^2 \Delta u + Vu = \varepsilon^{-\alpha}(I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

However, since the parameter is different, we need a different technique to obtain our result.

The proof of Theorem 1.5 is based on optimal energy estimates of e_c and $\int_{\mathbb{R}^N} |\nabla u_c|^2$ for each minimizer u_c. The main idea to prove Theorem 1.5 comes from [8], which was restricted to the case of local nonlinearities. But due to the fact that our nonlinearity is nonlocal and that the assumption imposed on V is more general than that in [8], the method used in [8] cannot be directly applied here. It needs some improvements and careful analysis. First, by choosing a suitable test function, we get that $0 < e_c \le C_1[1-(c/c_*)^{(2(\alpha+2))/N}]^{q/(q+2)}$ as $c \to (c_*)^-$ for some constant $C_1 > 0$ independent of c. The lower bound now is not optimal. The method in [8] which uses the perturbation term $\int_{\mathbb{R}^N} V(x)u^2$ to remove the local nonlinearity term does not work in our case. To overcome this difficulty, we notice that if there exists $\{\varepsilon_c\} \subset \mathbb{R}_+$ with $\lim\limits_{c\to(c_*)^-} \varepsilon_c = 0$ such that

$$C_2 \le \int_{\mathbb{R}^N} |\nabla \widetilde{w}_c(x)|^2 \le C_3, \tag{1.14}$$

where $\widetilde{w}_c(x) = \varepsilon_c^{N/2} u_c(\varepsilon_c x)$ and $C_3 > C_2 > 0$ are two constants independent of c, then there exist $\{y_c\} \subset \mathbb{R}^N$, $x_{j_0} \in \{x_1,\ldots,x_m\}$, a groundstate solution $W_0 \in H^1(\mathbb{R}^N)$ to (1.11) and a constant $\beta > 0$ such that $\varepsilon_c y_c \to x_{j_0}$ and $w_c(x) := \widetilde{w}_c(x+y_c) \to (\beta)^{N/2} W_0(\beta x)$ in $H^1(\mathbb{R}^N)$. Moreover, if

$$\lim_{c\to(c_*)^-} \frac{1-(c/c_*)^{2(\alpha+2)/N}}{\varepsilon_c^{q+2}} = 1, \tag{1.15}$$

then

$$\lim_{c\to(c_*)^-}\frac{e_c}{\varepsilon_c^q}=\frac{\lambda^2c_*^2(q+2)}{2q}\left(\frac{N}{\alpha+2}\right)^{q/(q+2)},$$

which implies that (1.13) holds. So it is enough to prove that (1.14) and (1.15) hold. However, we cannot directly obtain (1.14) if we just take

$$\varepsilon_c=\left[1-\left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}\right]^{1/(q+2)}$$

there. In other words, one should first obtain an optimal energy estimate of $\int_{\mathbb{R}^N}|\nabla u_c|^2$ (see Lemma 3.8 below). We succeeded in doing so by noticing that $\int_{\mathbb{R}^N}|\nabla u_c|^2\to+\infty$ as $c\to(c_*)^-$ and choosing suitable scalings. Such arguments were also used in [9], [10] (see also [6]) to deal with local equations.

Throughout this paper, we use standard notations. For simplicity, we write $\int_\Omega h$ to mean the Lebesgue integral of h over a domain $\Omega\subset\mathbb{R}^N$. $L^p:=L^p(\mathbb{R}^N)$ $(1\le p<+\infty)$ is the usual Lebesgue space with the standard norm $|\cdot|_p$. We use "$\to$ " and "$\rightharpoonup$ " to denote the strong and weak convergence in the related function space respectively. C will denote a positive constant unless specified. We use ":=" to denote definitions. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

The paper is organized as follows. In Section 2, we determine the best constant for the interpolation estimate (1.6) and give the proof of Theorem 1.1. In Section 3, we prove Theorems 1.4 and 1.5.

2. Proof of interpolation estimate (1.6) and Theorem 1.1

In this section, we first prove the interpolation estimate (1.6). It is enough to consider the following minimization problem:

$$S_p=\inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}}W_p(u),$$

where

$$W_p(u)=\frac{\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{(Np-(N+\alpha))/2}\left(\int_{\mathbb{R}^N}|u|^2\right)^{(N+\alpha-(N-2)p)/2}}{\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p}.$$

Lemma 2.1 ([22, Lemma 2.4]). *Let $N\ge1$, $\alpha\in(0,N)$, $p\in[1,2N/(N+\alpha))$ and $\{u_n\}$ be a bounded sequence in $L^{2Np/(N+\alpha)}(\mathbb{R}^N)$. If $u_n\to u$ almost everywhere in $\mathbb{R}^N$ as $n\to+\infty$, then*

$$\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}(I_\alpha*|u_n|^p)|u_n|^p-\int_{\mathbb{R}^N}(I_\alpha*|u_n-u|^p)|u_n-u|^p\right)=\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p.$$

Lemma 2.2. (a) ([29], Brezis Lemma) *Let Ω be an open subset of $\mathbb{R}^N$ and let $\{u_n\} \subset L^p(\Omega)$, $1 \le p < \infty$. If $\{u_n\}$ is bounded in $L^p(\Omega)$ and $u_n \to u$ almost everywhere on Ω, then*

$$\lim_{n\to+\infty}(|u_n|_p^p - |u_n - u|_p^p) = |u|_p^p.$$

(b) ([29], Vanishing Lemma) *Let $r > 0$ and $2 \le q < 2^*$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N}\int_{B_r(y)} |u_n|^q \to 0, \quad n \to +\infty,$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$.

Lemma 2.3. *Let $N \ge 1$, $\alpha \in (0,N)$ and $(N+\alpha)/N < p < (N+\alpha)/(N-2)_+$, then S_p has a minimizer $Q_p \in H^1(\mathbb{R}^N)$, where Q_p is a nontrivial solution of equation (1.7) and*

$$S_p = \frac{|Q_p|_2^{2p-2}}{p}.$$

Proof. The lemma can be viewed as a consequence of Proposition 2.1 in [22] and Theorem 9 in [14], but we give an alternative proof here. The idea of the proof comes from [28] (see also [2]), but some details are delicate.

Since $W_p(u) \ge 0$ for any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, S_p is well-defined. Let $\{v_n\} \subset H^1(\mathbb{R}^N) \setminus \{0\}$ be a minimizing sequence for S_p, without loss of generality, we can assume that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 = \int_{\mathbb{R}^N} |v_n|^2 = 1.$$

Up to a subsequence, let

$$\delta := \lim_{n\to+\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)} |v_n|^2.$$

If $\delta = 0$, then by Lemma 2.2, $v_n \to 0$ in $L^s(\mathbb{R}^N)$, $2 < s < 2^*$. Hence by the Hardy–Littlewood–Sobolev inequality (1.4),

$$W_p(v_n) = \frac{1}{\displaystyle\int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p} \to +\infty,$$

which is a contradiction. So $\delta > 0$ and there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |v_n|^2 \ge \frac{\delta}{2} > 0. \tag{2.1}$$

Up to translations, we may assume that $y_n = 0$. Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, by (2.1), there exists $v_p \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $v_n \rightharpoonup v_p$ in $H^1(\mathbb{R}^N)$.

Then by the Brezis Lemma and Lemma 2.1, we have

$$\begin{aligned} S_p \leq & W_p(v_p) \\ \leq & \lim_{n\to+\infty}\left[W_p(v_n)\frac{\int_{\mathbb{R}^N}(I_\alpha * |v_n|^p)|v_n|^p}{\int_{\mathbb{R}^N}(I_\alpha * |v_p|^p)|v_p|^p}\right. \\ & \left. - W_p(v_n - v_p)\frac{\int_{\mathbb{R}^N}(I_\alpha * |v_n - v_p|^p)|v_n - v_p|^p}{\int_{\mathbb{R}^N}(I_\alpha * |v_p|^p)|v_p|^p}\right] \\ \leq & S_p \lim_{n\to+\infty}\left(\frac{\int_{\mathbb{R}^N}(I_\alpha * |v_n|^p)|v_n|^p - \int_{\mathbb{R}^N}(I_\alpha * |v_n - v_p|^p)|v_n - v_p|^p}{\int_{\mathbb{R}^N}(I_\alpha * |v_p|^p)|v_p|^p}\right) = S_p, \end{aligned}$$

i.e. $W_p(v_p) = S_p$. Moreover, $|\nabla v_p|_2 = |v_p|_2 = 1$ and $S_p = 1/\int_{\mathbb{R}^N}(I_\alpha * |v_p|^p)|v_p|^p$. Therefore, for any $h \in H^1(\mathbb{R}^N)$, $\frac{d}{dt}\big|_{t=0} W_p(v_p + th) = 0$, i.e. v_p satisfies the following equation:

$$-[Np - (N+\alpha)]\Delta v_p + [N + \alpha - (N-2)p]v_p = 2pS_p(I_\alpha * |v|^p)|v_p|^{p-2}v_p,$$

in $\mathbb{R}^N$. Let $v_p = (1/pS_p)^{1(2p-2)}Q_p$, then Q_p is a nontrivial solution of (1.7) and $S_p = |Q_p|_2^{2p-2}/p$. $\square$

Next we give the proof of Theorem 1.1. For any $u \in H^1(\mathbb{R}^N)$, set

$$A(u) := \int_{\mathbb{R}^N}|\nabla u|^2, \qquad B(u) := \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p,$$

then $I_p(u) = A(u)/2 - B(u)/(2p)$. It follows from (1.6) and (1.7) that for $(N+\alpha)/N < p < (N+\alpha)/(N-2)_+$,

$$B(u) \leq \frac{p}{|Q_p|_2^{2p-2}} A(u)^{(Np-(N+\alpha))/2}|u|_2^{N+\alpha-(N-2)p} \tag{2.2}$$

with equality for $u = Q_p$ given in (1.7), moreover,

$$A(Q_p) = \frac{1}{p}B(Q_p) = |Q_p|_2^2. \tag{2.3}$$

LEMMA 2.4. *Let $N \geq 1$ and $\alpha \in (0, N)$.*

(a) *If $(N+\alpha)/N < p < (N+\alpha+2)/N$, then $I_p(u)$ is bounded from below and coercive on $S(c)$ for all $c > 0$, moreover, $I_p(c^2) < 0$.*

(b) *If $p = (N+\alpha+2)/N$, then*

$$I_{(N+\alpha+2)/N}(c^2) = \begin{cases} 0 & \text{if } 0 < c \leq c_* := |Q_{(N+\alpha+2)/N}|_2, \\ -\infty & \text{if } c > c_*. \end{cases}$$

(c) *If* $(N+\alpha+2)/N < p < (N+\alpha)/(N-2)_+$, *then* $I_p(c^2) = -\infty$ *for all* $c > 0$.

PROOF. (a) For any $c > 0$ and $u \in S(c)$, by (2.2), there exists $C := (c^{N+\alpha-(N-2)p})/|Q_p|_2^{2p-2}$ such that

$$I_p(u) \geq \frac{A(u) - CA(u)^{(Np-(N+\alpha))/2}}{2}. \tag{2.4}$$

Since $(N+\alpha)/N < p < (N+\alpha+2)/N$, $0 < Np-(N+\alpha) < 2$. Then (2.4) implies that $I_p(u)$ is bounded from below and coercive on $S(c)$ for any $c > 0$.

Set $u^t(x) := t^{N/2}u(tx)$ with $t > 0$, then $u^t \in S(c)$ and

$$I_p(u^t) = \frac{t^2}{2}A(u) - \frac{t^{Np-(N+\alpha)}}{2p}B(u) < 0 \quad \text{for } t > 0 \text{ small enough}, \tag{2.5}$$

since $0 < Np-(N+\alpha) < 2$, which implies that $I_p(c^2) < 0$ for each $c > 0$.

(b) When $p = (N+\alpha+2)/N$, $Np-(N+\alpha) = 2$, similarly to (2.4) and (2.5), we have

$$I_{(N+\alpha+2)/N}(u) \geq \frac{A(u)}{2}\left[1 - \left(\frac{c}{c_*}\right)^{(2(\alpha+2))/N}\right] \geq 0 \quad \text{if } 0 < c \leq c_*,$$

and $I_{(N+\alpha+2)/N}(c^2) \leq I_{(N+\alpha+2)/N}(u^t) \to 0$ as $t \to 0^+$ for all c. Then, if $0 < c \leq c_*$, $I_{(N+\alpha+2)/N}(c^2) = 0$.

If $c > c_*$, set $Q^t(x) := (ct^{N/2}/c_*)Q_{(N+\alpha+2)/N}(tx)$, then by (2.3),

$$I_{(N+\alpha+2)/N}(Q^t) = \frac{c^2t^2}{2c_*^2}\left[1 - \left(\frac{c}{c_*}\right)^{(2(\alpha+2))/N}\right] \to -\infty \quad \text{as } t \to +\infty,$$

then $I_{(N+\alpha+2)/N}(c^2) = -\infty$ for $c > c_*$.

(c) If $(N+\alpha+2)/N < p < (N+\alpha)/(N-2)_+$, then $Np-(N+\alpha) > 2$, hence by (2.5), we have $I_p(u^t) \to -\infty$ as $t \to +\infty$, so $I_p(c^2) = -\infty$ for all $c > 0$. □

LEMMA 2.5. *If* $(N+\alpha)/N < p < (N+\alpha+2)/N$, *then*

(a) *the function* $c \mapsto I_p(c^2)$ *is continuous on* $(0,+\infty)$;

(b) $I_p(c^2) < I_p(\alpha^2) + I_p(c^2-\alpha^2)$ *for all* $0 < \alpha < c < +\infty$.

PROOF. For any $c > 0$ and $u \in S(c)$, we have $\theta u \in S(\theta c)$ with $\theta > 0$ and

$$I_p(\theta u) - \theta^2 I(u) = \frac{\theta^2 - \theta^{2p}}{2p}B(u). \tag{2.6}$$

(a) If $\lim_{n\to+\infty} c_n = c$, let $\{u_n\} \subset S(c)$ be a minimizing sequence for $I_p(c^2)$, then by Lemma 2.4, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence by replacing u and θ in (2.6) by u_n and $\theta_n = c_n/c$, we see that $\lim_{n\to+\infty} I_p(c_n^2) \leq I_p(c^2)$. On the other hand, let $\{u_n\} \subset S(c_n)$ be such that $I_p(u_n) \leq I_p(c_n^2)+1/n < 1/n$, then similarly, $I_p(c^2) \leq \lim_{n\to+\infty} I_p(c_n^2)$, which implies the lemma.

(b) Let $\{u_n\} \subset S(c)$ be a bounded minimizing sequence for $I_p(c^2)$. Since $I_p(c^2) < 0$, there exists $K_1 > 0$ such that $B(u_n) \geq K_1$. Let $\theta > 1$ in (2.6), then we have $I_p(\theta u_n) - \theta^2 I(u_n) \leq (\theta^2 - \theta^{2p})K_1/(2p) < 0$, which implies that $I_p(\theta^2 c^2) < \theta^2 I_p(c^2)$ for each $\theta > 1$. Without loss of generality, we may assume that $0 < \alpha < \sqrt{c^2 - \alpha^2}$, then

$$\begin{aligned} I_p(c^2) &< \frac{c^2}{c^2 - \alpha^2} I_p(c^2 - \alpha^2) \\ &= I_p(c^2 - \alpha^2) + \frac{\alpha^2}{c^2 - \alpha^2} I_p(c^2 - \alpha^2) < I_p(c^2 - \alpha^2) + I_p(\alpha^2). \qquad \square \end{aligned}$$

LEMMA 2.6. *Let $N \geq 1$, $\alpha \in (0, N)$ and $(N+\alpha)/N < p < (N+\alpha)/(N-2)_+$. If u is a critical point of $I_p(u)$ constrained on $S(c)$, then there exists $\mu_c < 0$ such that $I_p'(u) - \mu_c u = 0$ in $H^{-1}(\mathbb{R}^N)$ and*

$$A(u) - \frac{Np - (N+\alpha)}{2p} B(u) = 0.$$

PROOF. Since $(I_p|_{S(c)})'(u) = 0$, there exists $\mu_c \in \mathbb{R}$ such that $I_p'(u) - \mu_c u = 0$ in $H^{-1}(\mathbb{R}^N)$. Then $A(u) - B(u) = \mu_c c^2$. By Proposition 3.5 in [23], u satisfies the following Pohozhaev identity:

$$\frac{N-2}{2} A(u) - \frac{N+\alpha}{2p} B(u) = \frac{N}{2} \mu_c c^2.$$

Hence

$$A(u) = \frac{Np - (N+\alpha)}{2p} B(u) \quad \text{and} \quad \mu_c = \frac{(N-2)p - (N+\alpha)}{2pc^2} B(u) < 0. \qquad \square$$

PROOF OF THEOREM 1.1. (a) If $p = (N+\alpha)/N$, for any $c > 0$ and $u \in S(c)$, by (1.8) we have

$$I_{(N+\alpha)/N}(u) \geq -\frac{N}{2(N+\alpha)} \left(\frac{c}{|Q_{(N+\alpha)/N}|_2} \right)^{(2(N+\alpha))/N}.$$

Set

$$Q^t_{(N+\alpha)/N}(x) := \frac{ct^{N/2}}{|Q_{(N+\alpha)/N}|_2} Q_{(N+\alpha)/N}(tx),$$

then, by (1.8) again, we see that

$$\begin{aligned} &I_{(N+\alpha)/N}(Q^t_{(N+\alpha)/N}) \\ &\quad = \frac{c^2 t^2}{2|Q_{(N+\alpha)/N}|_2^2} A(Q_{(N+\alpha)/N}) - \frac{N}{2(N+\alpha)} \left(\frac{c}{|Q_{(N+\alpha)/N}|_2} \right)^{(2(N+\alpha))/N}, \end{aligned}$$

letting $t \to 0^+$, then

$$I_{(N+\alpha)/N}(c^2) = -\frac{N}{2(N+\alpha)} \left(\frac{c}{|Q_{(N+\alpha)/N}|_2} \right)^{(2(N+\alpha))/N}.$$

By contradiction, if for some $c > 0$, there is $u \in S(c)$ such that $I_{(N+\alpha)/N}(u) = I_{(N+\alpha)/N}(c^2)$, then (1.8) shows that

$$0 \leq \frac{1}{2}A(u) = \frac{N}{2(N+\alpha)}\left[B(u) - \left(\frac{c}{|Q_{(N+\alpha)/N}|_2}\right)^{(2(N+\alpha))/N}\right] \leq 0,$$

which implies that $u = 0$. It is a contradiction. So $I_{(N+\alpha)/N}(c^2)$ has no minimizer for all $c > 0$.

(b) If $(N+\alpha)/N < p < (N+\alpha+2)/N$, for any $c > 0$, by Lemma 2.4, $I_p(c^2) < 0$. Let $\{u_n\} \subset S(c)$ be a minimizing sequence for $I_p(c^2)$, then Lemma 2.4 (a) implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and for some constant $C > 0$ independent of n, $B(u_n) \geq C$. Hence there exists $u \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^N), \qquad u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{2.7}$$

Moreover, by the Vanishing Lemma, up to translations, we may assume that $u \neq 0$. Then $0 < |u|_2 := \alpha \leq c$. We just suppose that $\alpha < c$, then $u \in S(\alpha)$. By (2.7) and the Brezis Lemma, we have

$$\lim_{n\to+\infty} |u_n - u|_2^2 = \lim_{n\to+\infty} |u_n|_2^2 - |u|_2^2 = c^2 - \alpha^2.$$

Then by Lemmas 2.1 and 2.5 (a), we have

$$I_p(c^2) = \lim_{n\to+\infty} I_p(u_n) = \lim_{n\to+\infty} I_p(u_n - u) + I_p(u) \geq I_p(c^2 - \alpha^2) + I_p(\alpha^2),$$

which contradicts Lemma 2.5 (b). So $|u|_2 = c$, i.e. $u_n \to u$ in $L^2(\mathbb{R}^N)$. By (2.2), we have $B(u_n) \to B(u)$. Then $I_p(c^2) \leq I_p(u) \leq \lim_{n\to+\infty} I_p(u_n) = I_p(c^2)$, i.e. u is minimizer for $I_p(c^2)$.

(c) (i) has been proved in Lemma 2.4 (b). To prove (b), suppose by contradiction that there exists $c_0 \in (0, c_*)$ such that $I_{(N+\alpha+2)/N}(c_0^2)$ has a minimizer $u_0 \in S(c_0)$, i.e. $I_{(N+\alpha+2)/N}(u_0) = I_{(N+\alpha+2)/N}(c_0^2) = 0$, then by (2.2),

$$A(u_0) = \frac{N}{N+\alpha+2}B(u_0) \leq \left(\frac{c_0}{c_*}\right)^{(2(\alpha+2))/N} A(u_0) < A(u_0),$$

which is impossible. So combining (i), we see that $I_{(N+\alpha+2)/N}(c^2)$ has no minimizer for all $c \neq c_*$.

By (2.3), we see that $I_{(N+\alpha+2)/N}(Q_{(N+\alpha+2)/N}) = 0 = I_{(N+\alpha+2)/N}(c_*^2)$, i.e. $Q_{(N+\alpha+2)/N}$ is a minimizer for $I_{(N+\alpha+2)/N}(c_*^2)$. Moreover, by Lemmas 3.1 (b) and 3.2 below, each groundstate solution of (1.7) is a minimizer of $I_{(N+\alpha+2)/N}(c_*^2)$. So we have proved (iii).

For any $c > 0$, suppose that u is a critical point of $I_{(N+\alpha+2)/N}(u)$ constrained on $S(c)$, then by (2.3) and Lemma 2.6, we have

$$A(u) = \frac{N}{N+\alpha+2}B(u) \leq \left(\frac{c}{c_*}\right)^{(2(\alpha+2))/N} A(u),$$

which implies that $c_* \le c$. Then, there exists no critical point for $I_{(N+\alpha+2)/N}(u)$ constrained on $S(c)$ if $0 < c < c_*$. So (iv) is proved.

(d) By Lemma 2.4 (c), $I_p(c^2)$ has no minimizer for all $c > 0$ if $(N+\alpha+2)/N < p < (N+\alpha)/(N-2)_+$. $\square$

3. Proof of Theorems 1.4 and 1.5

For $p = (N+\alpha+2)/N$, (2.2) turns to be

$$B(u) \le \frac{N+\alpha+2}{N}\left(\frac{1}{c_*}\right)^{(2(\alpha+2))/N} A(u)|u|_2^{(2(\alpha+2))/N}, \tag{3.1}$$

with equality for $u = Q_{(N+\alpha+2)/N}$ and $c_* := |Q_{(N+\alpha+2)/N}|_2$, where $Q_{(N+\alpha+2)/N}$ is a nontrivial solution of

$$\begin{aligned}(3.2)\qquad & -\Delta Q_{(N+\alpha+2)/N} + \frac{\alpha+2}{N} Q_{(N+\alpha+2)/N} \\ &= (I_\alpha * |Q_{(N+\alpha+2)/N}|^{(N+\alpha+2)/N})|Q_{(N+\alpha+2)/N}|^{(N+\alpha+2)/N-2}Q_{(N+\alpha+2)/N},\end{aligned}$$

in $\mathbb{R}^N$. Set

$$Q_{(N+\alpha+2)/N}(x) = \left(\sqrt{\frac{\alpha+2}{N}}\right)^{N/2} \widetilde{Q}_{(N+\alpha+2)/N}\left(\sqrt{\frac{\alpha+2}{N}}x\right),$$

then $\widetilde{Q}_{(N+\alpha+2)/N}$ satisfies the equation

$$\begin{aligned}(3.3)\qquad & -\Delta \widetilde{Q}_{(N+\alpha+2)/N} + \widetilde{Q}_{(N+\alpha+2)/N} \\ &= (I_\alpha * |\widetilde{Q}_{(N+\alpha+2)/N}|^{(N+\alpha+2)/N})|\widetilde{Q}_{(N+\alpha+2)/N}|^{(N+\alpha+2)/N-2}\widetilde{Q}_{(N+\alpha+2)/N},\end{aligned}$$

in $\mathbb{R}^N$. The following lemma is a direct consequence of Theorems 1–4 in [22].

Lemma 3.1. *Assume that $N \ge 1$ and $\alpha \in (0, N)$.*

(a) *There is at least one groundstate solution $u \in H^1(\mathbb{R}^N)$ to* (3.3) *with*

$$F(u) = d := \inf\{F(v \mid v \in H^1(\mathbb{R}^N) \setminus \{0\} \textit{ is a weak solution of } (3.3)\},$$

where

$$F(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2) - \frac{N}{2(N+\alpha+2)}\int_{\mathbb{R}^N} (I_\alpha * |v|^{(N+\alpha+2)/N})|v|^{(N+\alpha+2)/N}.$$

(b) *If $u \in H^1(\mathbb{R}^N)$ is a nontrivial solution of* (3.3)*, then $u \in L^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$, $u \in W^{2,s}(\mathbb{R}^N)$ for every $s > 1$ and $u \in C^\infty(\mathbb{R}^N \setminus u^{-1}(\{0\})$. Moreover,*

$$\frac{N+\alpha+2}{N} A(u) = \frac{N+\alpha+2}{\alpha+2}\int_{\mathbb{R}^N} |u|^2 = B(u). \tag{3.4}$$

(c) *If u is a groundstate solution of* (3.3)*, then u is either positive or negative and there exists $x_0 \in \mathbb{R}^N$ and a monotone function $v \in C^\infty(0,+\infty)$ such that*

$$u(x) = v(|x - x_0|), \quad \text{for all } x \in \mathbb{R}^N.$$

(d) *Let $N-2 \le \alpha < N$ if $N \ge 3$ and $0 < \alpha < N$ if $N = 1,2$. If u is a groundstate solution of* (3.3)*, then*

$$\lim_{|x|\to+\infty} |u(x)||x|^{(N-1)/2} e^{|x|} \in (0,+\infty).$$

Moreover, $|\nabla u(x)| = O(|x|^{-(N-1)/2} e^{-|x|})$ as $|x| \to +\infty$.

Lemma 3.2.

(a) $d = c_*^2/2$.

(b) *u is a nontrivial solution of* (3.3) *with $|u|_2 = c_*$ if and only if u is a groundstate solution.*

Proof. For any nontrivial solution u of (3.3), by Lemma 3.1 (a)–(b) and (3.1), we have

$$c_* \le |u|_2 \quad \text{and} \quad d \le F(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2,$$

where equality holds only if u is a groundstate solution. In particular, since $\widetilde{Q}_{(N+\alpha+2)/N}$ is a nontrivial solution of (3.3),

$$d \le F(\widetilde{Q}_{(N+\alpha+2)/N}) = \frac{|\widetilde{Q}_{(N+\alpha+2)/N}|_2^2}{2} = \frac{c_*^2}{2}.$$

Therefore, if u is a groundstate solution of (3.3), then, by Lemma 3.1 (c), u is nontrivial and

$$\frac{c_*^2}{2} \le \frac{|u|_2^2}{2} = F(u) = d \le \frac{c_*^2}{2},$$

which shows that $d = c_*^2/2$ and $|u|_2 = c_*$. On the other hand, if u is a nontrivial solution of (3.3) with $|u|_2 = c_*$, then

$$\frac{c_*^2}{2} = d \le F(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 = \frac{c_*^2}{2},$$

which implies that $F(u) = d$, i.e. u is a groundstate solution. □

Remark 3.3. $\widetilde{Q}_{(N+\alpha+2)/N}$ is a groundstate solution of (3.3).

Lemma 3.4 ([1]). *Suppose that $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $\lim_{|x|\to+\infty} V(x) = +\infty$, then the embedding $\mathcal{H} \hookrightarrow L^s(\mathbb{R}^N)$, $2 \le s < 2^*$, is compact.*

Proof of Theorem 1.4. Set

$$C(u) := \int_{\mathbb{R}^N} V(x)|u|^2 \ge 0, \quad \text{for all } u \in H^1(\mathbb{R}^N),$$

then

$$E(u) = \frac{A(u)}{2} + \frac{C(u)}{2} - \frac{N}{2(N+\alpha+2)} B(u).$$

(a) By (3.1), for any $0 < c \le c_*$ and $u \in \widetilde{S}(c)$,

$$E(u) \ge \frac{1}{2}\left[1 - \left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}\right] A(u) + \frac{1}{2} C(u) \ge 0, \tag{3.5}$$

then $e_c = \inf_{u \in \widetilde{S}(c)} E(u) \ge 0$ is well defined for $0 < c \le c_*$.

For each $0 < c < c_*$, let $\{u_n\} \subset \widetilde{S}(c)$ be a minimizing sequence for e_c, then by (3.5), $\{u_n\}$ is bounded in $\mathcal{H}$. Hence there exists $u_c \in \mathcal{H}$ such that $u_n \rightharpoonup u_c$ in $\mathcal{H}$. By Lemma 3.4, $u_n \to u_c$ in $L^s(\mathbb{R}^N)$, $2 \le s < 2^*$, which implies that $|u_c|_2 = c$ and $B(u_n) \to B(u_c)$. So $e_c \le E(u_c) \le \lim_{n\to+\infty} E(u_n) = e_c$, i.e. $u_c \in \widetilde{S}(c)$ is a minimizer of e_c. Moreover, by (3.5), $e_c > 0$. So $e_c > 0$ has at least one minimizer for all $0 < c < c_*$.

(b) Let $N - 2 \le \alpha < N$ if $N \ge 3$ and $0 < \alpha < N$ if $N = 1, 2$. For any $c > 0$, let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi(x) \le 1$, $\varphi(x) \equiv 1$ for $|x| \le 1$, $\varphi(x) \equiv 0$ for $|x| \ge 2$ and $|\nabla\varphi| \le 2$. For any $x_0 \in \mathbb{R}^N$ and any $t > 0$, set

$$\widetilde{Q}^t(x) = \frac{cA_t t^{N/2}}{c_*} \varphi(x - x_0) \widetilde{Q}_{(N+\alpha+2)/N}(t(x - x_0)), \tag{3.6}$$

where $A_t > 0$ is chosen to satisfy that $|\widetilde{Q}^t|_2 = c$. By the exponential decay of $\widetilde{Q}_{(N+\alpha+2)/N}$, we see that

$$\frac{1}{A_t^2} = 1 + \frac{1}{c_*^2} \int_{\mathbb{R}^N} \left(\varphi^2\left(\frac{x}{t}\right) - 1\right) |\widetilde{Q}_{(N+\alpha+2)/N}(x)|^2 \to 1$$

as $t \to +\infty$. Then A_t depends only on t and $\lim_{t\to+\infty} A_t = 1$. Since $V(x)\varphi^2(x - x_0)$ is bounded and has compact support, $C(\widetilde{Q}^t) \to (c^2/c_*^2)V(x_0)$.

$$\begin{aligned}
B(\widetilde{Q}^t) = {} & \left(\frac{cA_t}{c_*}\right)^{2(N+\alpha+2)/N} t^2 \Bigg\{ B(\widetilde{Q}_{(N+\alpha+2)/N}) \\
& + \int_{\mathbb{R}^N} \left\{ I_\alpha \left[\left(\left|\varphi\left(\frac{y}{t}\right)\right|^{(N+\alpha+2)/N} - 1 \right) |\widetilde{Q}_{(N+\alpha+2)/N}(y)|^{(N+\alpha+2)/N} \right] \right\} \\
& \cdot \left(\left|\varphi\left(\frac{x}{t}\right)\right|^{(N+\alpha+2)/N} + 1 \right) |\widetilde{Q}_{(N+\alpha+2)/N}(x)|^{(N+\alpha+2)/N} \Bigg\} \\
:= {} & \left(\frac{cA_t}{c_*}\right)^{2(N+\alpha+2)/N} t^2 [B(\widetilde{Q}_{(N+\alpha+2)/N}) + f_1(t)].
\end{aligned}$$

By the Hardy–Littlewood–Sobolev inequality (1.4) and the exponential decay of $\widetilde{Q}_{(N+\alpha+2)/N}$, there exists a constant $C>0$ such that

$$\begin{aligned}|f_1(t)| \le & C\bigg(\int_{\mathbb{R}^N}\bigg|\bigg[\varphi\bigg(\frac{x}{t}\bigg)\bigg]^{(N+\alpha+2)/N}-1\bigg|^{2N/(N+\alpha)} \\ & \cdot|\widetilde{Q}_{(N+\alpha+2)/N}(x)|^{2(N+\alpha+2)/(N+\alpha)}\bigg)^{(N+\alpha)/(2N)} \\ \le & C\bigg(\int_{|x|\ge t}|\widetilde{Q}_{(N+\alpha+2)/N}(x)|^{2(N+\alpha+2)/(N+\alpha)}\bigg)^{(N+\alpha)/(2N)} \\ \le & C\bigg(\int_t^{+\infty} r^{-2(N-1)/(N+\alpha)}e^{-2(N+\alpha+2)r/(N+\alpha)}\bigg)^{(N+\alpha)/(2N)} \\ \le & Ct^{-(N-1)/N}e^{-(N+\alpha+2)t/N}\end{aligned}$$

as $t\to+\infty$. Then by the exponential decay of $\widetilde{Q}_{(N+\alpha+2)/N}$ and $|\nabla\widetilde{Q}_{(N+\alpha+2)/N}|$, we have

$$E(\widetilde{Q}^t)=\frac{c^2}{2c_*^2}t^2A(\widetilde{Q}_{(N+\alpha+2)/N})\bigg[1-\bigg(\frac{c}{c_*}\bigg)^{2(\alpha+2)/N}\bigg]+t^2f_2(t)+\frac{c^2}{2c_*^2}V(x_0) \tag{3.7}$$

as $t\to+\infty$, where $f_2(t)$ denotes a function satisfying $\lim_{t\to+\infty}|f_2(t)|t^r=0$ for all $r>0$.

If $c>c_*$, then by (3.7), $e_c\le\lim_{t\to+\infty}E(\widetilde{Q}^t)=-\infty$, hence $e_c=-\infty$ and there exists no minimizer for e_c.

If $c=c_*$, then by (3.5) and (3.7), $0\le e_{c_*}\le V(x_0)/2$. Taking the infimum over x_0, $e_{c_*}=0$. We just suppose that there exists $u\in\widetilde{S}(c_*)$ such that $E(u)=e_{c_*}$, then it follows from (3.5) that

$$C(u)=0, \tag{3.8}$$

which along with condition (V_0) imply that u must have compact support. Then $A(u)=B(u)$, i.e. u is a minimizer of $S_{(N+\alpha+2)/N}$. So u satisfies equation (3.2). Set $u(x):=\big(\sqrt{(\alpha+2)/N}\big)^{N/2}w\big(\sqrt{(\alpha+2)/N}x\big)$, then w is a nontrivial solution of (3.3) with $|w|_2=c_*$, hence, by Lemma 3.2, w is a groundstate solution. So, by Lemma 3.1 (d), $\lim_{|x|\to+\infty}|u(x)||x|^{(N-1)/2}e^{|x|}\in(0,+\infty)$, which contradicts (3.8). Moreover, we conclude from (3.6) and (3.7) that

$$\limsup_{c\to(c_*)^-}e_c\le\frac{V(x_0)}{2}\quad\text{as } t\to+\infty.$$

As x_0 is arbitrary, we have $\lim_{c\to(c_*)^-}e_c=0=e_{c_*}$. □

In what follows, we consider the concentration behavior of minimizers u_c as c approaches c_* from below when $N-2\le\alpha<N$ if $N\ge3$ and $0<\alpha<N$ if $N=1,2$ and the potential V satisfies conditions (V_0) and (V_1).

Lemma 3.5. *Suppose that* (V_0) *and* (V_1) *hold, then there exists a positive constant* M_1 *independent of* c *such that*

$$0 < e_c \leq M_1\bigg[1 - \bigg(\frac{c}{c_*}\bigg)^{2(\alpha+2)/N}\bigg]^{q/(q+2)} \qquad \text{as } c \to (c_*)^-,$$

where $q = \max\{q_1, \ldots, q_m\}$.

Proof. Without loss of generality, we may assume that $q = q_{i_0}$ for some $1 \leq i_0 \leq m$. By (V_1), there exists $R > 0$ small enough such that $V(x) \leq 2\mu_{i_0}|x - x_{i_0}|^{q_{i_0}}$ for $|x - x_{i_0}| \leq R$. Similarly to (3.6), let

$$u(x) := \frac{cA_{R,t}t^{N/2}}{c_*}\varphi\bigg(\frac{2(x - x_{i_0})}{R}\bigg)\widetilde{Q}_{(N+\alpha+2)/N}(t(x - x_{i_0})) \in \widetilde{S}(c),$$

where $A_{R,t} > 0$ and $A_{R,t} \to 1$ as $t \to +\infty$. Then

$$C(u) \leq \frac{2\mu_{i_0}c^2A_{R,t}^2}{c_*^2}t^{-q_{i_0}}\int_{\mathbb{R}^N}|x|^{q_{i_0}}|\widetilde{Q}_{(N+\alpha+2)/N}(x)|^2.$$

Hence similarly to (3.7), for large t,

$$\begin{aligned} e_c \leq \frac{A(\widetilde{Q}_{(N+\alpha+2)/N})}{2}\,t^2\bigg[1 - \bigg(\frac{c}{c_*}\bigg)^{2(\alpha+2)/N}\bigg] \\ + 2\mu_{i_0}t^{-q_{i_0}}\int_{\mathbb{R}^N}|x|^{q_{i_0}}|\widetilde{Q}_{(N+\alpha+2)/N}(x)|^2 + t^2h(t), \end{aligned}$$

where $\lim_{t\to+\infty}|h(t)|t^2 = 0$.

Taking $t = [1 - (c/c_*)^{2(\alpha+2)/N}]^{-1/(q_{i_0}+2)}$, there exists a constant $M_1 > 0$ independent of c such that $0 < e_c \leq M_1[1 - (c/c_*)^{2(\alpha+2)/N}]^{q/q+2}$. $\square$

The following lemma is essential to obtain the optimal lower bound of e_c and to prove the main theorem.

Lemma 3.6. *Suppose that* (V_0) *and* (V_1) *hold and there exists* $\{\varepsilon_c\} \subset \mathbb{R}_+$ *with* $\varepsilon_c \to 0^+$ *as* $c \to (c_*)^-$ *such that* $C_1 \leq A(\varepsilon_c^{N/2}u_c(\varepsilon_c x)) \leq C_2$, *where* $C_2 > C_1 > 0$ *are two constants independent of* c, *then there exist* $\{y_c\} \subset \mathbb{R}^N$, $x_{j_0} \in \{x_1, \ldots, x_m\}$ *and* $y_0 \in \mathbb{R}^N$ *such that* $(\varepsilon_c y_c - x_{j_0})/\varepsilon_c \to y_0$ *as* $c \to (c_*)^-$. *Moreover,*

(a) *let* $w_c(x) := \varepsilon_c^{N/2}u_c(\varepsilon_c x + \varepsilon_c y_c)$, *then there exist a groundstate solution* $W_0 \in H^1(\mathbb{R}^N)$ *of the following equation:*

$$-\Delta W_0 + W_0 = (I_\alpha * |W_0|^{(N+\alpha+2)/N})|W_0|^{(N+\alpha+2)/N-2}W_0, \qquad x \in \mathbb{R}^N, \tag{3.9}$$

and a constant $\beta > 0$ *such that* $w_c(x) \to (\beta)^{N/2}W_0(\beta x)$ *in* $H^1(\mathbb{R}^N)$.

(b) $e_c \geq \dfrac{2+q_{j_0}}{2q_{j_0}} \left(\dfrac{c_*^2 \lambda^{q_{j_0}+2} C_1^{q_{j_0}/2}}{\beta^{q_{j_0}}} \right)^{2/(q_{j_0}+2)} \left[1 - \left(\dfrac{c}{c_*} \right)^{2(\alpha+2)/N} \right]^{q/(q+2)}$

as $c \to (c_*)^-$, *where* $q = \max\limits_{1 \leq i \leq m} q_i$ *and*

$$\lambda = \min_{1 \leq i \leq m} \left\{ \lambda_i \;\middle|\; \lambda_i = \left(\frac{q_i \mu_i}{2c_*^2} \int_{\mathbb{R}^N} |x|^{q_i} |W_0(x)|^2 \right)^{1/(q_i+2)} \right\}.$$

Moreover, if

$$\lim_{c \to (c_*)^-} \frac{1 - (c/c_*)^{2(\alpha+2)/N}}{\varepsilon_c^{q+2}} = 1,$$

then

$$\lim_{c \to (c_*)^-} \frac{e_c}{\varepsilon_c^q} = \frac{\lambda^2 c_*^2 (q+2)}{2q} \left(\frac{N}{\alpha+2} \right)^{q/(q+2)}$$

and $y_0 = 0$, $\beta = ((\alpha+2)/N)^{1/(q+2)} \lambda$.

PROOF. By (3.5) and Theorem 1.4, we see that

$$C(u_c) \to 0 \quad \text{as } c \to (c_*)^-. \tag{3.10}$$

Set $\widetilde{w}_c(x) := \varepsilon_c^{N/2} u_c(\varepsilon_c x)$. Then $|\widetilde{w}_c|_2 = c$. Up to a subsequence, let $\delta := \lim\limits_{c \to (c_*)^-} \sup\limits_{y \in \mathbb{R}^N} \int_{B_1(y)} |\widetilde{w}_c|^2$. If $\delta = 0$, then $\widetilde{w}_c \to 0$ in $L^s(\mathbb{R}^N)$ as $c \to (c_*)^-$, $2 < s < 2^*$, hence by (1.4), $B(\widetilde{w}_c) \to 0$. So

$$0 < \frac{C_1}{2} \leq \frac{A(\widetilde{w}_c)}{2} \leq e_c \varepsilon_c^2 + \frac{N}{2(N+\alpha+2)} B(\widetilde{w}_c) \to 0 \quad \text{as } c \to (c_*)^-,$$

which is impossible. Then $\delta > 0$ and there exists $\{y_c\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_c)} |\widetilde{w}_c| \geq \frac{\delta}{2} > 0.$$

Set $w_c(x) := \widetilde{w}_c(x + y_c) = \varepsilon_c^{N/2} u_c(\varepsilon_c x + \varepsilon_c y_c)$, then

$$\int_{B_1(0)} |w_c|^2 \geq \frac{\delta}{2}. \tag{3.11}$$

We claim that $\{\varepsilon_c y_c\}$ is uniformly bounded as $c \to (c_*)^-$. Indeed, if there exists a sequence $\{c_k\} \subset (0, c_*)$ with $c_k \to c_*$ such that $|\varepsilon_{c_k} y_{c_k}| \to +\infty$ as $k \to +\infty$, then by (V_0), (3.10) and (3.11) and the Fatou Lemma, we have

$$\begin{aligned} 0 &= \liminf_{k \to +\infty} B(u_{c_k}) = \liminf_{k \to +\infty} \int_{\mathbb{R}^N} V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k}) |w_{c_k}(x)|^2 \\ &\geq \int_{B_1(0)} \liminf_{k \to +\infty} [V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k}) |w_{c_k}(x)|^2] \geq (+\infty) \cdot \frac{\delta}{2} = +\infty, \end{aligned}$$

which is impossible. So $\{\varepsilon_c y_c\}$ is uniformly bounded as $c \to (c_*)^-$. Moreover, there exists $x_{j_0} \in \{x_1, \ldots, x_m\}$ such that

$$\left\{\frac{\varepsilon_c y_c - x_{j_0}}{\varepsilon_c}\right\} \quad \text{is uniformly bounded as } c \to (c_*)^-. \tag{3.12}$$

Indeed, by contradiction, we just suppose that for any $x_i \in \{x_1, \ldots, x_m\}$, there exists $c_k \to (c_*)^-$ such that $|(\varepsilon_{c_k} y_{c_k} - x_i)/\varepsilon_{c_k}| \to +\infty$ as $k \to +\infty$. By (V_1), (3.11) and the Fatou Lemma, for any positive constant C,

$$\begin{aligned}
&\liminf_{k\to+\infty} \frac{1}{\varepsilon_{c_k}^{q_i}} \int_{\mathbb{R}^N} V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k}) w_{c_k}^2(x) \\
&\geq \int_{\mathbb{R}^N} \liminf_{k\to+\infty} \frac{V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k})}{\varepsilon_{c_k}^{q_i}} |w_{c_k}(x)|^2 \\
&\geq \int_{\mathbb{R}^N} \liminf_{k\to+\infty} \frac{V(\varepsilon_{c_k} x + x_i)}{|\varepsilon_{c_k} x|^{q_i}} |x|^{q_i} \left| w_{c_k}\left(x + \frac{x_i - \varepsilon_{c_k} y_{c_k}}{\varepsilon_{c_k}}\right)\right|^2 \\
&\geq \mu_i \int_{B_1(0)} \liminf_{k\to+\infty} |x + \frac{\varepsilon_{c_k} y_{c_k} - x_i}{\varepsilon_{c_k}}|^{q_i} |w_{c_k}(x)|^2 \geq \frac{\mu_i \delta C}{2}.
\end{aligned} \tag{3.13}$$

Hence by (3.5),

$$\begin{aligned}
e_{c_k} &\geq \frac{A(w_{c_k})}{2\varepsilon_{c_k}^2}\left[1 - \left(\frac{c_k}{c_*}\right)^{2(\alpha+2)/N}\right] + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k}) |w_{c_k}|^2 \\
&\geq \frac{C_1}{2\varepsilon_{c_k}^2}\left[1 - \left(\frac{c_k}{c_*}\right)^{2(\alpha+2)/N}\right] + \frac{\mu_i \delta C}{4} \varepsilon_{c_k}^{q_i} \\
&\geq \frac{q_i + 2}{2q_i}\left(\frac{q_i \delta \mu_i C_1^{q_i/2}}{4}\right)^{2/q_i+2}\left[1 - \left(\frac{c_k}{c_*}\right)^{2(\alpha+2)/N}\right]^{q_i/q_i+2} C^{2/q_i+2} \\
&\geq \frac{q_i + 2}{2q_i}\left(\frac{q_i \delta \mu_i C_1^{q_i/2}}{4}\right)^{2/(q_i+2)} C^{2/(q_i+2)}\left[1 - \left(\frac{c_k}{c_*}\right)^{2(\alpha+2)/N}\right]^{q/(q+2)}
\end{aligned} \tag{3.14}$$

as $k \to +\infty$, which contradicts the upper bound obtained in Lemma 3.5 since $C > 0$ is arbitrary. Then (3.12) holds. So there exists some $y_0 \in \mathbb{R}^N$ such that $(\varepsilon_c y_c - x_{j_0})/\varepsilon_c \to y_0$ as $c \to (c_*)^-$.

Since $u_c \in \widetilde{S}(c)$ is a minimizer of e_c, $(E|_{\widetilde{S}(c)})'(u_c) = 0$, i.e. there exists a sequence $\{\lambda_c\} \subset \mathbb{R}$ such that $E'(u_c) - \lambda_c u_c = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of $\mathcal{H}$. Then

$$\varepsilon_c^2 \lambda_c = \frac{1}{c^2}\left(2\frac{N+\alpha+2}{N}\varepsilon_c^2 e_c - \frac{\alpha+2}{N}\varepsilon_c^2 C(u_c) - \frac{\alpha+2}{N} A(w_c)\right),$$

which along with (3.10) imply that there exists $\beta > 0$ such that $\varepsilon_c^2 \lambda_c \to -\beta^2$ as $c \to (c_*)^-$. By the definition of w_c, we see that w_c satisfies the following equation:

$$\begin{aligned}
&-\Delta w_c + \varepsilon_c^2 V(\varepsilon_c x + \varepsilon_c y_c) w_c \\
&\qquad - (I_\alpha * |w_c|^{(N+\alpha+2)/N}) |w_c|^{(N+\alpha+2)/N-2} w_c = \lambda_c \varepsilon_c^2 w_c
\end{aligned} \tag{3.15}$$

in $\mathbb{R}^N$. Since $\{w_c\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$, there exists $w_0 \in H^1(\mathbb{R}^N)$ such that

$$\begin{cases} w_c \rightharpoonup w_0 & \text{in } H^1(\mathbb{R}^N), \\ w_c \to w_0 & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N),\ 1 \le s < 2^*, \\ w_c(x) \to w_0(x) & \text{a.e. in } \mathbb{R}^N. \end{cases}$$

Moreover, (3.11) implies that $w_0 \neq 0$. Then w_0 is a nontrivial solution of $-\Delta w_0 + \beta^2 w_0 = (I_\alpha * |w_0|^{(N+\alpha+2)/N})|w_0|^{(N+\alpha+2)/N-2} w_0$ in $\mathbb{R}^N$. Set $w_0(x) := \beta^{N/2} W_0(\beta x)$, then W_0 is a nontrivial solution of

$$-\Delta W_0 + W_0 = (I_\alpha * |W_0|^{(N+\alpha+2)/N})|W_0|^{(N+\alpha+2)/N-2} W_0, \quad x \in \mathbb{R}^N.$$

Hence by Lemma 3.2, we have $c_* \le |W_0|_2 = |w_0|_2 \le \lim_{c \to (c_*)^-} |w_c|_2 = c_*$, i.e. $|w_0|_2 = |W_0|_2 = c_*$. Hence $w_c \to w_0$ in $L^2(\mathbb{R}^N)$ and then $B(w_c) \to B(w_0)$. So we conclude from the equations w_c and w_0 satisfy that $w_c \to w_0$ in $H^1(\mathbb{R}^N)$. Moreover, Lemma 3.2 shows that W_0 is a groundstate solution of (3.9). So by Lemma 3.1 (c)–(d), $W_0(x) = O(|x|^{-(N-1)/2} e^{-|x|})$ as $|x| \to +\infty$ and we may assume that, up to translations, W_0 is radially symmetric about the origin. Similarly to (3.13), we see that

$$\begin{aligned} (3.16) \qquad & \liminf_{c \to (c_*)^-} \frac{1}{\varepsilon_c^q} \int_{\mathbb{R}^N} V(\varepsilon_c x + \varepsilon_c y_c)|w_c(x)|^2 \\ & \ge \liminf_{c \to (c_*)^-} \frac{1}{\varepsilon_c^{q_{j_0}}} \int_{\mathbb{R}^N} V(\varepsilon_c x + \varepsilon_c y_c)|w_c(x)|^2 \\ & \ge \int_{\mathbb{R}^N} \liminf_{c \to (c_*)^-} \frac{V(\varepsilon_c x + x_{j_0})}{|\varepsilon_c x|^{q_{j_0}}} |x|^{q_{j_0}} \left| w_c\left(x + \frac{x_{j_0} - \varepsilon_c y_c}{\varepsilon_c}\right)\right|^2 \\ & \ge \frac{\mu_{j_0}}{\beta^{q_{j_0}}} \int_{\mathbb{R}^N} |x + \beta y_0|^{q_{j_0}} |W_0(|x|)|^2 \ge \frac{\mu_{j_0}}{\beta^{q_{j_0}}} \int_{\mathbb{R}^N} |x|^{q_{j_0}} |W_0(x)|^2, \end{aligned}$$

where the last inequality is strict if and only if $y_0 \neq 0$. Hence similarly to (3.14),

$$\begin{aligned} e_c & \ge \frac{C_1}{2\varepsilon_c^2}\left[1 - \left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}\right] + \frac{\mu_{j_0}}{2\beta^{q_{j_0}}} \int_{\mathbb{R}^N} |x|^{q_{j_0}} |W_0(x)|^2 \varepsilon_c^{q_{j_0}} \\ & = \frac{C_1}{2\varepsilon_c^2}\left[1 - \left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}\right] + \frac{c_*^2 \lambda_{j_0}^{q_{j_0}+2}}{q_{j_0} \beta^{q_{j_0}}} \varepsilon_c^{q_{j_0}} \\ & \ge \frac{2 + q_{j_0}}{2 q_{j_0}} \left(\frac{c_*^2 \lambda^{q_{j_0}+2} (\sqrt{C_1})^{q_{j_0}}}{\beta^{q_{j_0}}}\right)^{2/(q_{j_0}+2)} \left[1 - \left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}\right]^{q/(q+2)} \end{aligned}$$

as $c \to (c_*)^-$, where $\lambda = \min_{1 \le i \le m} \lambda_i$.

If $\lim_{c\to(c_*)^-}(1-(c/c_*)^{2(\alpha+2)/N}/\varepsilon_c^{q+2}=1$, then

$$\begin{aligned}
\liminf_{c\to(c_*)^-}\frac{e_c}{\varepsilon_c^q}&\geq\frac{1}{2}\bigg(A(w_0)+\frac{\mu_{j_0}}{\beta^{q_{j_0}}}\int_{\mathbb{R}^N}|x|^{q_{j_0}}|W_0(x)|^2\bigg)\\
&\geq c_*^2\bigg(\frac{\beta^2}{2}\frac{N}{\alpha+2}+\frac{\lambda_{j_0}^{q_{j_0}+2}}{q_{j_0}\beta^{q_{j_0}}}\bigg)\\
&\geq\frac{\lambda_{j_0}^2c_*^2}{2}\bigg(\frac{N}{\alpha+2}\bigg)^{q_{j_0}/(q_{j_0}+2)}\frac{q_{j_0}+2}{q_{j_0}}\\
&\geq\frac{\lambda^2c_*^2(q+2)}{2q}\bigg(\frac{N}{\alpha+2}\bigg)^{q/(q+2)}.
\end{aligned}\tag{3.17}$$

On the other hand, for any $x_i\in\{x_1,\ldots,x_m\}$ and $t>0$, let

$$v_c(x)=\frac{cA_c}{c_*}\bigg(\frac{t}{\varepsilon_c}\bigg)^{N/2}\varphi(x-x_i)W_0\bigg(\frac{t(x-x_i)}{\varepsilon_c}\bigg),$$

where φ is a cut-off function given as in (3.6) and $A_c>0$ is chosen to satisfy $v_c\in\widetilde{S}(c)$. Then $\lim_{c\to(c_*)^-}A_c=1$. By the Dominated Convergence Theorem, we see that

$$\begin{aligned}
\limsup_{c\to(c_*)^-}\frac{e_c}{\varepsilon_c^q}&\leq\limsup_{c\to(c_*)^-}\frac{E(v_c)}{\varepsilon_c^q}\\
&=\frac{t^2}{2}A(W_0)+\lim_{c\to(c_*)^-}\frac{t^N}{2\varepsilon_c^{N+q}}\int_{\mathbb{R}^N}V(x)\bigg|\varphi(x-x_i)W_0\bigg(\frac{t(x-x_i)}{\varepsilon_c}\bigg)\bigg|^2\\
&=\frac{1}{2}\bigg(\frac{t^2c_*^2N}{\alpha+2}+\frac{\overline{\mu}_i}{t^q}\int_{\mathbb{R}^N}|x|^q|W_0(x)|^2\bigg)\\
&=c_*^2\bigg(\frac{t^2}{2}\frac{N}{\alpha+2}+\frac{\overline{\lambda}_i^{q+2}}{qt^q}\bigg),
\end{aligned}\tag{3.18}$$

where

$$\overline{\mu}_i=\lim_{x\to x_i}\frac{V(x)}{|x-x_i|^q}=\begin{cases}\mu_i & \text{if } q=q_i,\\ +\infty & \text{if } q\neq q_i,\end{cases}$$

and

$$\overline{\lambda}_i=\bigg(\frac{\overline{\mu}_iq}{2c_*^2}\int_{\mathbb{R}^N}|x|^q|W_0(x)|^2\bigg)^{1/(q+2)}=\begin{cases}\lambda_i & \text{if } q=q_i,\\ +\infty & \text{if } q\neq q_i.\end{cases}$$

So, since $t>0$ is arbitrary, taking infimum over $\{\overline{\lambda}_i\}_{i=1}^m$ in (3.18) and combining with (3.17), we see that

$$\lim_{c\to(c_*)^-}\frac{e_c}{\varepsilon_c^q}=\frac{\lambda^2c_*^2(q+2)}{2q}\bigg(\frac{N}{\alpha+2}\bigg)^{q/(q+2)}.$$

Then (3.16)–(3.18) must be equalities, what implies that $y_0=0$ and $\beta=((\alpha+2)/N)^{1/(q+2)}\lambda$. $\square$

LEMMA 3.7. *Suppose that* (V_0) *and* (V_1) *hold, then there exists a constant* $M_2 > 0$ *independent of* c *such that*

$$e_c \geq M_2\bigg[1-\bigg(\frac{c}{c_*}\bigg)^{2(\alpha+2)/N}\bigg]^{q/(q+2)} \qquad \text{as } c \to (c_*)^-.$$

PROOF. Suppose that u_c is a minimizer of e_c, we first show that

$$A(u_c) \to +\infty \quad \text{as } c \to (c_*)^-. \tag{3.19}$$

In fact, by contradiction, if there exists a sequence $\{c_k\} \subset (0, c_*)$ with $c_k \to c_*$ as $k \to +\infty$ such that the sequence of minimizers $\{u_{c_k}\} \subset \widetilde{S}(c_k)$ is uniformly bounded in $\mathcal{H}$, then we may assume that for some $u \in \mathcal{H}$, $u_{c_k} \rightharpoonup u$ in $\mathcal{H}$ and, by Lemma 3.4 and (3.1),

$$u_{c_k} \to u \quad \text{in } L^2(\mathbb{R}^N) \quad \text{and} \quad B(u_{c_k}) \to B(u).$$

Hence $u \in \widetilde{S}(c_*)$ and $0 \leq e_{c_*} \leq E(u) \leq \lim_{k\to+\infty} E(u_{c_k}) = \lim_{k\to+\infty} e_{c_k} = 0$, i.e. u is a minimizer of e_{c_*}, what contradicts Theorem 1.4. Since

$$0 \leq \frac{1}{2}A(u_c) - \frac{N}{2(N+\alpha+2)}B(u_c) \leq e_c,$$

we see that

$$\lim_{c\to(c_*)^-} \frac{\dfrac{N}{N+\alpha+2}B(u_c)}{A(u_c)} = 1.$$

Then by (3.19), we have

$$\varepsilon_c^{-2} := \frac{N}{2(N+\alpha+2)}\, B(u_c) \to +\infty \quad \text{as} \quad c \to (c_*)^-$$

and

$$2 \leq A(\varepsilon_c^{N/2} u_c(\varepsilon_c x)) \leq 2 + 2\varepsilon_c^2 e_c \leq 4.$$

Hence our conclusion follows from Lemma 3.6. $\square$

LEMMA 3.8. *Suppose that* u_c *is a minimizer of* e_c *and* V *satisfies* (V_0) *and* (V_1), *then there exist two positive constants* $K_1 < K_2$ *independent of* c *such that*

$$K_1\bigg[1-\bigg(\frac{c}{c_*}\bigg)^{2(\alpha+2)/N}\bigg]^{-2/(q+2)} \leq A(u_c) \leq K_2\bigg[1-\bigg(\frac{c}{c_*}\bigg)^{2(\alpha+2)/N}\bigg]^{-2/(q+2)}$$

as $c \to (c_*)^-$.

PROOF. The idea of the proof comes from that of Lemma 4 in [8], but it needs more careful analysis. By (3.5) and Lemma 3.5, we see that

$$A(u_c) \leq \frac{2e_c}{1-\bigg(\dfrac{c}{c_*}\bigg)^{2(\alpha+2)/N}} \leq 2M_1\bigg[1-\bigg(\frac{c}{c_*}\bigg)^{2(\alpha+2)/N}\bigg]^{-2/(q+2)}$$

as $c \to (c_*)^-$, where M_1 is given in Lemma 3.5. For any fixed $b \in (0, c)$, there exist two functions $u_b \in \widetilde{S}(b)$, $u_c \in \widetilde{S}(c)$ such that $e_b = E(u_b)$ and $e_c = E(u_c)$, respectively. Then by (3.1), we see that

$$e_b \leq E\left(\frac{b}{c}u_c\right) < e_c + \frac{1}{2}\left[1 - \left(\frac{b}{c}\right)^{2(\alpha+2)/N}\right] A(u_c).$$

Let $\eta := (c-b)/(c_* - c) > 0$, then $\eta \to +\infty$ as $c \to (c_*)^-$. Then by Lemmas 3.5 and 3.7, we have

$$\begin{aligned}
\frac{1}{2}A(u_c) &> \frac{e_b - e_c}{1 - \left(\frac{b}{c}\right)^{2(\alpha+2)/N}} \\
&\geq \frac{M_2\left(1 - \left(\frac{b}{c_*}\right)^{2(\alpha+2)/N}\right)^{q/(q+2)} - M_1\left(1 - \left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}\right)^{q/(q+2)}}{1 - \left(\frac{b}{c}\right)^{2(\alpha+2)/N}} \\
&\geq \left[1 - \left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}\right]^{-2/(q+2)} \frac{M_2\left[\dfrac{1 - \left(\frac{b}{c_*}\right)^{2(\alpha+2)/N}}{1 - \left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}}\right]^{q/(q+2)} - M_1}{\left(1 - \left(\frac{b}{c}\right)^{2(\alpha+2)/N}\right)\left[1 - \left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}\right]^{-1}} \\
&\geq \left[1 - \left(\frac{c}{c_*}\right)^{2(\alpha+2)/N}\right]^{-2/(q+2)} \frac{M_2\left(\dfrac{N}{2(\alpha+2)}\right)^{q/(q+2)}(1+\eta)^{q/(q+2)} - M_1}{\eta},
\end{aligned}$$

what gives the desired positive lower bound as $c \to (c_*)^-$. □

PROOF OF THEOREM 1.5. Let $\{c_k\} \subset (0, c_*)$ be a sequence satisfying $c_k \to (c_*)^-$ as $k \to +\infty$ and $\{u_{c_k}\} \subset \widetilde{S}(c_k)$ be a sequence of minimizers for e_{c_k}. Set

$$\varepsilon_{c_k} := \left[1 - \left(\frac{c_k}{c_*}\right)^{2(\alpha+2)/N}\right]^{1/(q+2)} > 0.$$

By Lemma 3.8, we see that $K_1 \leq A(\varepsilon_{c_k}^{N/2} u_{c_k}(\varepsilon_{c_k} x)) \leq K_2$. Then by Lemma 3.6,

$$\lim_{k\to+\infty} \frac{e_{c_k}}{\varepsilon_{c_k}^q} = \frac{\lambda^2 c_*^2 (q+2)}{2q}\left(\frac{N}{\alpha+2}\right)^{q/(q+2)}.$$

Moreover, there exist a sequence $\{y_{c_k}\} \subset \mathbb{R}^N$ and $x_{j_0} \in \{x_1, \ldots, x_m\}$ such that $\varepsilon_{c_k} y_{c_k} \to x_{j_0}$ as $k \to +\infty$ and there is a groundstate solution $W_0 \in H^1(\mathbb{R}^N)$ of (3.9), which is, up to translations, radially symmetric about the origin and such that

$$\varepsilon_{c_k}^{N/2} u_{c_k}(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k}) = w_{c_k}(x) \to w_0(x) = \left(\left(\frac{\alpha+2}{N}\right)^{1/(q+2)} \lambda\right)^{N/2} W_0\left(\left(\frac{\alpha+2}{N}\right)^{1/(q+2)} \lambda x\right)$$

in $L^{2Ns/(N+\alpha)}(\mathbb{R}^N)$ for all $(N+\alpha)/N \le s < (N+\alpha)/(N-2)_+$. □

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