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# HOLOMORPHIC FUNCTIONAL CALCULUS ON UPPER TRIANGULAR FORMS IN FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. The decompositions of an element of a finite von Neumann algebra into the sum of a normal operator plus an s.o.t.-quasinilpotent operator, obtained using the Haagerup–Schultz hyperinvariant projections, behave well with respect to holomorphic functional calculus.

This note concerns the decomposition theorem for elements of a finite von Neumann algebra, recently proved in [2]. In that paper, given a von Neumann algebra $\mathcal{M}$ with a normal, faithful, tracial state $\tau$, by using the hyperinvariant subspaces found by Haagerup and Schultz [3] and their behavior with respect to Brown measure, for every element $T \in \mathcal{M}$ we constructed a decomposition $T = N + Q$ where $N \in \mathcal{M}$ is a normal operator whose Brown measure agrees with that of $T$ and where $Q$ is an s.o.t.-quasinilpotent operator. An element $Q \in \mathcal{M}$ is said to be s.o.t.-quasinilpotent if $((Q^*)^n Q^n)^{1/n}$ converges in the strong operator topology to the zero operator—by Corollary 2.7 in [3], this is equivalent to the Brown measure of $Q$ being concentrated at 0. In fact, $N$ is obtained as the conditional expectation of $T$ onto the (abelian) subalgebra generated by an increasing family of Haagerup–Schultz projections.

The Brown measure [1] of an element $T$ of a finite von Neumann algebra is a sort of spectral distribution measure, whose support is contained in the spectrum $\sigma(T)$ of $T$. We will use $\nu_T$ to denote the Brown measure of $T$. The Brown measure behaves well under holomorphic (or Riesz) functional calculus. Indeed, Brown proved (Theorem 4.1 of [1]) that if $h$ is holomorphic

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on a neighborhood of the spectrum of $T$, then $\nu_{h(T)} = \nu_T \circ h^{-1}$ (the push-forward measure by the function $h$).

In this note, we prove the following theorem.

THEOREM 1. *Let $T$ be an element of a finite von Neumann algebra $\mathcal{M}$ (with fixed normal, faithful tracial state $\tau$) and let $T = N + Q$ be a decomposition from [2], with $N$ normal, $\nu_N = \nu_T$ and $Q$ s.o.t.-quasinilpotent.*

(i) *Let $h$ be a complex-valued function that is holomorphic on a neighborhood of the spectrum of $T$. Then*

$$h(T) = h(N) + Q_h,$$

*where $Q_h$ is s.o.t.-quasinilpotent.*

(ii) *If $0 \notin \operatorname{supp} \nu_T$ (so that $N$ is invertible), then*

$$T = N\big(I + N^{-1}Q\big)$$

*and $N^{-1}Q$ is s.o.t.-quasinilpotent.*

The key result for the proof is Lemma 22 of [2], which allows us to reduce to the case when $N$ and $Q$ commute. Before using this, we require a few easy results about s.o.t.-quasinilpotent operators on Hilbert space.

LEMMA 2. *Let $\mathfrak{A}$ be a unital algebra and let $N, Q \in \mathfrak{A}$, $T = N + Q$ and suppose that both $N$ and $T$ are invertible. Then*

$$T^{-1} = N^{-1} - T^{-1}QN^{-1}.$$

*Proof.* We have

$$T^{-1} - N^{-1} = T^{-1}(N - T)N^{-1} = -T^{-1}QN^{-1}. \qquad \square$$

LEMMA 3. *Let $A$ and $Q$ be bounded operators on a Hilbert space $\mathcal{H}$ such that $AQ = QA$ and suppose $Q$ is s.o.t.-quasinilpotent. Then $AQ$ is s.o.t.-quasinilpotent.*

*Proof.* We have $(AQ)^n = A^n Q^n$ and

$$\big((AQ)^*\big)^n (AQ)^n = \big(Q^*\big)^n \big(A^*\big)^n A^n Q^n \le \|A\|^{2n} \big(Q^*\big)^n Q^n.$$

By Loewner's theorem, for $n \ge 2$ the function $t \mapsto t^{2/n}$ is operator monotone and we have

$$\Big(\big((AQ)^*\big)^n (AQ)^n\Big)^{2/n} \le \|A\|^4 \Big(\big(Q^*\big)^n Q^n\Big)^{2/n}.$$

Thus, for $\xi \in \mathcal{H}$, we have

$$\begin{aligned} \Big\|\Big(\big((AQ)^*\big)^n (AQ)^n\Big)^{1/n} \xi\Big\|^2 &= \Big\langle \Big(\big((AQ)^*\big)^n (AQ)^n\Big)^{2/n} \xi, \xi\Big\rangle \\ &\le \|A\|^4 \Big\langle \Big(\big(Q^*\big)^n Q^n\Big)^{2/n} \xi, \xi\Big\rangle \\ &= \|A\|^4 \Big\|\Big(\big(Q^*\big)^n Q^n\Big)^{1/n} \xi\Big\|^2. \end{aligned}$$

Since $Q$ is s.o.t.-quasinilpotent, this tends to zero as $n \to \infty$. $\square$

PROPOSITION 4. *Let $N$ and $Q$ be bounded operators on a Hilbert space and suppose $NQ=QN$ and $Q$ is s.o.t.-quasinilpotent. Let $T=N+Q$. Let $h$ be a function that is holomorphic on a neighborhood of the union $\sigma(T)\cup\sigma(N)$ of the spectra of $T$ and $N$. Then $h(T)$ and $h(N)$ commute, and $h(T)-h(N)$ is s.o.t.-quasinilpotent.*

*Proof.* If $\lambda$ is outside of $\sigma(T)\cup\sigma(N)$, then by Lemma 2,

$$(T-\lambda)^{-1}=(N-\lambda)^{-1}-(T-\lambda)^{-1}Q(N-\lambda)^{-1}. \tag{1}$$

Let $C$ be a contour in the domain of the complement $\sigma(T)\cup\sigma(N)$, with winding number 1 around each point in $\sigma(T)\cup\sigma(N)$. Then

$$h(T)=\frac{1}{2\pi i}\int_C h(\lambda)(\lambda-T)^{-1}\,d\lambda,$$
$$h(N)=\frac{1}{2\pi i}\int_C h(\lambda)(\lambda-N)^{-1}\,d\lambda.$$

For any complex numbers $\lambda_1$ and $\lambda_2$ outside of $\sigma(T)\cup\sigma(N)$, the operators $(\lambda_1-T)^{-1}$, $(\lambda_2-N)^{-1}$ and $Q$ commute; thus, $h(T)$ and $h(N)$ commute with each other. Using (1), we have

$$h(T)-h(N)=\frac{1}{2\pi i}\int_C h(\lambda)(\lambda-T)^{-1}Q(\lambda-N)^{-1}\,d\lambda=AQ,$$

where

$$A=\frac{1}{2\pi i}\int_C h(\lambda)(\lambda-T)^{-1}(\lambda-N)^{-1}\,d\lambda.$$

We have $AQ=QA$. By Lemma 3, $AQ$ is s.o.t.-quasinilpotent. □

For the remainder of this note, $\mathcal{M}$ will be a finite von Neumann algebra with specified normal, faithful, tracial state $\tau$.

LEMMA 5. *Let $T\in\mathcal{M}$. Suppose $p\in\mathcal{M}$ is a $T$-invariant projection with $p\notin\{0,1\}$.*

(i) *If $T$ is invertible, then $p$ is $T^{-1}$-invariant. Moreover, we have*

$$T^{-1}p=(pTp)^{-1},$$
$$(1-p)T^{-1}=\big((1-p)T(1-p)\big)^{-1},$$

*where the inverses on the right-hand-sides are in $p\mathcal{M}p$ and $(1-p)\mathcal{M}(1-p)$, respectively.*

(ii) *The union of the spectra of $pTp$ and $(1-p)T(1-p)$ (in $p\mathcal{M}p$ and $(1-p)\mathcal{M}(1-p)$, respectively) equals the spectrum of $T$.*

(iii) *If $h$ is a function that is holomorphic on a neighborhood of $\sigma(T)$, then $p$ is $h(T)$-invariant. Moreover, $h(T)p=h(pTp)$.*

*Proof.* For (i), a key fact is that one-sided invertible elements of $\mathcal{M}$ are always invertible. Thus, writing $T = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ with respect to the projections $p$ and $(1-p)$ (so that $a = pTp$, $b = pT(1-p)$ and $c = (1-p)T(1-p)$) writing $T^{-1} = \left(\begin{smallmatrix} x & y \\ w & z \end{smallmatrix}\right)$ and multiplying, we easily see that $a$ and $c$ must be invertible and

$$T^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}bc^{-1} \\ 0 & c^{-1} \end{pmatrix}. \tag{2}$$

Thus, $p$ is $T^{-1}$-invariant.

For (ii) we use (i) and the fact that the formula (2) shows that $T$ is invertible whenever $pTp$ and $(1-p)T(1-p)$ are invertible.

For (iii), writing

$$h(T) = \frac{1}{2\pi i} \int_C h(\lambda)(\lambda - T)^{-1}\, d\lambda \tag{3}$$

for a suitable contour $C$, where this is a Riemann integral that converges in norm, the result follows by applying part (i). □

For a von Neumann subalgebra $\mathcal{D}$ of $\mathcal{M}$, let $\mathrm{Exp}_{\mathcal{D}}$ and $\mathrm{Exp}_{\mathcal{D}'}$, respectively denote the $\tau$-preserving conditional expectations onto $\mathcal{D}$ and, respectively, the relative commutant of $\mathcal{D}$ in $\mathcal{M}$.

Lemma 6. *Let $T \in \mathcal{M}$.*

(i) *Suppose $0 = p_0 \le p_1 \le \cdots \le p_n = 1$ are $T$-invariant projections and let $\mathcal{D} = \mathrm{span}\{p_1, \ldots, p_n\}$. Then the spectra of $T$ and of $\mathrm{Exp}_{\mathcal{D}'}(T)$ agree. If $T$ is invertible, then $\mathrm{Exp}_{\mathcal{D}'}(T^{-1}) = \mathrm{Exp}_{\mathcal{D}'}(T)^{-1}$.*

(ii) *Suppose $(p_t)_{0\le t\le 1}$ is an increasing family of $T$-invariant projections in $\mathcal{M}$ with $p_0 = 0$ and $p_1 = 1$, that is right-continuous with respect to strong operator topology. Let $\mathcal{D}$ be the von Neumann algebra generated by the set of all $p_t$. If $T$ is invertible, then so is $\mathrm{Exp}_{\mathcal{D}'}(T)$ and $\mathrm{Exp}_{\mathcal{D}'}(T^{-1}) = \mathrm{Exp}_{\mathcal{D}'}(T)^{-1}$.*

*Proof.* For (i), we have

$$\mathrm{Exp}_{\mathcal{D}'}(T) = \sum_{j=1}^{n} (p_j - p_{j-1})T(p_j - p_{j-1}).$$

The assertions now follow from repeated application of Lemma 5.

For (ii), using the right-continuity of $p_t$ it is easy to choose an increasing family of finite dimensional subalgebras $\mathcal{D}_n$ of $\mathcal{D}$ whose union is strong operator topology dense in $\mathcal{D}$. Then $\mathrm{Exp}_{\mathcal{D}_n'}(T)$ and $\mathrm{Exp}_{\mathcal{D}_n'}(T^{-1})$ converge in strong operator topology to $\mathrm{Exp}_{\mathcal{D}'}(T)$ and $\mathrm{Exp}_{\mathcal{D}'}(T^{-1})$, respectively, and both sequences are bounded. From (i), we have the equality

$$\mathrm{Exp}_{\mathcal{D}_n'}(T)\mathrm{Exp}_{\mathcal{D}_n'}(T^{-1}) = I,$$

and taking the limit as $n \to \infty$ yields the desired result. □

LEMMA 7. *Let $T \in \mathcal{M}$ and let $p_t$ and $\mathcal{D}$ be as in either part* (i) *or part* (ii) *of Lemma 6. Suppose a function $h$ is holomorphic on a neighborhood of the spectrum of $T$. Then* $\mathrm{Exp}_{\mathcal{D}'}(h(T)) = h(\mathrm{Exp}_{\mathcal{D}'}(T))$.

*Proof.* Using that the Riemann integral (3) converges in norm, that $\mathrm{Exp}_{\mathcal{D}'}$ is norm continuous and applying Lemma 6, we get

$$\begin{aligned}\mathrm{Exp}_{\mathcal{D}'}\big(h(T)\big) &= \frac{1}{2\pi i}\int_C h(\lambda)\mathrm{Exp}_{\mathcal{D}'}\big((\lambda - T)^{-1}\big)\,d\lambda \\ &= \frac{1}{2\pi i}\int_C h(\lambda)\big(\lambda - \mathrm{Exp}_{\mathcal{D}'}(T)\big)^{-1}\,d\lambda = h\big(\mathrm{Exp}_{\mathcal{D}'}(T)\big). \qquad \square\end{aligned}$$

For convenience, here is the statement of Lemma 22 of [2] and an immediate consequence.

LEMMA 8. *Let $T \in \mathcal{M}$. For any increasing, right-continuous family of $T$-invariant projections $(q_t)_{0\le t\le 1}$ with $q_0 = 0$ and $q_1 = 1$, letting $\mathcal{D}$ be the von Neumann algebra generated by the set of all the $q_t$, the Fuglede–Kadison determinants of $T$ and* $\mathrm{Exp}_{\mathcal{D}'}(T)$ *agree. Since the same is true for $T - \lambda$ and* $\mathrm{Exp}_{\mathcal{D}'}(T) - \lambda$ *for all complex numbers $\lambda$, we have that the Brown measures of $T$ and* $\mathrm{Exp}_{\mathcal{D}'}(T)$ *agree.*

Now we have all the ingredients to prove our main result.

*Proof of Theorem 1.* In Theorem 6 of [2] the decomposition $T = N + Q$ is constructed by considering an increasing, right-continuous family $(p_t)_{0\le t\le 1}$ of Haagerup–Schultz projections, with $p_0 = 0$ and $p_1 = 1$, that are $T$-invariant, letting $\mathcal{D}$ be the von Neumann algebra generated by the set of projections in this family and taking $N = \mathrm{Exp}_{\mathcal{D}}(T)$. In particular, each $p_t$ is also $Q$-invariant.

For (i), we need to show that the Brown measure of $h(T) - h(N)$ is the Dirac mass at 0. By Lemma 5(iii), each $p_t$ is $h(T)$-invariant. So by Lemma 8, the Brown measures of $h(T) - h(N)$ and $\mathrm{Exp}_{\mathcal{D}'}(h(T) - h(N))$ agree. Since $h(N) \in \mathcal{D}$, we have $\mathrm{Exp}_{\mathcal{D}'}(h(N)) = h(N)$ and by Lemma 7, we have $\mathrm{Exp}_{\mathcal{D}'}(h(T)) = h(\mathrm{Exp}_{\mathcal{D}'}(T))$. Combining these facts we get

$$\nu_{h(T)-h(N)} = \nu_{h(\mathrm{Exp}_{\mathcal{D}'}(T))-h(N)}. \tag{4}$$

We have

$$\mathrm{Exp}_{\mathcal{D}'}(T) = N + \mathrm{Exp}_{\mathcal{D}'}(Q)$$

and $\mathrm{Exp}_{\mathcal{D}'}(Q)$ is s.o.t.-quasinilpotent. This last statement follows formally from Lemma 8 and the fact that $Q$ is s.o.t.-quasinilpotent. However, we should mention that the fact that $\mathrm{Exp}_{\mathcal{D}'}(Q)$ is s.o.t.-quasinilpotent was actually proved directly in [2] as a step in the proof that $Q$ is s.o.t.-quasinilpotent.

In any case, since $N$ and $\mathrm{Exp}_{\mathcal{D}'}(T)$ commute and $\mathrm{Exp}_{\mathcal{D}'}(Q)$ is s.o.t.-quasinilpotent, by Proposition 4 it follows that $h(\mathrm{Exp}_{\mathcal{D}'}(T)) - h(N)$ is s.o.t.-quasinilpotent. Using (4), we get that $h(T) - h(N)$ is s.o.t.-quasinilpotent, as desired.

For (ii), the projections $p_t$ form a right-continuous family, each of which is invariant under $N^{-1}Q$. By Lemma 8, the Brown measure of $N^{-1}Q$ equals the Brown measure of

$$\mathrm{Exp}_{\mathcal{D}'}\big(N^{-1}Q\big) = N^{-1}\mathrm{Exp}_{\mathcal{D}'}(Q). \tag{5}$$

But since $N^{-1}$ and $\mathrm{Exp}_{\mathcal{D}'}(Q)$ commute and since the latter is s.o.t.-quasinilpotent, by Lemma 3, their product (5) is s.o.t.-quasinilpotent. □

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