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ISOMETRIC DEFORMATIONS OF MINIMAL SURFACES IN $\mathbb{S}^4$

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ABSTRACT. We provide an elementary proof of the fact that the space of all isometric minimal immersions $f : M \to \mathbb{S}^4$ of a 2-dimensional Riemannian manifold M into $\mathbb{S}^4$ with the same normal curvature is, up to congruence, either finite or a circle. Furthermore, we show that if M is compact and the Euler number of the normal bundle of f is nonzero, then there exist at most finitely many noncongruent isometric minimal immersions of M into $\mathbb{S}^4$ with the same normal curvature.

1. Introduction

A classical question about isometric immersions is to decide if given an isometric immersion $f : M \to N$, this is, up to ambient isometries, the unique way of immerse isometrically the Riemannian manifold M into the Riemannian manifold N. When f is a minimal immersion, one can ask if this is the unique isometric minimal immersion of M into N. If this is the case, f is called *minimally rigid*. The rigidity aspects of minimal hypersurfaces in space forms have drawn several author's attention. A conclusive result was given Dajczer and Gromoll [5] for complete minimal hypersurfaces.

It is interesting to determine whether a given minimal surface can be deformed in a nontrivial way. Choi, Meeks and White [4] proved a rigidity result for complete minimal surfaces in $\mathbb{R}^3$. The case where the Euclidean space is replaced by a sphere is more difficult. Barbosa [2] proved that minimally immersed 2-spheres in a sphere are minimally rigid, while according to Ramanathan [12] any compact minimal surface in $\mathbb{S}^3$, allows at most finitely many noncongruent isometric minimal surfaces.

We are interested in isometric deformations of oriented minimal surfaces $f : M \to \mathbb{S}^4$ which preserve the normal curvature. If M is simply connected,

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then there exists a 2π-periodic one-parameter family f_θ of isometric minimal immersions preserving the normal curvature, the *associated family*. This family is obtained by rotating the second fundamental form of f and then integrate the system of Gauss, Codazzi and Ricci equations. It is trivial, in the sense that each f_θ is congruent to f, if f is superminimal (cf. [14]). Thus the rigidity for simply connected minimal surfaces fails in a natural way, and consequently the rigidity problem for minimal surfaces has a global nature. The above procedure cannot be carried out if the fundamental group is nontrivial. Inspired by a recent paper due to Smyth and Tinaglia [13], we provide elementary proofs of the following results concerning the space of isometric deformations of minimal surfaces in $\mathbb{S}^4$.

THEOREM 1. *Let $f : M \to \mathbb{S}^4$ be an isometric minimal immersion of a 2-dimensional Riemannian manifold M into $\mathbb{S}^4$ with normal curvature K_N. Then, up to congruence, the space of all isometric minimal immersions of M into $\mathbb{S}^4$ with normal curvature K_N is either finite or a circle.*

THEOREM 2. *Let $f : M \to \mathbb{S}^4$ be an isometric minimal immersion of a compact oriented 2-dimensional Riemannian manifold M into $\mathbb{S}^4$ with normal curvature K_N. If the Euler number of the normal bundle of f is nonzero, then there exist at most finitely many noncongruent isometric minimal immersions of M into $\mathbb{S}^4$ with normal curvature K_N.*

The Euler number of the normal bundle of f is nonzero if and only if its self-intersection number q is nonzero ([11, Cor. 3.2]). It is known (cf. [9]) that for any compact nonsuperminimal surface of genus g we have $|q| \leq 2(g-1)$. Thus $g \geq 2$ if the Euler number of its normal bundle is nonzero. All compact superminimal surfaces in $\mathbb{S}^4$, which are not totally geodesic, have normal bundle with nonzero Euler number. For all these surfaces, the number N of noncongruent isometric minimal immersions into $\mathbb{S}^4$ is $N = 1$. We do not know any examples with $N \geq 2$.

An immediate application of Theorem 2 is the following.

COROLLARY 3. *Let $f : M \to \mathbb{S}^4$ be compact minimal surface whose normal bundle has nonzero Euler number. If M admits a one parameter group of isometries φ_t, that preserve the normal curvature, then there exists a one parameter group of isometries τ_t of $\mathbb{S}^4$ such that $f \circ \varphi_t = \tau_t \circ f$ for all $t \in \mathbb{R}$.*

2. Preliminaries

Let $f : M \to \mathbb{S}^4$ be a minimal surface, that is, an isometric minimal immersion of a connected oriented 2-dimensional Riemannian manifold M, with normal bundle Nf and second fundamental form B. We view M as a Riemann surface with complex structure determined by the metric and the orientation. The complexified tangent bundle $TM \otimes \mathbb{C}$ is decomposed into the eigenspaces of the complex structure J, called $T'M$ and $T''M$, corresponding to the eigenvalues i and $-i$. The second fundamental form can be complex linearly extended to $TM \otimes \mathbb{C}$ with values in the complexified bundle $Nf \otimes \mathbb{C}$ and then decomposed into its (p,q)-components, $p+q=2$, which are tensor products of p many 1-forms vanishing on $T''M$ and q many 1-forms vanishing on $T'M$. Since f is minimal, for any local complex coordinate $z=u+iv$, we have

$$B = B^{(2,0)} + B^{(0,2)},$$

where

$$B^{(2,0)} = B\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) dz^2, \qquad B^{(0,2)} = \overline{B^{(2,0)}} \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right).$$

The Hopf differential is the differential form of type $(4,0)$

$$\Phi := \langle B^{(2,0)}, B^{(2,0)} \rangle.$$

For any local orthonormal frame field $\{e_j\}$ along f, such that $\{e_1, e_2\}$ are tangent with dual coframe $\{\omega_1, \omega_2\}$, we put

$$H_\alpha = \langle B(e_1,e_1), e_\alpha \rangle + i\langle B(e_1,e_2), e_\alpha \rangle, \quad \alpha = 3,4 \quad \text{and} \quad \varphi = \omega_1 + i\omega_2.$$

We easily obtain

$$\Phi = \frac{1}{4}\big(\overline{H}_3^2 + \overline{H}_4^2\big)\varphi^4.$$

The *curvature ellipse* of f at $x \in M$, is

$$\mathcal{E}(x) = \big\{B(X,X) : X \in T_xM, |X| = 1\big\}.$$

The zeros of Φ are precisely the points where the curvature ellipse is a circle. A minimal surface is called *superminimal* if $\Phi \equiv 0$. The Codazzi equation implies that Φ is holomorphic (cf. [3]). Thus either f is superminimal, or the points where the curvature ellipse is a circle are isolated.

The *normal curvature* K_N (cf. [1]) is given by

$$d\omega_{34} = -K_N \omega_1 \wedge \omega_2,$$

or equivalently,

$$K_N = i(H_3\overline{H}_4 - \overline{H}_3 H_4). \tag{1}$$

We note that $|K_N| = 2\kappa\mu$, where $\kappa \geq \mu \geq 0$ are the length of the semi-axes of the curvature ellipse. The length of the second fundamental form satisfies

$$\|B\|^2 = 2\big(|H_3|^2 + |H_4|^2\big). \tag{2}$$

Using the null frame field $\eta = e_3 + ie_4, \bar{\eta} = e_3 - ie_4$ of the complexified bundle $Nf \otimes \mathbb{C}$, we have

$$\langle B^{(2,0)}, B^{(2,0)} \rangle = \langle B^{(2,0)}, \eta \rangle \langle B^{(2,0)}, \bar{\eta} \rangle.$$

Therefore, we obtain

$$\Phi = \frac{1}{4}\big(\overline{H}_3^2 + \overline{H}_4^2\big)\varphi^4 = \frac{1}{4}k_+ k_- \varphi^4,$$

where $k_\pm := \overline{H}_3 \pm i\overline{H}_4$. The functions $a_\pm := |k_\pm|$ determine the geometry of the curvature ellipse. Indeed, since the Gaussian curvature K of M is given by $K = 1 - \|B\|^2/2$, from (1) and (2) we deduce that $a_\pm = (1 - K \pm K_N)^{1/2} = \kappa \pm \varepsilon\mu$, where $\varepsilon = \pm 1$, depending on the sign of K_N.

3. Isometric deformations of minimal surfaces in $\mathbb{S}^4$

3.1. Associated family of simply connected minimal surfaces. Let $f : M \to \mathbb{S}^4$ be a simply connected minimal surface. For each $\theta \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, we consider the orthogonal and parallel tensor field

$$J_\theta = \cos\theta I + \sin\theta J,$$

where I is the identity map and J is the complex structure. The symmetric section $\varGamma_\theta$ of the homomorphism bundle $\mathrm{Hom}(TM \times TM, Nf)$ given by

$$\varGamma_\theta(X,Y) := B(J_\theta X, Y), \quad X, Y \in TM$$

satisfies the Gauss, Codazzi and Ricci equations with respect to the normal connection $\nabla^\perp$ of f (cf. [14], [5], [6]). Hence there exists an isometric minimal immersion $f_\theta : M \to \mathbb{S}^4$ with second fundamental form

$$B^{f_\theta}(X,Y) = T_\theta \circ B(J_\theta X, Y),$$

where $T_\theta : Nf \to Nf_\theta$ is a parallel and orthogonal vector bundle isomorphism. The 2π-periodic family f_θ is the *associated family* of f. It is trivial, in the sense that each f_θ is congruent to f, if f is superminimal (cf. [14]).

It is obvious that f and f_θ have the same normal curvature. Eschenburg and Tribuzy [9, Th. 2] proved that any other minimal immersion of M into $\mathbb{S}^4$ with normal curvature K_N is congruent to some f_θ.

3.2. Deformations of nonsimply connected minimal surfaces. Let $f : M \to \mathbb{S}^4$ be a nonsimply connected minimal surface with normal curvature K_N. Since superminimal surfaces are rigid among superminimal surfaces (cf. [15]), we may assume hereafter that f is not superminimal. Let $g : M \to \mathbb{S}^4$ be another immersed minimal surface with normal curvature K_N. We consider the covering map $p : \tilde{M} \to M$, $\tilde{M}$ being the universal cover of M equipped with the metric and the orientation that makes p an orientation preserving local isometry. Corresponding objects on $\tilde{M}$ are denoted with tilde. The minimal surfaces $\tilde{f} := f \circ p$ and $\tilde{g} := g \circ p$ have normal curvature $\tilde{K}_N = K_N \circ p$. According to [9], $\tilde{g}$ is congruent to some $\tilde{f}_\theta$ in the associated family of $\tilde{f}$. Thus, the space of all isometric minimal immersions of M into $\mathbb{S}^4$ with normal curvature K_N is the set

$$\mathcal{S}(f) := \{\theta \in [0, 2\pi] : \text{there exists } f_\theta : M \to \mathbb{S}^4 \text{ so that } \tilde{f}_\theta = f_\theta \circ p\}.$$

Clearly $0 \in \mathcal{S}(f)$ and, for each $\theta \in \mathcal{S}(f)$, the normal curvature of f_θ is K_N.

Lemma 4. *For any σ in the group $\mathcal{D}$ of deck transformations of the covering map $p : \tilde{M} \to M$, the minimal surfaces $\tilde{f}_\theta$ and $\tilde{f}_\theta \circ \sigma$ are congruent.*

Proof. It is enough to prove the existence of an orthogonal and parallel isomorphism between the normal bundles of $\tilde{f}_\theta$ and $\tilde{f}_\theta \circ \sigma$ that preserves the second fundamental forms. If T_θ is the isomorphism between the normal bundles of $\tilde{f}$ and $\tilde{f}_\theta$, then we define the bundle isomorphism

$$\varSigma_\theta : N\tilde{f}_\theta \to N(\tilde{f}_\theta \circ \sigma)$$

so that $\varSigma_\theta|_{\tilde{x}} : N_{\tilde{x}}\tilde{f}_\theta \to N_{\tilde{x}}(\tilde{f}_\theta \circ \sigma)$ is given at any point $\tilde{x} \in \tilde{M}$ by

$$\varSigma_\theta|_{\tilde{x}}(\xi) := T_\theta|_{\sigma(\tilde{x})}\big(T_\theta^{-1}|_{\tilde{x}}(\xi)\big), \quad \xi \in N_{\tilde{x}}\tilde{f}_\theta.$$

The second fundamental forms of $\tilde{f}_\theta \circ \sigma$ and $\tilde{f}_\theta$ are related by

$$B^{\tilde{f}_\theta \circ \sigma}|_{\tilde{x}}(\tilde{v}, \tilde{w}) = T_\theta \circ B^{\tilde{f}}|_{\sigma(\tilde{x})}\big(\tilde{J}_\theta \circ d\sigma_{\tilde{x}}(\tilde{v}), d\sigma_{\tilde{x}}(\tilde{w})\big), \quad \tilde{v}, \tilde{w} \in T_{\tilde{x}}\tilde{M},$$

where $\tilde{J}_\theta = \cos\theta \tilde{I} + \sin\theta \tilde{J}$. Since σ is a deck transformation, it follows that

$$B^{\tilde{f}_\theta \circ \sigma}|_{\tilde{x}}(\tilde{v}, \tilde{w}) = \varSigma_\theta \circ B^{\tilde{f}_\theta}|_{\tilde{x}}(\tilde{v}, \tilde{w}).$$

Let $\xi = T_\theta(\eta)$, where η is a section of $N\tilde{f}$. Since $\varSigma_\theta(\xi) = T_\theta(\eta \circ \sigma^{-1}) \circ \sigma$, for any $\tilde{X}$ tangent to $\tilde{M}$, we have

$$\begin{aligned}
\big(\nabla^\perp_{\tilde{X}} \varSigma_\theta\big)\xi &= \nabla^\perp_{\tilde{X}}\big(T_\theta\big(\eta \circ \sigma^{-1}\big) \circ \sigma\big) - T_\theta\big(\nabla^\perp_{\tilde{X}}\big(\eta \circ \sigma^{-1}\big)\big) \circ \sigma \\
&= \big(\nabla^\perp_{d\sigma(\tilde{X})} T_\theta\big(\eta \circ \sigma^{-1}\big)\big) \circ \sigma - T_\theta\big(\nabla^\perp_{\tilde{X}}\big(\eta \circ \sigma^{-1}\big)\big) \circ \sigma \\
&= T_\theta\big(\nabla^\perp_{d\sigma(\tilde{X})}\big(\eta \circ \sigma^{-1}\big) - \nabla^\perp_{\tilde{X}}\big(\eta \circ \sigma^{-1}\big)\big) \circ \sigma,
\end{aligned}$$

where, by abuse of notation, $\nabla^\perp$ stands for the normal connection of both $\tilde{f}_\theta$ and $\tilde{f}_\theta \circ \sigma$. Let δ be a local section of Nf such that $\eta \circ \sigma^{-1} = \delta \circ p$. Now observe that

$$\nabla^\perp_{d\sigma(\tilde{X})}\big(\eta \circ \sigma^{-1}\big) - \nabla^\perp_{\tilde{X}}\big(\eta \circ \sigma^{-1}\big) = \nabla^\perp_{dp \circ d\sigma(\tilde{X})}\delta - \nabla^\perp_{dp(\tilde{X})}\delta = 0.$$

Therefore $\varSigma_\theta$ is parallel, and this completes the proof. ☐

Proof of Theorem 1. Lemma 4 allows us to define a homomorphism $\varPhi_\theta : \mathcal{D} \to \mathrm{Isom}(\mathbb{S}^4)$ for each $\theta \in [0, 2\pi]$, such that

$$\tilde{f}_\theta \circ \sigma = \varPhi_\theta(\sigma) \circ \tilde{f}_\theta, \quad \sigma \in \mathcal{D}.$$

We observe that $\theta \in \mathcal{S}(f)$ if and only if $\varPhi_\theta(\mathcal{D}) = \{I\}$. Assume that $\mathcal{S}(f)$ is infinite. Then there exists a sequence $\{\theta_m\}$ in $\mathcal{S}(f)$ which converges to some $\theta_0 \in [0, 2\pi]$. From $\varPhi_{\theta_m}(\mathcal{D}) = \{I\}$ for all $m \in \mathbb{N}$, we immediately obtain $\varPhi_{\theta_0}(\mathcal{D}) = \{I\}$. Let $\sigma \in \mathcal{D}$. By applying the Mean Value Theorem to each entry $(\varPhi_\theta(\sigma))_{jk}$ of the corresponding matrix, we have

$$\frac{d}{d\theta}\big(\varPhi_\theta(\sigma)\big)_{jk}(\mathring{\theta}_m) = 0$$

for some $\mathring{\theta}_m$ which lies between θ_0 and θ_m. By continuity, we obtain

$$\frac{d}{d\theta}\big(\varPhi_\theta(\sigma)\big)_{jk}(\theta_0) = 0.$$

Applying repeatedly the Mean Value Theorem, we conclude that

$$\frac{d^n}{d\theta^n}\big(\Phi_\theta(\sigma)\big)_{jk}(\theta_0)=0$$

any integer $n\geq 1$. Since $\Phi_\theta(\sigma)$ is an analytic curve (cf. [8]) in $\mathrm{Isom}(\mathbb{S}^4)$, we infer that $\Phi_\theta(\sigma)=I$ for each $\sigma\in\mathcal{D}$, and so $\mathcal{S}(f)=[0,2\pi]$. □

4. Isometric deformations of compact minimal surfaces

For the proof of Theorem 2, we need some auxiliary lemmas.

LEMMA 5. *Let $f:M\to\mathbb{S}^4$ be a minimal surface which is not contained in any totally geodesic $\mathbb{S}^3$. For any $\theta\in\mathcal{S}(f)$ there exists a parallel and orthogonal bundle isomorphism $T_\theta: Nf\to Nf_\theta$ such that the second fundamental forms of f and f_θ are related by*

$$B^{f_\theta}(X,Y)=T_\theta\circ B^f(J_\theta X,Y),\quad X,Y\in TM.$$

Proof. Since f and f_θ have the same normal curvature, for any simply connected subset U of M there exists a parallel and orthogonal bundle isomorphism $T_\theta^U: Nf|_U\to Nf_\theta|_U$ such that the second fundamental forms of $f|_U$ and $f_\theta|_U$ are related by

$$B^{f_\theta|_U}(X,Y)=T_\theta^U\circ B^{f|_U}(J_\theta X,Y),\quad X,Y\in TM.$$

Let U,V be simply connected subsets of M with $U\cap V\neq\varnothing$. Then we have $T_\theta^U=T_\theta^V$ on $U\cap V\setminus M_0$, where M_0 is the set of points where the normal curvature vanishes. Since $M\setminus M_0$ is dense in M, by continuity, we see that $T_\theta^U=T_\theta^V$ on $U\cap V$. Thus T_θ^U is globally well-defined. □

LEMMA 6 ([15]). *Assume that $f:M\to\mathbb{S}^4$ is not superminimal and let M_1 be the zero set of Φ. Around each point in $M\setminus M_1$, there exist a local complex coordinate (U,z), $U\subset M\setminus M_1$ and orthonormal frames $\{e_1,e_2\}$ in $TM|U$, $\{e_3,e_4\}$ in $Nf|U$ which agree with the given orientations such that*

(i) *the Riemannian metric of M is given by*

$$ds^2=\frac{|dz|^2}{(\kappa_1^2-\mu_1^2)^{1/2}}\quad\text{and}\quad\frac{\partial}{\partial z}=\frac{e_1-ie_2}{2(\kappa_1^2-\mu_1^2)^{1/4}},$$

(ii) *e_3 and e_4 give respectively the directions of the major and the minor axes of the curvature ellipse, and*

(iii) *$H_3=\kappa_1, H_4=i\mu_1$, where κ_1 and μ_1 are smooth real functions with $\kappa=|\kappa_1|,\mu=|\mu_1|$. Moreover, the connection and the normal connection forms, with respect to this frame, are given respectively, by*

$$\omega_{12}=-\frac{1}{4}*d\log\big(\kappa_1^2-\mu_1^2\big),\qquad\omega_{34}=*\frac{\kappa_1d\mu_1-\mu_1d\kappa_1}{\kappa_1^2-\mu_1^2},\tag{3}$$

where $*$ stands for the Hodge operator.

Let $f: M \to \mathbb{S}^4$ be a minimal surface which is not contained in any totally geodesic $\mathbb{S}^3$. Assume hereafter that f is not superminimal. For each point $x \in M \setminus M_1$, we consider an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on a neighborhood $U_x \subset M \setminus M_1$ of x as in Lemma 6. We note that ω_{34} cannot vanish on any open subset of $M \setminus M_1$. For any $\theta \in \mathcal{S}(f)$, $\{e_1, e_2, T_\theta e_3, T_\theta e_4\}$ is a frame along f_θ, where $T_\theta : Nf \to Nf_\theta$ is the bundle isomorphism of Lemma 5. We observe that H_3, H_4 and the corresponding functions H_3^θ, H_4^θ for f_θ, associated to the frame $\{e_1, e_2, T_\theta e_3, T_\theta e_4\}$, satisfy

$$H_3^\theta = \exp(-i\theta) H_3 \quad \text{and} \quad H_4^\theta = \exp(-i\theta) H_4. \tag{4}$$

Using (4) and the Weingarten formula for f_θ, we obtain

$$\tilde{\nabla}_E (T_\theta e_3) = -\kappa_1 \exp(i\theta) df_\theta(\overline{E}) + \omega_{34}(E) T_\theta e_4, \tag{5}$$

$$\tilde{\nabla}_E (T_\theta e_4) = i\mu_1 \exp(i\theta) df_\theta(\overline{E}) - \omega_{34}(E) T_\theta e_3, \tag{6}$$

where $E = e_1 - ie_2$ and $\tilde{\nabla}$ stands for the usual connection in the induced bundle $(i_1 \circ f)^*(T\mathbb{R}^5)$, $i_1 : \mathbb{S}^4 \to \mathbb{R}^5$ being the inclusion map.

Lemma 7. *Assume that for $\theta_j \in \mathcal{S}(f), j = 1, \ldots, n$, there exist vectors $v_j \in \mathbb{R}^5$, such that*

$$\sum_{j=1}^n \langle f_{\theta_j}, v_j \rangle = 0 \quad \textit{on } U_x. \tag{7}$$

Then the following hold:

$$\sum_{j=1}^n \exp(i\theta_j) \big(\kappa_1 \langle T_\theta e_3, v_j \rangle - i\mu_1 \langle T_\theta e_4, v_j \rangle \big) = 0, \tag{8}$$

$$\kappa_1 \sum_{j=1}^n \exp(2i\theta_j) \langle f_{\theta_j}, v_j \rangle = L \sum_{j=1}^n \exp(i\theta_j) \langle T_\theta e_4, v_j \rangle \tag{9}$$

away from the zeros of ω_{34}, where $L = -E(\omega_{34}(E)) - 3i\omega_{12}(E)\omega_{34}(E)$, and

$$\overline{E} \left(\sum_{j=1}^n \exp(i\theta_j) \langle T_\theta e_4, v_j \rangle \right) = -\omega_{34}(\overline{E}) \sum_{j=1}^n \exp(i\theta_j) \langle T_\theta e_3, v_j \rangle. \tag{10}$$

Proof. Our assumption implies that

$$\sum_{j=1}^n \langle df_{\theta_j}, v_j \rangle = 0. \tag{11}$$

Differentiating, using the Gauss formula, (4) and (7), we see that

$$\sum_{j=1}^n \exp(i\theta_j) \big(\overline{H}_3 \langle T_\theta e_3, v_j \rangle + \overline{H}_4 \langle T_\theta e_4, v_j \rangle \big) = 0.$$

Since $H_3 = \kappa_1, H_4 = i\mu_1$, the above immediately implies (8).

Equations (3) yield

$$E(\kappa_1) = -2i\kappa_1\omega_{12}(E) + i\mu_1\omega_{34}(E), \tag{12}$$

$$E(\mu_1) = -2i\mu_1\omega_{12}(E) + i\kappa_1\omega_{34}(E). \tag{13}$$

Differentiating (8) with respect to E, using (5), (6), (12) and (13), we obtain

$$\begin{aligned}\sum_{j=1}^{n}&\exp(i\theta_j)\big(r\langle T_\theta e_3, v_j\rangle - s\langle T_\theta e_4, v_j\rangle\big)\\ &= \frac{1}{2}\big(\kappa_1^2-\mu_1^2\big)\sum_{j=1}^{n}\exp(2i\theta_j)\big\langle df_{\theta_j}(\overline{E}), v_j\big\rangle,\end{aligned}$$

where $r = -i\kappa_1\omega_{12}(E) + i\mu_1\omega_{34}(E)$ and $s = \mu_1\omega_{12}(E) - \kappa_1\omega_{34}(E)$. Using (8), we have

$$\sum_{j=1}^{n}\exp(i\theta_j)\langle T_\theta e_4, v_j\rangle = \frac{\kappa_1}{2\omega_{34}(E)}\sum_{j=1}^{n}\exp(2i\theta_j)\big\langle df_{\theta_j}(\overline{E}), v_j\big\rangle. \tag{14}$$

Differentiating (14) with respect to E, using (6), (8), the Gauss formula and (14), we find

$$\begin{aligned}&\left(E\left(\frac{\kappa_1}{\omega_{34}(E)}\right) - \frac{i\kappa_1\omega_{12}(E)}{\omega_{34}(E)} - i\mu_1\right)\sum_{j=1}^{n}\exp(2i\theta_j)\big\langle df_{\theta_j}(\overline{E}), v_j\big\rangle\\ &= \frac{2\kappa_1}{\omega_{34}(E)}\sum_{j=1}^{n}\exp(2i\theta_j)\langle f_{\theta_j}, v_j\rangle.\end{aligned}$$

By virtue of (12) and since $\kappa_1^2 > \mu_1^2$ on U_x, the above is written as

$$2\omega_{34}(E)\sum_{j=1}^{n}\exp(2i\theta_j)\langle f_{\theta_j}, v_j\rangle = L\sum_{j=1}^{n}\exp(2i\theta_j)\big\langle df_{\theta_j}(\overline{E}), v_j\big\rangle, \tag{15}$$

where $L = -E(\omega_{34}(E)) - 3i\omega_{12}(E)\omega_{34}(E)$. Now (9) follows from (14) and (15). Moreover, we obtain using (6) that

$$\begin{aligned}\overline{E}\left(\sum_{j=1}^{n}\exp(i\theta_j)\langle T_\theta e_4, v_j\rangle\right) &= -i\mu_1\sum_{j=1}^{n}\big\langle df_{\theta_j}(E), v_j\big\rangle\\ &\quad - \omega_{34}(\overline{E})\sum_{j=1}^{n}\exp(i\theta_j)\langle T_\theta e_3, v_j\rangle,\end{aligned}$$

which in view of (11) immediately yields (10). □

We need the topological restrictions for minimal surfaces in $\mathbb{S}^4$ obtained by Eschenburg and Tribuzy [9]. To state their result, we recall that a nonzero function $a\colon M \to [0, +\infty)$ is called of absolute value type [7], [9] if locally $a = a_0|h|$, where a_0 is smooth and positive and h is holomorphic. The zero set of such a function is isolated, and outside its zeros the function is smooth. The order $k \geq 1$ of any $x \in M$ with $a(x)=0$ is the order of h at x. Let $N(a)$ be the sum of all orders for all zeros of a. Then $\Delta \log a$ is bounded on $M \setminus \{a=0\}$ and its integral satisfies

$$\int_M \Delta \log a \, dA = -2\pi N(a).$$

LEMMA 8 ([9]). *Let $f : M \to \mathbb{S}^4$ be a compact minimal nonsuperminimal surface with Gaussian curvature K and normal curvature K_N. Then the functions $a_\pm = (1-K \pm K_N)^{1/2}$ are of absolute value type and the Euler numbers $\chi(M), \chi(Nf)$ of the tangent and the normal bundle satisfy*

$$2\chi(M) \pm \chi(Nf) = -N(a_\mp).$$

Proof of Theorem 2. We assume that f is not superminimal. According to Theorem 1, either $\mathcal{S}(f)$ is finite or $\mathcal{S}(f)=[0,2\pi]$. Suppose to the contrary that $\mathcal{S}(f)=[0,2\pi]$. We claim that the coordinate functions of the minimal surfaces $f_\theta, \theta \in [0,\pi)$, are linearly independent. Since these functions are eigenfunctions of the Laplace operator of M with corresponding eigenvalue 2, this contradicts the fact the eigenspaces of the Laplace operator are finite dimensional. To show that the coordinate functions are linearly independent, it is enough to prove that if

$$\sum_{j=1}^{n} \langle f_{\theta_j}, v_j \rangle = 0, \tag{16}$$

for $0<\theta_1<\cdots<\theta_n<\pi$, then $v_j=0$ for all $1 \leq j \leq n$.

Assume to the contrary that $v_j \neq 0$ for all $1 \leq j \leq n$. Let $M_1=\{x_1,\ldots,x_k\}$ be the zero set of Φ. Around each point $x \in M \setminus M_1$, we choose local complex coordinate (U_x, z) and an orthonormal frame $\{e_1,e_2,e_3,e_4\}$ on $U_x \subset M \setminus M_1$ as in Lemma 6. We consider the complex valued function

$$\psi := \left(\sum_{j=1}^{n} \exp(i\theta_j) \langle T_{\theta_j} e_4, v_j \rangle \right)^2,$$

where $T_{\theta_j} : Nf \to Nf_{\theta_j}$ is the bundle isomorphism of Lemma 5. Obviously ψ is well-defined on $M \setminus M_1$. The second equation of (3) yields

$$\omega_{34}(\overline{E}) = \frac{i}{\kappa_1^2 - \mu_1^2} \big(\kappa_1 \overline{E}(\mu_1) - \mu_1 \overline{E}(\kappa_1) \big).$$

Then (8) and (10) imply that $\overline{E}(\psi(1-\mu^2/\kappa^2))=0$. Hence, the function $\psi(1-\mu^2/\kappa^2) : M \setminus M_1 \to \mathbb{C}$ is holomorphic. Since Ψ is bounded, its isolated singularities are removable and consequently there exists a constant c such that

$$\psi\big(\kappa^2 - \mu^2\big) = c\kappa^2 \quad \text{on } M \setminus M_1. \tag{17}$$

We claim that $c = 0$. Indeed, if $\kappa(x_l) = \mu(x_l) > 0$, for some l, then taking the limit in (17) along a sequence of points in $M \setminus M_1$ which converges to x_l, we deduce that $c = 0$.

Suppose now that $\kappa(x_l) = \mu(x_l) = 0$ for all $1 \le l \le k$. Let (V, z) be a local complex coordinate around x_l with $z(x_l) = 0$. It is a well-known consequence of the Codazzi equation that

$$B^{(2,0)} = B\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) dz^2$$

is holomorphic as a $Nf \otimes \mathbb{C}$-valued tensor field (cf. [10]). Since $B^{(2,0)}$ is not identically zero and $B|_{x_l} = 0$, we may write

$$B^{(2,0)} = z^{m_l} \tilde{B}^{(2,0)} \quad \text{on } V, \tag{18}$$

where m_l is a positive integer and $\tilde{B}^{(2,0)}$ is a tensor field of type $(2,0)$ with $\tilde{B}^{(2,0)}|_{x_l} \neq 0$. We define the Nf-valued tensor field $\tilde{B} := \tilde{B}^{(2,0)} + \overline{\tilde{B}^{(2,0)}}$. Since its $(1,1)$-part vanishes, it follows easily that $\tilde{B}$ maps the unit circle on each tangent plane into an ellipse, on the corresponding normal space, whose length of the semi-axes are denoted by $\tilde{\kappa} \ge \tilde{\mu} \ge 0$. We also consider the differential form of type $(4,0)$

$$\tilde{\Phi} := \left\langle \tilde{B}^{(2,0)}, \tilde{B}^{(2,0)} \right\rangle$$

which, in view of (18), is related to the Hopf differential of f by $\Phi = z^{2m_l}\tilde{\Phi}$. We split Φ and $\tilde{\Phi}$, with respect to arbitrary orthonormal frames $\{\xi_1, \xi_2\}$ and $\{\xi_3, \xi_4\}$ of $TM|V$ and $Nf|V$, respectively as

$$\Phi = \frac{1}{4}\left(\overline{H}_3^2 + \overline{H}_4^2\right)\varphi^4 = \frac{1}{4}k_+ k_- \varphi^4,$$
$$\tilde{\Phi} = \frac{1}{4}\left(\overline{\tilde{H}}_3^2 + \overline{\tilde{H}}_4^2\right)\varphi^4 = \frac{1}{4}\tilde{k}_+ \tilde{k}_- \varphi^4,$$

where

$$k_\pm = \overline{H}_3 \pm i\overline{H}_4, \qquad \tilde{k}_\pm = \overline{\tilde{H}}_3 \pm i\overline{\tilde{H}}_4,$$
$$H_\alpha = \left\langle B(\xi_1, \xi_1), \xi_\alpha \right\rangle + i\left\langle B(\xi_1, \xi_2), \xi_\alpha \right\rangle,$$
$$\tilde{H}_\alpha = \left\langle \tilde{B}(\xi_1, \xi_1), \xi_\alpha \right\rangle + i\left\langle \tilde{B}(\xi_1, \xi_2), \xi_\alpha \right\rangle, \quad \alpha = 3, 4.$$

From (18), we obtain $\overline{H}_\alpha = z^{m_l}\overline{\tilde{H}}_\alpha$, or equivalently, $k_\pm = z^{m_l}\tilde{k}_\pm$. Hence

$$\kappa = |z|^{m_l}\tilde{\kappa}, \qquad \mu = |z|^{m_l}\tilde{\mu}. \tag{19}$$

Now (17) yields

$$\psi\left(\tilde{\kappa}^2 - \tilde{\mu}^2\right) = c\tilde{\kappa}^2 \quad \text{on } V \setminus \{x_l\}. \tag{20}$$

If $\tilde{\kappa}(x_l) > \tilde{\mu}(x_l)$ for all $1 \le l \le k$, then (19) implies that $N(a_+) = \sum_{l=1}^{k} m_l = N(a_-)$. Hence, Lemma 8 yields $\chi(Nf) = 0$, which contradicts our assumption. Thus, $\tilde{\kappa}(x_l) = \tilde{\mu}(x_l)$ for some $1 \le l \le k$. Taking the limit in (20), along a sequence of points in $V \setminus \{x_l\}$ which converges to x_l, we obtain $c\tilde{\kappa}^2(x_l) = 0$. Since $\tilde{B}|_{x_l} \neq 0$, we infer that $c = 0$.

From (17), we obtain $\psi = 0$ on $M \setminus M_1$. Hence, (9) implies

$$\sum_{j=1}^{n} \exp(2i\theta_j)\langle f_{\theta_j}, v_j\rangle = 0.$$

Combining this with (16), we have

$$\sum_{j=2}^{n} \langle f_{\theta_j}, w_j\rangle = 0,$$

where $w_j := \lambda_j v_j \neq 0, j = 2, \ldots, n$, and $\lambda_j = \cos 2\theta_n - \cos 2\theta_1$ or $\lambda_j = \sin 2\theta_n - \sin 2\theta_1$. By induction, we finally conclude that $\langle f_{\theta_n}, w\rangle = 0$, for some nonzero vector w. So f_{θ_n} lies in a totally geodesic $\mathbb{S}^3$, contradiction. This concludes the proof of the theorem. $\square$

Proof of Corollary 3. Let $f_{\theta_j}, j = 1, \ldots, n, 0 = \theta_0 < \theta_1 < \cdots < \theta_n \leq 2\pi$ is the maximal family of noncongruent minimal surfaces in $\mathbb{S}^4$ which are isometric to f and have the same normal curvature. Since the second fundamental form of the minimal surfaces $f_t := f \circ \varphi_t$ depends continuously on the parameter, we deduce that f_t is congruent to exactly one of f_{θ_j} for all t. Since $f \circ \varphi_0 = f$, we conclude that f_t is congruent to f for all t. $\square$

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