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NUMBERS ASSOCIATED WITH STIRLING NUMBERS AND X^x

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Dedicated to the memory of my good friend E. G. Strauss

ABSTRACT We discuss two infinite triganular matrices $b(n, k)$ and $B(n, k)$ of rational integers that are associated with the matrices $s(n, k)$ and $S(n, k)$ of the Stirling numbers of the first and second kind. The numbers $b(n, k)$ were introduced in 1974 by Comtet in treating the nth derivative of x^x. They are generated by powers of the function $(1 + x)\log(1 + x)$. The numbers $B(n, k)$ are generated by powers of the inverse function.

All four matrices are treated together and numerous properties and relations are presented. In particular it is shown that $b(4h + 1, 2h) = 0$ for all integers $h > 0$. The values of the elements in a particular row of a matrix as well as the row sum when reduced modulo a prime p are also considered.

In 1974 Comtet introduced the numbers $b(n, k)$ defined by

$$\sum_{n=1}^{\infty} b(n, k)x^n/n! = \{(1 + x)\log(1 + x)\}^k/k!.$$

He used these numbers in the formula

$$\frac{d^n(x^x)}{dx^n} = x^x \sum_{j=0}^{n} (\log x)^j \binom{n}{j} \sum_{h=0}^{n-j} b(n - j, n - k - j)x^{-h}.$$

It is my purpose to show that these numbers are closely related to the Stirling numbers of the first and second kind and that they have a number of interesting properties. In fact it is important to introduce a second set of numbers $B(n, k)$ in order to treat the whole subject adequately.

We begin by introducing four infinite lower triangular matrices s, S, b, B. The elements on the nth row and kth column we denote by

(1) $$s(n, k),\ S(n, k),\ b(n, k),\ B(n, k)$$

with initial conditions

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$$s(0, 0) = S(0, 0) = b(0, 0) = B(0, 0) = 1$$
$$s(n, 0) = S(n, 0) = b(n, 0) = B(n, 0) = 0 \text{ if } n \neq 0$$
$$s(0, k) = S(0, k) = b(0, k) = B(0, k) = 0 \text{ if } n \neq 0.$$

If $n \neq 0$ and $k \neq 0$ these elements are generated by the following generating functions.

$$k!\sum_{n=1}^{\infty} s(n, k)x^n/n! = \{\log(1 + x)\}^k \tag{2}$$

$$k!\sum_{n=1}^{\infty} S(n, k)x^n/n! = (e^x - 1)^k \tag{3}$$

$$k!\sum_{n=1}^{\infty} b(n, k)x^n/n! = \{(1 + x)\log(1 + x)\}^k \tag{4}$$

$$k!\sum_{n=1}^{\infty} B(n, k)x^n/n! = \{\psi(x)\}^k. \tag{5}$$

Here we use the notation $\psi(x)$ to denote the function

$$\psi(x) = \sum_{\nu=1}^{\infty} (-1)^{\nu-1}(\nu - 1)^{\nu-1}x^\nu/\nu! \tag{6}$$

which is the inverse of the function $(1 + x)\log(1 + x)$ in the sense that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Since the four basic functions $\log(1 + x)$, $e^x - 1$, $(1 + x)\log(1 + x)$, $\psi(x)$ all vanish at the origin, all four matrix elements vanish whenever $k > n$ and moreover all elements on the main diagonals of the four matrices are equal to 1.

The elements $s(n, k)$ and $S(n, k)$ are called Stirling numbers of the first and second kind respectively. We call the elements $b(n, k)$ and $B(n, k)$ Comtet numbers of the first and second kind.

The matrices s and S are mutually inverse and so are b and B. The first ten rows and columns of these four matrices are shown in Tables 1 to 4.

We find it convenient to introduce two more matrices

$$M(n, k) = (-1)^{n+k}n^{n-1}\binom{n-1}{k-1} \tag{7}$$

$$m(n, k) = k^{n-k}\binom{n}{k}. \tag{8}$$

The fact these are mutually inverse is easy to establish.

Inversion Lemma. We first prove the useful result.

LEMMA. *Let ω and Ω be two mutually inverse matrices and let $g_m(x)$ be a sequence of functions of x. Finally let*

Table 1

$s(n, k)$

	1	2	3	4	5	6	7	8	9	10
1	1									
2	−1	1								
3	2	−3	1							
4	−6	11	−6	1						
5	24	−50	35	−10	1					
6	−120	274	−225	85	−15	1				
7	720	−1764	1624	−735	175	−21	1			
8	−5040	13068	−13132	6769	−1960	322	−28	1		
9	40320	−109584	118124	−67284	22449	−4536	546	−36	1	
10	−362880	1026576	−1172700	723680	−269325	63273	−9450	870	−45	1

Table 2
$S(n,k)$

	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

Table 3
$b(n,k)$

	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	−1	3	1							
4	2	−1	6	1						
5	−6	0	5	10	1					
6	24	4	−15	25	15	1				
7	−120	−28	49	−35	70	21	1			
8	720	188	−196	49	0	154	28	1		
9	−5040	−1368	944	0	−231	252	294	36	1	
10	40320	11016	−5340	−820	1365	−987	1050	510	45	1

Table 4
$B(n,k)$

	1	2	3	4	5	6	7	8	9	10
1	1									
2	−1	1								
3	4	−3	1							
4	−27	19	−6	1						
5	256	−175	55	−10	1					
6	−3125	2101	−660	125	−15	1				
7	46656	−31031	9751	−1890	245	−21	1			
8	−823543	543607	−170898	33621	−4550	434	−28	1		
9	16777216	−11012415	3463615	688506	95781	−9702	714	−36	1	
10	−387420489	253202761	−79669320	15958405	−2263065	238287	−18900	1110	−45	1

$$\sum_{\nu=1} \omega(k, \nu) g_\nu(x) = G_k(x).$$

Then

$$g_n(x) = \sum_{k=1} \Omega(n, k) G_k(x).$$

PROOF. Substituting from the first equation into the second we get

$$\begin{aligned}\sum_{k=1} \Omega(n, k) G_k(x) &= \sum_{k=1} \Omega(n, k) \sum_{\nu=1} \omega(k, \nu) g_\nu(x) \\ &= \sum_{\nu=1} g_\nu(x) \sum_{k=1} \Omega(n, k)\omega(k, \nu) = \sum_{\nu=1} g_\nu(x)\, \delta_\nu^n = g_n(x),\end{aligned}$$

where δ_ν^n is Kronecker's delta. This proves the lemma.

If we multiply both sides of (2) by $t^k/k!$ and then sum over k we obtain

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} t^k s(n, k) x^n/n! = e^{t \log(1+x)} = (1 + x)^t.$$

If we identify the coefficients of $x^n/n!$ on both sides we get

$$\sum_{k=0}^{n} s(n, k) t^k = t(t-1)(t-2) \cdots (t - n + 1) = t^{[n]} \tag{9}$$

a well known identity. If we use the lemma we get the familiar

$$\sum_{k=0}^{n} S(n, k) t^{[k]} = t^n. \tag{10}$$

If we let σ denote the row sum function of b, so that

$$\sum_{k=0}^{n} b(n, k) t^k = \sigma_n(t) \tag{11}$$

then by the lemma

$$\sum_{k=0}^{n} B(n, k) \sigma_n(t) = t^n.$$

The row sum function of b

THEOREM 1. *The function $\sigma_n(t)$ is generated by*

$$(1 + x)^{t(1+x)} = \sum \sigma_n(t) x^n/n!.$$

PROOF.

$$\begin{aligned}(1+x)^{t(1+x)} &= e^{t(1+x)\log(1+x)} = \sum_{k=0}^{\infty} t^k[(1 + x) \log(1 + x)]^k/k! \\ &= \sum_{=0k}^{\infty} t^k \sum_{n=1}^{\infty} b(n, k) x^n/n! = \sum_{n=1}^{\infty} x^n \Big(\sum_{k=0}^{\infty} b(n, k) t^k\Big)/n! \\ &= \sum \sigma_n(t) x^n/n!.\end{aligned}$$

Table 5

n	σ_n	n	σ_n	n	σ_n
0	1	5	10	10	47160
1	1	6	54	11	−419760
2	2	7	−42	12	4297512
3	3	8	944	13	−47607144
4	8	9	−5112	14	575023344

For $t = 1$ we get $\sigma_n(1)$ as the sum of the elements of the nth row of b and it is the coefficient of $x^n/n!$ in the expanison of $(1 + x)^{1+x}$ in powers of x. Table 5 gives a small table of $\sigma_n(1) = \sigma_n$.
The number

$$\sigma_{30}(1) = 357611376476800486783526273280$$

has 30 digits.

The numbers $\sigma_n(1)$ can be expressed in terms of Stirling numbers of the first kind by means of the following theorem.

THEOREM 2.

$$\sigma_n(1) = n! \sum_{\lambda=1}^{n-1} [s(n - \lambda - 1, \lambda) + s(n - \lambda - 1), \lambda - 1)]/(n - \lambda)!.$$

PROOF. By (2) we can write

$$\begin{aligned}(1 + x)^x = e^{x \log(1+x)} &= \sum_{k=0}^{\infty} x^k(\log(1 + x))^k/k! \\ &= \sum_{k=0}^{\infty} x^k \sum_{m=0}^{\infty} x^m s(m, k)/m! \\ &= \sum_{n=0}^{\infty} x^n \sum_{m=0}^{\infty} s(m, n - m)/(n - m)!.\end{aligned}$$

Multiplying both sides by $1 + x$ we have

$$\begin{aligned}(1 + x)^{1+x} &= \sum_{m=1}^{\infty} x^m \sum_{k=0}^{\infty} s(m-k, k)/(m-k)! + \sum_{m=1}^{\infty} x^{m+1}s(m-k, k)/(m-k)! \\ &= \sum_{m=1}^{\infty} x^m \sum_{k=0}^{\infty} \{s(m - k, k) + (m - k)s(m - k - 1, k)\}/(m - \mathrm{k})! \\ &= \sum_{m=1}^{\infty} x^m \sum_{k=1}^{[m/2]} \frac{s(m - k - 1, k) + s(m - k - 1, k - 1)}{(m - k)!}.\end{aligned}$$

But by Theorem 1 with $t = 1$ we see that $\sigma_n(1)/m!$ is the coefficient of x^m on the right side of the last equality. This proves the theorem.

Connection b with s. Comtet gave the following equation (12) which connects the elements of b with those of s. If we write

$$\{(1 + x)\log(1 + x)\}^k = (1 + x)^k \{\log(1 + x)\}^k$$

and identify the coefficients of $x^n/n!$ on both sides we get

$$\begin{aligned} b(n, k) &= \sum_{\nu=0}^{k} \nu! \binom{k}{\nu}\binom{n}{\nu} s(n - \nu, k) \\ &= \sum_{\nu=0}^{k} \binom{n}{\nu} \sum_{\lambda=0}^{\nu} k^{\lambda} s(\nu, \lambda) s(n - \nu, k) \qquad (12) \\ &= \sum_{\nu=k}^{n} \binom{\nu}{k} k^{\nu-k} s(n, \nu) \end{aligned}$$

where use is made of the known identity

$$\binom{a}{b} s(c, a) = \sum_{d=0} \binom{c}{d} s(c - d, b) s(d, a - b).$$

More facts about b. A main result is the following theorem.

THEOREM 3. If $n > 1$

$$\sum_{k=1}^{n} (-1)^k (k - 1)^{k-1} b(n, k) = 0.$$

PROOF. We make use of Abel's generalization of the binomial theorem (see Riordan [2], p. 18, (13a))

$$x^{-1}(x + y + n)^n = \sum_{k=0}^{n} (k + x)^{k-1} \binom{n}{k} (y + n - k)^{n-k}. \qquad (13)$$

If we put $n = \nu$, $x = -1$, $y = -\nu$ and divide both sides by $(-1)^\nu$ we get

$$\sum_{k=0}^{\nu} (-1)^k (k - 1)^{k-1} \binom{\nu}{k} k^{\nu-k} = -1.$$

Now we write

$$\begin{aligned} \sum_{k=1}^{n} (-1)^k (k - 1)^{k-1} b(n, k) &= \sum_{k=1}^{n} (-1)^k (k - 1)^{k-1} \sum_{\nu=k}^{n} \binom{\nu}{k} k^{\nu-k} s(n, \nu) \\ &= \sum_{\nu=0}^{n} s(n, \nu) \sum_{k=0}^{n} (-1)^k (k - 1)^{k-1} \binom{\nu}{k} k^{\nu-k} \\ &= -\sum_{\nu=0}^{n} s(n, \nu) = 0. \end{aligned}$$

This proves the theorem.

This theorem can be used to prove that $\psi(x)$ and $(1 + x)\log(1 + x)$ are mutually inverse as follows. Let x be chosen so that $|x|$ is so small that $(1 + x)\log(1 + x)$ is inside the circle of convergence of ψ, that is

$$|1 + x|\,|\log(1 + x)| < 1/e.$$

Then

$$\psi[(1+x)\log(1+x)] = \sum_{\nu=1}^{\infty}(-1)^{\nu-1}(\nu-1)^{\nu-1}[(1+x)\log(1+x)]^{\nu}/\nu!$$

$$= \sum_{\nu=1}^{\infty}(-1)^{\nu-1}(\nu-1)^{\nu-1}\sum_{n=1}^{\infty} b(n,\nu)x^n/n!$$

$$= -\sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{\nu=1}^{\infty}(-1)^{\nu}(\nu-1)^{\nu-1}b(n,\nu).$$

By Theorem 3 the inner sum is 0 if $n > 1$ and it is -1 if $n = 1$.
That is

$$\psi[(1+x)\log(1+x)] = x.$$

A more general result than Theorem 3 is the following.

THEOREM 4.

$$\sum_{n=1}^{n}(-1)^k(k+x)^{k-1}b(n,k) = (-1)^n\binom{n+x-1}{n-1}(n-1)!$$

PROOF. If, we do not fix x at -1 in (13), we get

$$x^{\nu-1} = \sum_{k=0}^{\nu}\binom{\nu}{k}(x+k)^{k-1}(-1)^{\nu-k}k^{\nu-k}.$$

Using this identity with (12) we find that

$$\sum_{k=0}^{n}(-1)^k(k+x)^{k-1}b(n,k) = \sum_{\nu=0}^{n} s(n,\nu)(-1)^{\nu}x^{\nu-1} = \frac{1}{x}\sum_{\nu=0}^{n}(-x)^{\nu}s(n,\nu)$$

$$= (-1)^n(x+1)(x+2)\cdots(x+n-1)$$

$$= (-1)^n(n-1)!\binom{n+x-1}{n-1}.$$

This proves Theorem 4.

If we apply the lemma to Theorem 4 we obtain

$$(14) \qquad \sum_{k=1}(-1)^k\binom{k+x-1}{k-1}(k-1)!B(n,k) = (-1)^n(x+n)^{n-1}.$$

If we use the fact developed in the proof of Theorem 4 that

$$(-1)^k(k-1)!\binom{k+x-1}{k-1} = \sum_{\nu=0}^{n} s(k,\nu)(-1)^{\nu}x^{\nu-1}$$

then (14) gives us, on identifying the coefficients of $x^{\nu-1}$ on both sides,

$$(15) \qquad \sum_{k=1}^{n} B(n,k)s(k,\nu) = (-1)^{\nu+n}n^{n-\nu}\binom{n-1}{\nu-1} = M(n,\nu).$$

THEOREM 5.

$$\sum_{k=1}^{n} S(n, k)b(k, \nu) = n^{n-\nu}\binom{n}{\nu}.$$

PROOF. The relation (15) can be written $Bs = M$. Taking the inverse of both sides gives us $(Bs)^{-1} = s^{-1}B^{-1} = Sb = M^{-1} = m$. Since

$$m(n, \nu) = \nu^{n-\nu}\binom{n}{\nu},$$

the theorem is poved.

Another theorem about B is the following:

THEOREM 6.

$$\sum_{k=2}^{n} (-1)^k(k-2)!\, B(n, k) = (-1)^n(n-1)^{n-1}.$$

PROOF. Since the functions $\psi(x)$ and $(1+x)\log(1+x)$ are inverse, we have

$$(1 + \psi(x))\log(1 + \psi(x)) = x.$$

Hence

$$\begin{aligned} x &= \psi(x) + \sum_{\lambda=2}^{\infty} (-1)^{\lambda-1}\{(\psi(x))^{\lambda}/\lambda + (\psi(x))^{\lambda+1}/\lambda\} \\ &= \psi(x) + \sum_{k=2}^{\infty} (-1)^k(\psi(x))^k/(k(k-1)) \\ &= \psi(x) + \sum_{k=2}^{\infty} (-1)^k(k-2)! \sum_{n=k} B(n, k)x^n/n! \\ &= \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)^{n-1}x^n/n! + \sum_{n=2}^{\infty} \frac{x^n}{n!} \sum_{k=2}^{\infty} (-1)^k(k-2)!B(n, k). \end{aligned}$$

If $n > 1$ the coefficient of $x^n/n!$ on both sides is zero. Transposing the first term on the right gives the theorem.

The b-counterpart of Theorem 6 is Theorem 3. Another way of proving Theorem 3 along similar lines starts with the relation $\psi[(1+x)\log(1+x)] = x$.

Further identities. Another set of four identities comes from (2), (3), (4), (5) by writing

$$\{F(x)\}^k/k! = \{F(x)/k\}\ \{F(x)\}^{k-1}/(k-1)!$$

for

$$F(x) = \log(1 + x),\ e^x - 1,\ (1 + x)\log(1 + x),\ \psi(x).$$

Identifying coefficients of $x^n/n!$ on both sides gives the following results, with $\nu = n - k + 1$.

(16) $$ks(n, k) = \sum_{\lambda=1}^{v} (-1)^{\lambda-1}(\lambda - 1)!\binom{n}{\lambda}s(n - \lambda, k - 1)$$

(17) $$kS(n, k) = \sum_{\lambda=1}^{v} \binom{n}{\lambda}S(n - \lambda, k - 1)$$

(18) $$kb(n, k) = b(n - 1, k - 1) + \sum_{\lambda=2}^{v} (-1)^{\lambda}(\lambda - 2)!\binom{n}{\lambda}b(n - \lambda, k - 1)$$

(19) $$kB(n, k) = \sum_{\lambda=1}^{v} (-1)^{\lambda-1}\binom{n}{\lambda}(\lambda - 1)^{\lambda-1}B(n - \lambda, k - 1).$$

These will be used later.

Diagonal polynomials. If one examines the elements of the matrix b, say, that lie on a diagonal of slope -1, that is the elements

$$b(n, n - r) \quad (n = r + 1, r + 2, \ldots)$$

one finds that these are the values of a polynomial with rational coefficients. More precisely, there is a polynomial $P_r(b, x)$, of degree $r - 1$ in x, with integer coefficients and an integer d_r such that

(20) $$d_r b(n, n - r) = \binom{n}{r + 1}P_r(b, n).$$

The other three polynomials $P_r(s, x)$, $P_r(S, x)$, $P_r(B, x)$ enjoy the same denominator d_r, the first six values are displayed below.

r	1	2	3	4	5	6
d_r	1	4	2	48	16	576

The corresponding polynomials can be listed as follows.

$P_1(s, x) = -1$ $\quad$ $P_1(S, x) = 1$

$P_1(b, x) = 1$ $\quad$ $P_1(B, x) = -1$

$P_2(s, x) = 3x - 1$ $\quad$ $P_2(S, x) = 3x - 5$

$P_2(b, x) = 3x - 13$ $\quad$ $P_2(B, x) = 3x + 7$

$P_3(s, x) = -x(x - 1)$ $\quad$ $P_3(S, x) = (x - 2)(x - 3)$

$P_3(b, x) = (x - 5)(x - 8)$ $\quad$ $P_3(B, x) = -(x + 2)(x + 5)$

$P_4(s, x) = 15x^3 - 30x^2 + 5x + 2$

$P_4(S, x) = 15x^3 - 150x^2 + 485x - 502$

$P_4(b, x) = 15x^3 - 390x^2 + 3245x - 8638$

$P_4(B, x) = 15x^3 + 210x^2 + 845x + 938$

$P_5(s, x) = -x(x - 1)(3x^2 - 7x - 2)$

$P_5(S, x) = (x - 4)(x - 5)(3x^2 - 23x + 38)$

$$P_5(b, x) = (x - 9)(3x^3 - 103x^2 + 1118x - 3876)$$
$$P_5(B, x) = -(x + 4)(3x^3 + 58x^2 + 313x + 386).$$

All the above polynomials have no complex roots. However

$$P_6(s, x) = 63x^5 - 315x^4 + 315x^3 + 91x^2 - 42x - 16$$

has the pair $-.2835345 \pm .2696825i$ of complex roots.

Recurrences. The two term recurrences for s and S

$$s(n + 1, k) = s(n, k - 1) - ns(n, k) \tag{21}$$

and

$$S(n + 1, k) = S(n, k - 1) + kS(n, k) \tag{22}$$

are well known. The first follows from identifying the coefficients of $x^n/n!$ on both sides of

$$(1 + x)\frac{d}{dx}[\log(1 + x)]^k/k! = [\log(1 + x)]^{k-1}/(k - 1)!$$

The second recurrence (22) follows via (10) from

$$\sum_{k=1}^{n+1} S(n + 1, k)t^{[k]} = t^{n+1} = t \cdot t^n = t\sum_{h=1}^{n} S(n, h)t^{[h]}$$
$$= \sum_{h=1}^{n+1} S(n, h)\{t^{[h+1]} + ht^{[h]}\}$$

by indentifying coefficients of $t^{[k]}$ on both sides. Comtet gave the following recurrence for b

$$b(n + 1, k) = nb(n - 1, k - 1) + b(n, k - 1) - (n - k)b(n, k). \tag{23}$$

This follows from the identity

$$(1 + x)\frac{d}{dx}([(1 + x)\log(1 + x)]^k)/k! = (1 + x)[(1 + x)\log(1 + x)]^{k-1}/(k - 1)!$$
$$+ k[(1 + x)\log(1 + x]^k/k!$$

Whether Comtet numbers of the second kind have a recurrence with a fixed number of terms I don't know.

The central $b(n, k)$. Perhaps the most striking features of the matrix b are the three zero values

$$b(5, 2) = b(8, 5) = b(9, 4) = 0.$$

Are there any more occurrences of zero or are these three the only ones? This question is answered by the following

THEOREM 7. *Let h be any positive integer. Then*

(24) $$b(4h+1, 2h) = 0$$

(25) $$2hb(4h, 2h) = -b(4h, h-1)$$

(26) $$b(4h-1, 2h-1) = b(4h, 2h)$$

(27) $$(2h-1)b(4h-1, 2h) = (4h-1)b(4h-2, 2h-1).$$

PROOF. We begin by proving (27). We define polynomials F_1 and F_2 by

$$F_1(x) = \prod_{k=1}^{4h-2} (x+2h-k),\ F_2(x) = \prod_{k=0}^{4h-2} (x+2h-k)$$

and consider the polynomial

$$F(x) = (2h-1)F_2(x) - x(4h-1)F_1(x).$$

Since $F_2(x) = (x+2h)F_1(x)$, we have

$$F(x) = -2h(x-2h+1)F_1(x) = -2hx \prod_{k=1}^{2h-1} (x^2-k^2)$$

and so $F(x)$ is an odd function of x.

By (9), the polynomials F_1 and F_2 have the expansions.

$$F_1(x) = \sum_{k=0}^{4h-2} s(4h-2, k)\,(x+2h-1)^k$$

$$F_2(x) = \sum_{k=0}^{4h-1} s(4h-1, k)\,(x+2h)^k.$$

If we ask for the coefficient of x^{2h} in $xF_1(x)$ we obtain in view of (12)

$$\sum_{m=0} \binom{m}{2h-1}(2h-1)^{m-2h+1}s(4h-2, m) = b(4h-2, 2h-1).$$

Similarly, the coefficient of x^{2h} in $F_2(x)$ is

$$\sum_{m=0} \binom{m}{2h}(2h)^{m-2h}s(4h-1, m) = b(4h-1, 2h-1).$$

The coefficient of x^{2h} in $F(x)$ is therefore

$$(2h-1)b(4h-1, 2h) - (4h-1)b(4h-2, 2h-1).$$

But F is an odd function so this must be zero. Thus (27) is established.

We next prove (26). We define F_3 and F_4 by

$$F_3(x) = \prod_{k=0}^{4h-1} (x+2h-k),\ F_4(x) = \prod_{k=1}^{4h-1} (x+2h-k)$$

so that $F_3(x) = xF_4(x) + 2hF_4(x)$. Since $F_4(x)$ is an odd function of x, the coefficients of x^{2h} in $F_3(x)$ and $xF_4(x)$ are identical.

Now

$$F_3(x) = \sum_{m=0}^{4h} s(4h, m)\,(x + 2h)^m$$
$$= \sum_{k=0} x^k \sum_{m=k}^{4h} \binom{m}{k} (2h)^{m-k} s(4h, m).$$

Hence the coefficient of x^{2h} in $F_3(x)$ is

$$\sum_{m=2h}^{4h} \binom{m}{2h} (2h)^{m-2h}\, s(4h, m) = b(4h, 2h).$$

Similarly,

$$xF_4(x) = x \sum_{m=0}^{4h-1} s(4h - 1, m)\,(x + 2h - 1)^m$$
$$= \sum_{k=1}^{4h} x^{k+1} \sum_{m=k}^{4h-1} \binom{m}{k} (2h - 1)^{m-k} s(4h - 1, m).$$

The coefficient of x^{2h} in $xF_4(x)$ is therefore

$$\sum_{m=2h-1}^{4h-1} \binom{m}{2n - 1} (2h - 1)^{m-2h+1} s(4h - 1, m) = b(4h - 1, 2h - 1).$$

Equating these two coefficients gives us (26). Next we prove (24). We define $F_5(x)$ by

$$F_5(x) = \prod_{k=0}^{4h} (x + 2h - k).$$

Hence

$$F_5(x) = \prod_{m=0}^{4m+1} s(4m + 1, m)(x + 2h)^m$$
$$= \sum_{k=0} x^k \sum_{m=k}^{4h+1} \binom{m}{k} (2h)^{m-k} s(4h + 1, m),$$

Since $F_5(x)$ is an odd function of x the coefficient of x^{2h} must vanish. That is,

$$0 = \sum_{m=2h}^{4h+1} \binom{m}{2h} (2h)^{m-2h} s(4h + 1, m) = b(4h + 1, 2h).$$

This proves (24). The relation (25) is now an easy consequence of (24), (26) and the recurrence (23).

The fact that $b(8, 5) = 0$ is easily explained since from (16) we have $P_3(b, x) = (x - 5)(x - 8)$. However, this gives us little hope of finding further zeros in the b matrix besides the ones we already know about.

Congruence properties. We give a few properties of the Stirling and

Comtet numbers modulo a prime p. The first of these shows that these numbers behave like the binomial coefficients.

THEOREM 8. *If p is a prime and if $1 < k < p$ then p divides $s(p, k)$, $S(p, k)$, $b(p, k)$ and $B(p, k)$.*

PROOF. If we inspect formulas (16), (17), (18), (19), we observe the ubiquitous factor $\binom{n}{\lambda}$. For $n = p$ this becomes a multiple of p, except when $\lambda = p$. Since $1 < k < p$, this never happens. This proves Theorem 8.

If $k = p$ all four numbers are equal to 1. If $k = 1$ with the help of Wilson's Theorem we find

$$\begin{aligned} s(p, 1) &= (-1)^{p-1}(p-1)! \equiv -1 \pmod p \\ S(p, 1) &= 1 \\ b(p, 1) &= (-1)^{p}(p-2)! \equiv -1 \pmod p \\ B(p, 1) &= (-1)^{p-1}(p-1)^{p-1} \equiv 1 \pmod p. \end{aligned}$$

Theorem 8 can be extended as follows:

THEOREM 9. *Let p be a prime and let $r + 1 < k < p$. Then $s(p + r, k)$, $S(p + r, k)$, $b(p + r, k)$ and $B(p + r, k)$ are all divisible by p.*

PROOF. We prove the theorem for the number $S(p + r, k)$. The same proof works for the three other numbers. For $r = 0$ we have Theorem 8 and we use induction on r. Suppose the theorem is true for all $r < h$. If we set $n = p + h$ in (17) we get

$$kS(p+h, k) = \sum_{\lambda=1}^{p+h-k+1} \binom{p+h}{\lambda} S(p+h-\lambda, k-1).$$

Because $h + 1 < k < p$ we have

$$\lambda \leqq p + h - k + 1 < p.$$

If $\lambda > h$ then $p + h - \lambda < p$ and hence

$$\binom{p+h}{\lambda} \equiv 0 \pmod p.$$

That is

$$kS(p+h, k) \equiv \sum_{\lambda=1}^{h} \binom{p+h}{\lambda} S(p-\lambda+h, k-1) \pmod p.$$

Now $h < k - 1 < p$ and so by hypothesis of induction each value of $S(p - \lambda + h, k - 1)$ is a multiple of p and so the theorem holds for $r = h$.

Row sum congruences. Congruences for the row sums of the matrices s, S, b, B follow from Theorem 9. In conclusion we give a few results of this kind.

In the first place the n-th row sum for the matrix s is zero if $n > 1$.

For the matrix S the row sum is usually denoted by B_n

$$B_n = \sum_{k=0}^{n} S(n, k)$$

and is called the Bell number. The row sums for b we have denoted by σ_n and those for B we call $\sum_n$. That is

$$\sigma_n = \sum_{k=0}^{n} b(n, k), \sum_n = \sum_{k=0}^{n} B(n, k).$$

The values of these functions modulo p for $n = p + i$ are tabulated below for $i = 0(1)5$.

n	B_n	σ_n	$\sum_n$
p	2	0	2
$p+1$	3	-1	2
$p+2$	7	-2	1
$p+3$	20	-6	-1
$p+4$	67	-12	23
$p+5$	255	-40	-345

The Stirling numers of the second kind have a so called explicit formula

$$k!S(n, k) = \sum_{\nu=0}^{k} (-1)^{k-\nu}\binom{k}{\nu}\nu^n$$

while no such formula seems to hold for $s(n, k)$. The same situation prevails for the Comtet numbers. In fact we have the following result.

THEOREM 10.

$$(k - 1)!B(n, k) = \sum_{\nu=0}^{k=1} (-1)^{n-k-\nu}\binom{k-1}{\nu}(n - \nu - 1)^{n-1}.$$

PROOF. That the theorem holds for $k = 1$) follows from (5) and (6). In fact

$$\sum_{n=1}^{\infty} B(n, 1)x^n/n! = \phi(x) = \sum_{=1n}^{\infty} (-1)^{n-1}(n - 1)^{n-1}x^n/n!$$

so that

$$B(n, 1) = (-1)^{n-1}(n - 1)^{n-1}$$

which is Theorem 10 when $k = 1$.

The proof now proceeds by induction on k. If the theorem holds for all n and for $k - 1$ we have

$$\begin{aligned}(-1)^{n-\lambda+k-1}(k-2)!B(n-\lambda,k-1)\\ =\sum_{\nu=0}^{k-2}(-1)^{\nu}\binom{k-2}{\nu}(n-\lambda-\nu-1)^{n-\lambda-1}.\end{aligned} \tag{28}$$

Multiplying (19) by $(k - 2)!$ and substituting (28) into this product gives

$$\begin{aligned}&(-1)^{n-k}\,k(k-2)!B\,(n,k)\\ &=\sum_{\lambda=1}^{n-k+1}\binom{n}{\lambda}(\lambda-1)^{\lambda-1}\sum_{\nu=0}^{k-2}(-1)^{\nu}\binom{k-2}{\nu}(n-\lambda-\nu-1)^{n-\lambda-1}\\ &=\sum_{\nu=0}^{k-2}(-1)^{\nu}\binom{k-\nu}{\nu}\sum_{\lambda=1}^{n-h+1}\binom{n}{\lambda}(\lambda-1)^{\lambda-1}(n-\lambda-\nu-1)^{n-\lambda-1}.\end{aligned}$$

To evaluate the inner sum we make use of the identity found in [**2**], p. 23.

$$\sum_{x=0}^{n}\binom{n}{\lambda}(x+\lambda)^{\lambda-1}(n+y-\lambda)^{n-\lambda-1}=(x^{-1}+y^{-1})(n+x+y)^{n-1}$$

with $x = -1$, $y = -\nu - 1$. If we let $\lambda = 0(1)\,n$ we get

$$-\frac{\nu+2}{\nu+1}(n-\nu-2)^{n-1}.$$

From this we must subtract the terms for $\lambda = 0$, $\lambda = n$ and $\lambda = (n-k+2)$ $(1)\,(n - 1)$. Hence the inner sum is

$$\begin{aligned}-\frac{\nu+2}{\nu+1}(n-\nu-2)^{n-1}+(n-\nu-1)^{n-1}+(\nu+1)^{-1}(n-1)^{n-1}\\ -\sum_{\lambda=n-k+2}^{n-1}\binom{n}{\lambda}(\lambda-1)^{\lambda-1}(n-\lambda-\nu-1)^{n-\lambda-1}.\end{aligned}$$

We now have

$$\begin{aligned}&(-1)^{n-k}\,k(k-2)!B(n,k)\\ &=\sum_{q=1}^{k}(n-q)^{n-1}c_q-\sum_{t=1}^{k-2}\binom{n}{t}(n-t-1)^{n-t-1}\\ &\qquad\sum_{\nu=0}^{k-2}(-1)^{\nu}\binom{k-2}{\nu}(t-\nu-1)^{t-1}\end{aligned} \tag{30}$$

where

$$c_q=(-1)^{q-1}\frac{k}{k-q}\binom{k-2}{q-1}.$$

The inner sum in (30) vanishes. Multiplying both sides by $(k - 1)/k$ we get

$$(-1)^{n-k}(k-1)!B(n,k) = \sum_{q=1}^{k} (n-q)^{n-1}(-1)^{q}\binom{k-2}{q-1}\frac{k-1}{k-q}$$
$$= \sum_{q=1}^{k} (-1)^{q}\binom{k-1}{q-1}(n-q)^{n-1}.$$

Thus the theorem holds for k.

The following corollary results from setting $k = n - r - 1$ and $m = n - 1$.

$$\sum_{\lambda=0}^{m-r} (-1)^{\lambda}\binom{m-r}{\lambda}(m-\lambda)^{m} = \frac{(-1)^{r}P_{r}(B, m+1)}{d(r_{r}+1)!}(n+1)!$$

Examples of this corollary, the first of which is well known, are

$$\sum_{\lambda=0}^{m} (-1)^{\lambda}\binom{m}{\lambda}(m-\lambda)^{m} = m!$$
$$\sum_{\lambda=0}^{m-1} (-1)^{\lambda}\binom{m-1}{\lambda}(m-\lambda)^{m} = \frac{1}{2}(m+1)!$$
$$\sum_{\lambda=0}^{m-2} (-1)^{\lambda}\binom{m-2}{\lambda}(m-\lambda)^{m} = \frac{3m+10}{6}(m+1)!$$
$$\sum_{\lambda=0}^{m-3} (-1)^{\lambda}\binom{m-3}{\lambda}(m-\lambda)^{m} = \frac{(m+3)(m+6)}{48}(m+1)!$$

We have not taken the time and space to discuss the numerical analysis and combinatorial meanings of the matrices b and B. This we hope to do in a future note.

REFERENCES

1. Louis Comtet, *Advanced Combinatorics*. Reidel, Dordrecht, 1974, p. 139–140.
2. John Riordan. *Combinatorial Identities*. Wiley, New York, 1968.

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