Illinois Journal of Mathematics
Volume 54, Number 4, Winter 2010, Pages 1313–1328
S 0019-2082

ON THE ASSOCIATION AND CENTRAL LIMIT THEOREM FOR SOLUTIONS OF THE PARABOLIC ANDERSON MODEL

M. CRANSTON AND G. MUELLER

Abstract. We consider large scale behavior of the solution set $\{u(t,x) : x \in \mathbf{Z^d}\}$ of the parabolic Anderson equation

$$u(t,x) = 1 + \kappa \int_0^t \Delta u(s,x)\,ds + \int_0^t u(s,x)\,\partial W_x(s), \quad x \in \mathbf{Z^d}, t \geq 0,$$

where $\{W_x : x \in \mathbf{Z^d}\}$ is a field of i.i.d. standard, one-dimensional Brownian motions, Δ is the discrete Laplacian and $\kappa > 0$. We establish that the properly normalized sum, $\sum_{x\in\Lambda_L} u(t,x)$, over spatially growing boxes $\Lambda_L = \{x \in \mathbf{Z^d} : \|x\| < L\}$ has an asymptotically normal distribution if the box Λ_L grows sufficiently quickly with t and provided κ is sufficiently small depending on dimension. The asymptotic distribution of properly normalized sums over spatially growing disjoint boxes Λ_L^1, Λ_L^2 is asymptotically independent. Thus, on sufficiently large scales the field of solutions averaged over disjoint large boxes looks like an i.i.d. Gaussian field. We identify the variance of this Gaussian distribution in terms of the eigenfunction of the positive eigenvalue of the operator $2\kappa\Delta + \delta_0$.

1. Introduction

In this paper, we consider the property of association for the solutions of the discrete space parabolic Anderson model and its application to a central limit theorem for the field of solutions of this equation.

A collection of random variables $\{X_k, k \in S\}$ where S is a countable set, is said to be *associated* if for any d and coordinate-wise increasing functions

Received April 9, 2010; received in final form January 14, 2011.
Research of first author supported by a Grant NSF DMS 10-07176.
2010 *Mathematics Subject Classification.* Primary 60F05, 60H15. Secondary 60G60.

$f, g : \mathbf{R^d} \to \mathbf{R}$, and any finite subcollection $X_{k_1}, X_{k_2}, \ldots, X_{k_d}$, it holds that

$$\operatorname{Cov}\bigl(f(X_{k_1}, X_{k_2}, \ldots, X_{k_d}), g(X_{k_1}, X_{k_2}, \ldots, X_{k_d})\bigr) \geq 0.$$

This notion was introduced in [6] and is of course related to the FKG inequality. One important aspect of this property was developed in [7] where a central limit theorem was derived for the collection $\{X_k, k \in \mathbf{Z^d}\}$ under the assumptions that the $\{X_k, k \in \mathbf{Z^d}\}$ are stationary and satisfy finite susceptibility

$$\sum_{k \in \mathbf{Z^d}} \operatorname{Cov}(X_0, X_k) < \infty. \tag{1}$$

A classical application of this is to take the $X_k = \sigma(k) \in \{0,1\}^{\mathbf{Z^d}}$, the spins of a stochastic Ising model and derive a central limit theorem for sums, $\sum_{k \in \Lambda} \sigma(k)$ over growing boxes Λ, with respect to a Gibbs state. The spins are correlated, but they possess the property of being associated and stationary with respect to the Gibbs state.

The parabolic Anderson model is defined as a parabolic differential equation with white noise potential. Let $\{W_x : x \in \mathbf{Z^d}\}$ be i.i.d. standard, one-dimensional Brownian motions defined on some probability space (Ω, Q). We say that this field is *δ-correlated* since $E[W_x(s)W_y(s)] = (s \wedge t)\delta_0(x - y)$. In what follows, $\circ\, dW_x(t)$ denotes the Stratonovitch differential of $W_x(t)$ while $dW_x(t)$ denotes the Itô differential. The parabolic Anderson equation is a Cauchy initial-value problem with random potential $\circ\, dW_x(t)$, given by

$$\frac{\partial u}{\partial t}(t,x) = \kappa \Delta u(t,x) + u(t,x) \circ dW_x(t), \quad u(0,x) \equiv 1,$$

where Δ is the discrete Laplacian and $\kappa > 0$. To be precise, by the discrete Laplacian we mean $\Delta f(x) = \sum_{|y-x|=1}(f(y) - f(x))$. This differential equation makes sense in its integral form

$$u(t,x) = 1 + \kappa \int_0^t \Delta u(s,x)\, ds + \int_0^t u(s,x) \circ dW_x(s), \quad x \in \mathbf{Z^d}, t \geq 0. \tag{2}$$

The reader is referred to [2] for fundamental information on this equation and its applications. Our interest is in the behavior of the field of solutions $\{u(t,x) : x \in \mathbf{Z^d}\}$. We stress that the random variables in this field are dependent and their correlation structure is examined in detail in Theorem 3.1. A crucial property of the solutions of (2) is that of intermittency which is defined in terms of the Lyapunov exponents which are the a.s. limits

$$\gamma(p) = \lim_{t \to \infty} \frac{1}{t} \ln E\bigl[u^p(t,x)\bigr].$$

Full intermittency is then the property that $\gamma(p)$ is strictly convex on $[1, \infty)$, that is

$$\frac{\gamma(p)}{p} < \frac{\gamma(p+1)}{p+1} \quad \forall p \geq 1.$$

It is by now classical and was proven in [2] that in dimensions $d = 1, 2$ full intermittency holds for all $\kappa > 0$ but in dimensions $d \geq 3$, full intermittency only holds for $0 < \kappa < \kappa_c(d)$ where $\kappa_c(d)$ is a dimension dependent constant. In [4], it was shown in the case of full intermittency that a CLT holds for sums, namely

$$\frac{\sum_{x \in \Lambda_L}(u(t,x) - e^{t/2})}{\sqrt{\operatorname{Var} \sum_{x \in \Lambda_L} u(t,x)}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \tag{3}$$

with $\Lambda_L = \{x \in \mathbf{Z^d} : \|X\| < L\}$ provided that $L = L(t) \to \infty$ sufficiently rapidly. The proof followed Bernstein's method of decomposing the sum (3) into sums over disjoint, slightly separated boxes. The proof was quite technical and relied on approximation of the solution $u(t,x)$ to obtain some degree of independence and a difficult large deviation result from [5].

In this paper, we give a new proof of a stronger result, Theorem 4.1, using the ideas of Newman about associated random variables. The proof is much simpler than the previous one. It also yields more information about the variance of $\sum_{x \in \Lambda_L} u(t,x)$, relating it to the first eigenfunction of $2\kappa\Delta + \delta_0$ and gives the joint distribution of these sums over disjoint growing boxes. Moreover, the technique extends easily to cover related models. For example, the technique applies equally to parabolic Anderson equations for some other choices of covariance structure for the potentials than the δ spatial correlation case that we consider here. This follows from the fact that the field of solutions will still be associated and stationary under appropriate conditions on the correlation function. Indeed, let $\{\widetilde{W}_x : x \in \mathbf{R^d}\}$ be a field of standard, one-dimensional Brownian motions on some probability space (Ω, Q) with correlation

$$\operatorname{Cov}\bigl(\widetilde{W}_x(s), \widetilde{W}_y(t)\bigr) = (s \wedge t)\Gamma(x - y)$$

for which Γ is a nonnegative. This of course includes the special case $\Gamma(x) = \delta_0(x)$, covered in (2). The condition Γ nonnegative ensures that the increments $\widetilde{W}_{x_i}(t_i) - \widetilde{W}_{x_i}(s_i), t_i > s_i, i = 1, 2, \ldots, n$, are associated. This is a result of Pitt, [8] which states that a necessary and sufficient condition for the associativity of a Gaussian vector is the point-wise nonnegativity of its correlation function. It's an easy matter to check that for any $x, y \in \mathbf{Z^d}$ and $t > s, u > v$, one has

$$\operatorname{Cov}\bigl(\widetilde{W}_x(t) - \widetilde{W}_x(s), \widetilde{W}_y(u) - \widetilde{W}_y(v)\bigr) \geq 0.$$

What is also needed is that the field of solutions of an equation analogous to (2)

$$u(t,x) = 1 + \kappa \int_0^t \Delta u(s,x)\, ds + \int_0^t u(s,x) \circ d\widetilde{W}_x(s), \quad x \in \mathbf{Z^d}, t \geq 0, \tag{4}$$

satisfy the finite susceptibility condition (1). This will occur when the discrete Schrödinger operator $2\kappa\Delta + \Gamma$ possesses a simple positive eigenvalue and that the eigenfunction ψ associated to this eigenvalue decay exponentially, $\psi(x) \leq ce^{-c|x|}$. Given a Γ with compact support, this will be satisfied for sufficiently small positive κ depending on Γ by arguments similar to those in [2].

This idea also applies to the stationary case considered in [1] (though we won't carry this out here) and even may be adapted to derive a CLT in the continuous space setting under appropriate conditions. If one now takes Δ to be the Laplacian on $\mathbf{R^d}$ and $\{W_x : x \in \mathbf{R^d}\}$ to be a field of Brownian motions with a smooth, positive correlation function Σ, then we are interested in the solutions of (2) with this operator and potential on continuous space. The field of solutions will be correlated and the finite susceptibility condition will hold provided that the operator $\Delta + \Sigma$ possesses a positive eigenvalue with exponentially decreasing eigenfunction.

The paper is organized as follows. In Section 2, we introduce the notion of association, some basic results about it and prove that the field of solutions of (4) is associated. In Section 3, we derive necessary estimates on the covariance of the solutions to (2) at different spatial points. Section 4 contains the main result and its proof.

2. Association

We turn now to cover some background information on associated random variables. An infinite collection of random variables is associated if any finite subcollection is associated. The following theorem was noted in [6].

THEOREM 2.1 (Esary, Proschan, Walkup). *If $\{Y_k : k = 1, 2, \ldots, d\}$ are independent random variables and $f_1, f_2, \ldots, f_n$ are real valued, coordinate-wise increasing functions on $\mathbf{R^d}$ then denoting $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$ the variables $f_1(\mathbf{Y}), f_2(\mathbf{Y}), \ldots, f_n(\mathbf{Y})$ are associated.*

If $T_1^n, T_2^n, \ldots, T_k^n$ are associated for all n and $T_j^n \xrightarrow{\mathcal{L}} T_j, j = 1, 2, \ldots, k$, then $T_1, T_2, \ldots, T_k$ are associated.

We now use this result to show solutions of the parabolic Anderson equation (4) are associated.

THEOREM 2.2. *Let $\{\widetilde{W}_x : x \in \mathbf{Z^d}\}$ be a Gaussian field of standard, one-dimensional Brownian motions on some probability space (Ω, Q) with correlation*

$$\mathrm{Cov}\big(\widetilde{W}_x(s), \widetilde{W}_y(s)\big) = (s \wedge t)\Gamma(x - y),$$

where $\Gamma(x) \geq 0, x \in \mathbf{Z^d}$. Let $\mathcal{U}_t = \{u(t, x) : x \in \mathbf{Z^d}\}$ be the field of solutions of (4). Then $\mathcal{U}_t$ is associated.

Proof. We begin by recalling the Feynman–Kac representation of solutions of (4). Namely, the solutions may be expressed as an average over continuous time random walk paths. Let $(\{X(s) : s \geq 0\}, \mathcal{F}_s, P_x^\kappa)$ be the continuous time Markov process on $\mathbf{Z^d}$ with generator $\kappa\Delta$. The σ-fields $\mathcal{F}_s$ are given by $\mathcal{F}_s =$

$\sigma(X(u): u \le s)$ and P_x^κ is the measure on paths started at x. This process is assumed to be independent of the field $\{\widetilde{W}_x : x \in \mathbf{Z^d}\}$. Then by the Feynmen–Kac formula we may write

$$u(t,x) = E_x^\kappa\left[\exp\left\{\int_0^t d\widetilde{W}_{X(t-s)}(s)\right\}\right].$$

Since the property of association is a property of the distribution of random variables, we may work instead with the field of random variables $\mathcal{V}_t = \{v(t,x) : x \in \mathbf{Z^d}\}$ defined by

$$v(t,x) = E_x^\kappa\left[\exp\left\{\int_0^t d\widetilde{W}_{X(s)}(s)\right\}\right], \tag{5}$$

which has the same distribution as the field $\mathcal{U}_t$.

Let $\mathcal{D}_t = D([0,t], \mathbf{Z^d})$ be the space of paths which are right continuous, possess left limits and have a finite number of jumps of size one only. Given a finite collection of such paths $X^i \in D([0,t],\mathbf{Z^d}), i \in \{1,2,\dots,n\}$, define the joint Gaussian random variables

$$H_t(X^i) = \int_0^t d\widetilde{W}_{X^i(s)}(s), \quad i \in \{1,2,\dots,n\}.$$

The covariance of this vector is given by

$$C^{ij} = E\left[H_t(X^i)H_t(X^j)\right] = \int_0^t \Gamma\left(X^i(s) - X^j(s)\right) ds$$

and since $\Gamma(x) \ge 0$, it follows from Pitt's theorem that the random variables $H_t(X^i)$ are associated. Since increasing functions of associated random variables are associated, the variables $\exp\{H_t(X^i)\} \wedge N, i \in \{1,2,\dots,n\}$, are associated for any integer N. Going a step further, given $p_j \ge 0$, $j = 1,2,\dots$, with $\sum_{j=1}^\infty p_j = 1$ and collections $X_k^i \in D([0,t],\mathbf{Z^d}), i \in \{1,2,\dots,n\}, k \in \{1,2,\dots, m\}$, similar reasoning implies that the random variables $\sum_{i=1}^n p_i \times \exp\{H_t(X_k^i)\} \wedge N, k \in \{1,2,\dots,m\}$, are associated. These can be written

$$\sum_{i=1}^n p_i \exp\left\{H_t\left(X_k^i\right)\right\} \wedge N = \int_{\mathcal{D}_t} \exp\left\{H_t(\gamma)\right\} \wedge N \, d\mu_k(\gamma)$$

for measures defined by $\mu_k = \sum_{i=1}^n p_i \delta_{X_k^i}$. Now take sequences of such measures $\mu_k^n, k \in \{1,2,\dots,m\}$, converging weakly to $P_{x_k}^\kappa, k \in \{1,2,\dots,m\}$. Since the functions $\exp\{H_t(X_k^i)\} \wedge N$ are continuous on $\mathcal{D}_t$ in the metric ρ where $\rho^2(X,Y) = \Gamma(0)t - \int_0^t \Gamma(X(s) - Y(s))\, ds$. $\square$

REMARK 2.1. A heuristic argument the random variables $\{u(t,x) : x \in \mathbf{Z^d}\}$ are associated can be given using the Malliavin calculus. This relies on showing that $\{v(t,x) : x \in \mathbf{Z^d}\}$ are increasing functions of the increments of $\{W_x : x \in \mathbf{Z^d}\}$. Since increasing functions of independent random variables are associated, this gives that the field of solutions of (2) is associated. The

Malliavin derivative, denoted $D_{t,x}$, is heuristically equal to $\frac{\partial}{\partial(dW_x(t))}$. Express the integral in the exponential in the representation of $v(t,x)$ at (5) as

$$H_T(X) = \sum_{x\in\mathbf{Z^d}} \int_0^T W_x(t)\delta_x(X_t).$$

This has the form

$$W(h) = \int_0^T h(t,x)\,dW_x(t),$$

where $h = h(t,X) = \delta_x(X(t))$ and obviously, $h \in L^2([0,T]) \times \mathbf{Z^d}$. The Malliavin derivative of $W(h)$ is thus the element of $L^2([0,T]) \times \mathbf{Z^d}$ defined by

$$D_{t,x}H_T = \delta_x\bigl(X(t)\bigr).$$

Then taking $f(y) = e^y$ and applying the chain rule, we find the Malliavin derivative of $f(H_T)$ is given by

$$D_{t,x}f(H_T) = f(H_T)\delta_x\bigl(X(t)\bigr).$$

Taking the average over paths X and then differentiating yields

$$D_{t,x}v(T,x) = E\bigl[\delta_x\bigl(X(t)\bigr)e^{H_t(X)}\bigr] > 0.$$

Thus, one expects the association property to hold for the field $\{v(t,x) : x \in \mathbf{Z^d}\}$.

The field $\{v(t,x) : x \in \mathbf{Z^d}\}$ satisfies the strictly weaker property of being positively correlated (in the computation below, X and Y are independent copies of the Markov process with generator $\kappa\Delta$)

$$\begin{aligned}
\operatorname{Cov}\bigl(v(t,x), v(t,y)\bigr) &= E\bigl[v(t,x)v(t,y)\bigr] - E\bigl[v(t,x)\bigr]E\bigl[v(t,y)\bigr] \\
&= E^{\kappa}_{x,y}E\bigl[\exp\bigl\{H_t(X) + H_t(Y)\bigr\}\bigr] - e^t \\
&= E^{\kappa}_{x,y}\left[\exp\left\{\frac{1}{2}E\bigl[\bigl(H_t(X) + H_t(Y)\bigr)^2\bigr]\right\}\right] - e^t \\
&= E^{\kappa}_{x,y}\left[\exp\left\{t + \int_0^t \delta_0\bigl(X(s) - Y(s)\bigr)\,ds\right\}\right] - e^t \\
&= e^t\left(E^{\kappa}_{x,y}\left[\exp\left\{\int_0^t \delta_0\bigl(X(s) - Y(s)\bigr)\,ds\right\}\right] - 1\right) \\
&\geq 0.
\end{aligned}$$

We now give a description of the result of [7] which is of principal interest to us. It is an inequality for the characteristic function of sums of associated random variables.

THEOREM 2.3 (Newman). *Suppose $X_1, X_2, \ldots, X_k$ have finite variance and are associated. Then, for any $r_1, r_2, \ldots, r_k \in \mathbf{R}$,*

$$\left| E\left[\exp\left\{ i \sum_{j=1}^{k} r_j X_j \right\}\right] - \prod_{j=1}^{k} E\left[\exp\{ i r_j X_j\}\right] \right| \leq \frac{1}{2} \sum_{l \neq m} |r_l||r_m| \operatorname{Cov}(X_l, X_m).$$

The content of this theorem is that if the sum of the covariances can be controlled, then the distribution of associated random variables can be compared to the distribution of independent random variables.

3. Moment properties

We now discuss the moment properties of solutions $u(t,x)$ to (2), again working instead with their identically distributed analogs $v(t,x)$ defined by (5). First, as noted above,

$$E\left[v(t,x)\right] = e^{\frac{t}{2}},$$

and so the first moment Lyapunov exponent is easy to compute,

$$\lim_{t\to\infty} \frac{1}{t} \ln E\left[v(t,x)\right] = \frac{1}{2} \equiv \lambda(1).$$

The second Lyapunov exponent exists

$$\lim_{t\to\infty} \frac{1}{t} \ln E\left[v^2(t,x)\right] = \lambda(2).$$

By Hölder's inequality

$$2\lambda(1) \leq \lambda(2),$$

but strict inequality, which is called intermittency, holds under certain restrictions on the parameter κ. As developed in [2], one has

$$2\lambda(1) < \lambda(2), \qquad 0 < \kappa < \infty, \quad d = 1, 2, \tag{6}$$

but for dimensions $d \geq 3$, there is a dimension dependent $\kappa_c(d) > 0$ such that

$$2\lambda(1) < \lambda(2), \qquad 0 < \kappa < \kappa_c(d), \quad d \geq 3. \tag{7}$$

In the rest of this paper we shall always assume we are in the intermittent regime, that is $\kappa < \kappa_c(d)$ so that $2\lambda(1) < \lambda(2)$. The mixed second moments $E[u(t,x)u(t,y)]$ are significant in the present work. They are given by

$$\begin{aligned} E\left[u(t,x)u(t,y)\right] &= e^t E_{x,y}^{\kappa}\left[\exp\left\{ \int_0^t \delta_0\big(X(s) - Y(s)\big)\, ds \right\}\right] \\ &= e^t E_{x-y}^{2\kappa}\left[\exp\left\{ \int_0^t \delta_0\big(X(s)\big)\, ds \right\}\right]. \end{aligned}$$

The asymptotics of the function $E[u(t,x)u(t,y)]$ can be evaluated as follows. Define

$$Z_{\beta,t}^{\kappa}(x) = E_x^{\kappa}\left[\exp\left\{ \beta \int_0^t \delta_0\big(X(s)\big)\, ds \right\}\right].$$

Then observe that

$$e^{-t}E\big[u(t,x)u(t,y)\big] = Z^{2\kappa}_{1,t}(x-y).$$

But there is a scaling relation

$$\begin{aligned}
Z^{2\kappa}_{1,t}(x-y) &= E^{2\kappa}_{x-y}\left[\exp\left\{\int_0^t \delta_0\big(X(s)\big)\,ds\right\}\right]\\
&= E^{2\kappa}_{x-y}\left[\exp\left\{\int_0^{2\kappa t} \delta_0\big(X(s/2\kappa)\big)\,d(s/2\kappa)\right\}\right]\\
&= E^{1}_{x-y}\left[\exp\left\{\frac{1}{2\kappa}\int_0^{2\kappa t} \delta_0\big(X(s)\big)\,ds\right\}\right]\\
&= Z^1_{\frac{1}{2\kappa},2\kappa t}(x-y),
\end{aligned}$$

since $Y(\cdot) = X((2\kappa)^{-1}\cdot)$ is rate 1 simple symmetric random walk on $\mathbf{Z}^{\mathbf{d}}$ with respect to $P^{2\kappa}_x$. This shows that

$$Z^{2\kappa}_1(t,x) = Z^1_{\frac{1}{2\kappa},2\kappa t}(x). \tag{8}$$

The function $Z_{\beta,t}(x) = Z^1_{\beta,t}(x)$ arises as the partition function of a homopolymer, [3], and by the Feyman–Kac formula, $Z_{\beta,t}(x)$ solves

$$\frac{\partial}{\partial t}Z_{\beta,t}(x) = \Delta Z_{\beta,t}(x) + \beta\delta_0(x)Z_{\beta,t}(x), \quad Z_{\beta,0}(x) \equiv 1.$$

The same proof that shows (6) and (7) also shows that the spectrum of the operator

$$H_\beta = \Delta + \beta\delta_0$$

satisfies

$$\text{spectrum}\{H_\beta\} = [-4d,0] \cup \big\{\lambda_0(\beta)\big\}, \quad \beta > 0, d = 1,2,$$

and for $d \geq 3$, there is a dimension dependent $\beta_c(d)$ such that

$$\text{spectrum}\{H_\beta\} = [-4d,0], \quad 0 < \beta < \beta_c(d), d \geq 3,$$

and

$$\text{spectrum}\{H_\beta\} = [-4d,0] \cup \big\{\lambda_0(\beta)\big\}, \quad \beta_c(d) < \beta, d \geq 3.$$

In the above, $\lambda_0(\beta) > 0$ is a simple eigenvalue for H_β. In fact, one now sees that $\beta_c(d) = 1/2\kappa_c(d)$. We denote the corresponding eigenfunction by ψ_β and note that it is given by, see [3],

$$\psi_\beta(x) = \frac{1}{(2\pi)^d}\int_{\mathbf{T}^{\mathbf{d}}} \frac{e^{i\langle\phi,x\rangle}}{\lambda_0(\beta)+\Phi(\phi)}\,d\phi, \tag{9}$$

where

$$\Phi(\phi) = 2\sum_{j=1}^{d}(1-\cos\phi_j)$$

is the symbol (Fourier transform) of Δ and $\mathbf{T^d}$ is the d-dimensional torus. The representation (9) can be used to establish the existence of a positive constant $c = c(\beta, d)$ such that

$$\psi_\beta(x) \le ce^{-c|x|}, \quad x \in \mathbf{Z^d}.$$

By the spectral theorem, letting E_λ be the resolution of the identity for the operator $H_{1/2\kappa}$ one has

$$Z_{1/2\kappa,t}(x) = e^{\lambda_0(1/2\kappa)t}\psi_{1/2\kappa}(x)\|\psi_{1/2\kappa}\|_{L^1} + \int_{-4d}^{0} e^{\lambda t}\langle dE_\lambda 1_x, 1\rangle.$$

Note that $Z_{1/2\kappa,t}(x) = E_x^{1/2\kappa}[\exp\{\int_0^t \delta_0(X(s))\,ds\}] \ge 1$.

THEOREM 3.1. *For $d = 1, 2$ and any $\kappa > 0$ or for $d \ge 3$ and $\kappa < \kappa_c(d) = 1/2\beta_c(d)$,*

$$\begin{aligned} Z_{1/2\kappa,t}(x) - 1 \\ &= \exp\bigl(\lambda_0(1/2\kappa)t\bigr) \\ &\quad \times \bigl(\psi_{1/2\kappa}(x)\|\psi_{1/2\kappa}\|_{L^1(\mathbb{R}^d)} + q(x)O\bigl(\exp(-\varepsilon t)\bigr)\bigr), \quad t \to \infty, \end{aligned} \tag{10}$$

where

$$\sum_{x \in \mathbf{Z^d}} q(x) < \infty.$$

In addition, when there is a positive eigenvalue for $H_{1/2\kappa}$, this eigenfunction satisfies

$$\psi_{1/2\kappa}(x) \le ce^{-c|x|}, \quad x \in \mathbf{Z^d},$$

where $c = c(1/2\kappa, d) > 0$ depends on d and κ.

Proof. Denoting $\beta = 1/2\kappa$, consider the fundamental solution of the heat equation

$$\begin{aligned} &\frac{\partial}{\partial t} p_\beta(t, x, y) = \Delta p_\beta(t, x, y) + \beta\delta_0(x)p_\beta(t, x, y), \\ &\lim_{t \to 0} p_\beta(t, x, y) = \delta_x(y). \end{aligned}$$

Then

$$Z_{\beta,t}(x) = \sum_{y \in \mathbf{Z^d}} p_\beta(t, x, y).$$

The corresponding resolvent has the kernel

$$R_{\lambda,\beta}(x, y) = \int_0^\infty e^{-\lambda t} p_\beta(t, x, y)\,dt$$

which satisfies

$$R_{\lambda,\beta}(x, y) = \frac{R_{\lambda,0}(x, y)}{1 - \beta I(\lambda)}, \tag{11}$$

where

$$I(\lambda) = R_{\lambda,0}(0,0).$$

From (11), we can see that $\lambda_0(\beta)$ satisfies $I(\lambda_0(\beta)) = \frac{1}{\beta}$. The partition function $Z_{\beta,t}(x)$ can be expressed in terms of the resolvent kernel by inverting the Laplace transform. Since we are in the case $\beta > \beta_c(d) = \frac{1}{I(0)}$, the kernel $R_{\lambda,\beta}(x,y)$ is easily seen to have a simple pole at $\lambda_0(\beta)$. Choosing the contour to be $\Gamma(a) = \{a + is : s \in \mathbf{R^d}\}$ with $a > \lambda_0(\beta)$, we have

$$Z_{\beta,t}(x) - 1 = \frac{-\beta}{2\pi i} \int_{\Gamma(a)} \frac{e^{\lambda t}}{\lambda} R_{\lambda,\beta}(x,0)\, d\lambda.$$

By (11), the residue of the simple pole of $R_{\lambda,\beta}(x,0)$ at $\lambda = \lambda_0(\beta)$ is $R_{\lambda_0(\beta),0}(x, 0)$, and since $\psi_\beta(x) = R_{\lambda_0(\beta),0}(x,0)$, on selecting $\epsilon \in (0, \lambda_0(\beta))$, we get

$$Z_{\beta,t}(x) - 1 = \frac{\beta e^{\lambda_0(\beta)t}}{\lambda_0(\beta)} \psi_\beta(x)\psi_\beta(0) - \frac{\beta}{2\pi i} \int_{\Gamma(\lambda_0(\beta)-\epsilon)} \frac{e^{\lambda t}}{\lambda} R_{\beta,\lambda}(x,0)\, d\lambda.$$

Since $(\Delta + \beta\delta_0 - \lambda_0(\beta))\psi_\beta = 0$, we have

$$\sum_{x\in\mathbf{Z^d}} \beta\delta_0(x)\psi_\beta(x) = \sum_{x\in\mathbf{Z^d}} \big(\lambda_0(\beta) - \Delta\big)\psi_\beta(x) = \sum_{x\in\mathbf{Z^d}} \lambda_0(\beta)\psi_\beta(x).$$

That is,

$$\lambda_0(\beta) \sum_{y\in\mathbf{Z^d}} \psi_\beta(y) = \lambda_0(\beta)\psi_\beta(0).$$

Consequently,

$$\begin{aligned} &Z_{\beta,t}(x) - 1 \\ &\quad = e^{\lambda_0(\beta)t}\psi_\beta(x)\|\psi_\beta\|_{L^1} - \frac{\beta}{2\pi i} \int_{\Gamma(\lambda_0(\beta)-\epsilon)} \frac{e^{\lambda t}}{\lambda} R_{\beta,\lambda}(x,0)\, d\lambda \\ &\quad = e^{\lambda_0(\beta)t}\bigg(\psi_\beta(x)\|\psi_\beta\|_{L^1} - \frac{\beta}{2\pi i} \int_{\Gamma(\lambda_0(\beta)-\epsilon)} \frac{e^{(\lambda-\lambda_0(\beta))t}}{\lambda} R_{\beta,\lambda}(x,0)\, d\lambda\bigg). \end{aligned}$$

Note that $e^{(\lambda-\lambda_0(\beta))t} = O(e^{-\epsilon t})$ on the contour $\Gamma(\lambda_0(\beta) - \epsilon)$, while

$$\begin{aligned} \sum_{x\in\mathbf{Z^d}} R_{\beta,\lambda}(x,0) &= \int_0^\infty \sum_{x\in\mathbf{Z^d}} \frac{e^{-\lambda t} p_0(t,0,x)}{\lambda(1-\beta I(\lambda))}\, dt \\ &= \frac{1}{\lambda^2(1-\beta I(\lambda))}. \end{aligned}$$

Thus,

$$-\frac{\beta}{2\pi i} \int_{\Gamma(\lambda_0(\beta)-\epsilon)} \frac{e^{(\lambda-\lambda_0(\beta))t}}{\lambda} R_{\beta,\lambda}(x,0)\, d\lambda = q(x) O\big(e^{-\epsilon t}\big),$$

where

$$\sum_{x\in\mathbf{Z^d}} q(x) < \infty.$$

This completes the proof of the theorem. $\square$

Recalling the equivalence in law stated in (8) it follows from (10) that

$$Z^{2\kappa}_{1,t}(x) \sim e^{\lambda_0(1/2\kappa)2\kappa t}\big(\psi_{1/2\kappa}(x)\|\psi_{1/2\kappa}\|_{L^1} + q(x)O\big(e^{-\epsilon 2\kappa t}\big)\big).$$

This leads to the following corollary.

COROLLARY 3.1. *For $d = 1, 2$ and any $\kappa > 0$ or for $d \geq 3$ and $0 < \kappa < \kappa_c(d)$,*

$$\begin{aligned}
&\operatorname{Cov}\big(u(t,x), u(t,y)\big)\\
&\quad = e^t\big(Z^{2\kappa}_{1,2\kappa t}(x-y) - 1\big)\\
&\quad \sim e^t e^{\lambda_0(1/2\kappa)2\kappa t}\big(\psi_{1/2\kappa}(x-y)\|\psi_{1/2\kappa}\|_{L^1} + q(x-y)O\big(e^{-\epsilon 2\kappa t}\big)\big).\\
&E\big[u^2(t,x)\big]\\
&\quad = e^t\big(Z^{2\kappa}_{1,t}(0) - 1\big)\\
&\quad \sim e^t e^{\lambda_0(1/2\kappa)2\kappa t}\big(\psi_{1/2\kappa}(0)\|\psi_{1/2\kappa}\|_{L^1} + q(0)O\big(e^{-\epsilon 2\kappa t}\big)\big).
\end{aligned} \tag{12}$$

Consequently,

$$\begin{aligned}
\lambda(2) &= 2\lambda(1) + \lambda_0(1/2\kappa)2\kappa\\
&= 1 + \lambda_0(1/2\kappa)2\kappa.
\end{aligned} \tag{13}$$

Note that (13) gives a quantitative expression for the intermittency condition $\lambda(2) > 2\lambda(1)$ since $\lambda(1) = \frac{1}{2}$ and so we see that

$$\lambda(2) - 2\lambda(1) = \lambda_0(1/2\kappa)2\kappa.$$

We note that $\lambda_0(1/2\kappa) \to 0$ as $\kappa \nearrow \kappa_c(d)$. Its rate of decay depends on the dimension.

4. Main result

Instead of t we shall now switch to n to denote time. We will be concerned with sums of the variables $u(n,x)$ over boxes

$$B^n_k = \big\{x \in \mathbf{Z^d} : k_i n \leq x_i \leq (k_i+1)n, i = 1, 2, \ldots, d\big\}$$

for $k \in \mathbf{Z^d}$. Our main result is the following theorem.

THEOREM 4.1. *Suppose $\kappa < \kappa_c(d)$ and let $\{u(n,j) : j \in \mathbf{Z^d}\}$ be the solution of the parabolic Anderson equation (2). Define the random variables*

$$\begin{aligned}
X^n_j &= e^{-\frac{1}{2}\lambda(2)n}u(n,j), \quad j \in \mathbf{Z^d},\\
Y^{m,n}_k &= m^{-\frac{d}{2}}\sum_{j\in B^m_k}\big(X^n_j - E\big[X^n_j\big]\big), \quad k \in \mathbf{Z^d}
\end{aligned}$$

and

$$Y^{m(n)}_k = m(n)^{-\frac{d}{2}}\sum_{j\in B^{m(n)}_k}\big(X^n_j - E\big[X^n_j\big]\big), \quad k \in \mathbf{Z^d}.$$

If $m(n) = e^{\gamma n}$ *with* $\gamma > \frac{1}{d}(2\lambda'(2) - \lambda(2))$, *then*

$$Y_k^{m(n)} \to Z_k, \quad k \in \mathbf{Z^d},$$

where the field

$$\{Z_k : k \in \mathbf{Z^d}\}$$

is composed of i.i.d. $\mathcal{N}(0, A)$ *random variables with*

$$A = \sum_{j \in \mathbf{Z^d}} \psi_{1/2\kappa}(j) \|\psi_{1/2\kappa}\|_{L^1}.$$

We start with some preliminary remarks. Using (12), since

$$\begin{aligned}&\sum_{y \in B_0^n} \operatorname{Cov}\bigl(u(n,0), u(n,y)\bigr)\\ &\quad \sim e^{\lambda(2)n} \sum_{y \in B_0^n} \bigl(\psi_{1/2\kappa}(y)\|\psi_{1/2\kappa}\|_{L^1} + q(y) O\bigl(e^{-\epsilon 2\kappa n}\bigr)\bigr)\end{aligned}$$

it follows that

$$\sum_{y \in B_0^n} \operatorname{Cov}\bigl(X_0^n, X_y^n\bigr) \sim \sum_{y \in B_0^n} \bigl(\psi_{1/2\kappa}(y)\|\psi_{1/2\kappa}\|_{L^1} + q(y) O\bigl(e^{-\epsilon 2\kappa n}\bigr)\bigr).$$

Then, since

$$\begin{aligned}Y_k^{m(n)} &= m(n)^{-\frac{d}{2}} \sum_{j \in B_k^{m(n)}} \bigl(X_j^n - E\bigl[X_j^n\bigr]\bigr)\\ &= m(n)^{-\frac{d}{2}} \sum_{j \in B_k^{m(n)}} \bigl(X_j^n - e^{(\lambda(1) - \frac{\lambda(2)}{2})n}\bigr),\end{aligned}$$

one gets,

$$\begin{aligned}(14)\quad &\operatorname{Var}\bigl(Y_k^{m(n)}\bigr)\\ &= m(n)^{-d}\left(E\left[\left(\sum_{j \in B_k^{m(n)}} X_j^n\right)^2\right] - m(n)^{2d} e^{-n(\lambda(2) - 2\lambda(1))}\right)\\ &\sim m(n)^{-d} \sum_{j,k \in B_k^{m(n)}} \bigl(\psi_{1/2\kappa}(j-k)\|\psi_{1/2\kappa}\|_{L^1} + q(j-k) O\bigl(e^{-\epsilon 2\kappa n}\bigr)\bigr)\\ &\qquad - m(n)^d e^{-n(\lambda(2) - 2\lambda(1))}\\ &\sim \sum_{j \in B_k^{m(n)}} \bigl(\psi_{1/2\kappa}(j)\|\psi_{1/2\kappa}\|_{L^1} + q(j) O\bigl(e^{-\epsilon 2\kappa n}\bigr)\bigr)\\ &\qquad - m(n)^d e^{-\lambda_0(\sqrt{2\kappa}) 2\kappa n}.\end{aligned}$$

Define

$$A_{m(n)} = \operatorname{Var}\bigl(Y_k^{m(n)}\bigr),$$

which, by stationarity, does not depend on $k \in \mathbf{Z^d}$ and

$$A = \sum_{j \in \mathbf{Z^d}} \psi_{1/2\kappa}(j) \|\psi_{1/2\kappa}\|_{L^1}.$$

Note that we have just established by (14), since $\kappa < \kappa_c(d)$ implies $\lambda_0(\sqrt{2\kappa}) > 0$, that

$$\lim_{n\to\infty} A_{m(n)} = A.$$

This gives our lemma.

LEMMA 4.1. *For any* $k \in \mathbf{Z^d}$,

$$\lim_{n\to\infty} \operatorname{Var}\bigl(Y_k^{m(n)}\bigr) = \sum_{j \in \mathbf{Z^d}} \psi_{1/2\kappa}(j) \|\psi_{1/2\kappa}\|_{L^1}. \tag{15}$$

If, in addition,

$$\lim_{n\to\infty} \frac{m(n)}{\tilde{m}(n)} = 1$$

then

$$\lim_{n\to\infty} E\bigl[\bigl(Y_k^{m(n)} - Y_k^{\tilde{m}(n)}\bigr)^2\bigr] = 0. \tag{16}$$

Also, for $j \neq k$,

$$\lim_{n\to\infty} \operatorname{Cov}\bigl(Y_j^{m(n)}, Y_k^{\tilde{m}(n)}\bigr) = 0. \tag{17}$$

Proof. The limit in (15) follows from (14) as outlined before the lemma. For (16), if $m(n) \geq \tilde{m}(n)$, then

$$\begin{aligned}
&E\bigl[\bigl(Y_k^{m(n)} - Y_k^{\tilde{m}(n)}\bigr)^2\bigr] \\
&\quad = \operatorname{Var}\bigl(Y_k^{m(n)}\bigr) + \operatorname{Var}\bigl(Y_k^{\tilde{m}(n)}\bigr) \\
&\qquad - 2\bigl(m(n)\tilde{m}(n)\bigr)^{-d/2} \operatorname{Cov}\bigl(X_0^{m(n)}, X_0^{\tilde{m}(n)}\bigr) \\
&\quad \leq 7 \operatorname{Var}\bigl(Y_k^{m(n)}\bigr) + \operatorname{Var}\bigl(Y_k^{\tilde{m}(n)}\bigr) - 2\left(\frac{\tilde{m}(n)}{m(n)}\right)^{-d/2} \operatorname{Var}\bigl(Y_k^{\tilde{m}(n)}\bigr)
\end{aligned}$$

while if $m(n) < \tilde{m}(n)$, then

$$\begin{aligned}
E\bigl[\bigl(Y_k^{m(n)} - Y_k^{\tilde{m}(n)}\bigr)^2\bigr] \leq \operatorname{Var}\bigl(Y_k^{m(n)}\bigr) &+ \operatorname{Var}\bigl(Y_k^{\tilde{m}(n)}\bigr) \\
&- 2\left(\frac{m(n)}{\tilde{m}(n)}\right)^{-d/2} \operatorname{Var}\bigl(Y_k^{\tilde{m}(n)}\bigr).
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{m(n)}{\tilde{m}(n)} = 1$, by (15),

$$\lim_{n\to\infty} E\bigl[\bigl(Y_k^{m(n)} - Y_k^{\tilde{m}(n)}\bigr)^2\bigr] = 0. \qquad \square$$

Proof of Theorem 4.1. This is essentially a triangular array version of the central limit theorem for associated random variables proved in [7]. The independence of the variables $Z_k, k \in \mathbf{Z^d}$ follows from Theorem 2.3 and (17) since for finite $\Lambda \subset \mathbf{Z^d}$,

$$\begin{aligned}
&\lim_{n\to\infty}\left|E\left[\exp\left\{i\sum_{j\in\Lambda} ir_j Y_j^{m(n)}\right\}\right] - \exp\left\{-\frac{1}{2}\sum_{j\in\Lambda} Ar_j^2\right\}\right| \\
&\quad = \lim_{n\to\infty}\left|E\left[\exp\left\{i\sum_{j\in\Lambda} ir_j Y_j^{m(n)}\right\}\right] - \prod_{j\in\Lambda} E\left[\exp\left\{ir_j Y_j^{m(n)}\right\}\right]\right| \\
&\quad \le \lim_{n\to\infty}\frac{1}{2}\sum_{k,j\in\Lambda}|r_j||r_k|\operatorname{Cov}\left(Y_j^{m(n)}, Y_k^{m(n)}\right) \\
&\quad = 0.
\end{aligned}$$

Observe as well that, for any nonnegative integer l and using $[\cdot]$ to denote the greatest integer function, since $\lim_{n\to\infty}\frac{l[m(n)/l]}{m(n)} = 1$, it follows that

$$\begin{aligned}
&\left|E\left[\exp\left\{irY_0^{m(n)}\right\}\right] - E\left[\exp\left\{irY_0^{l[m(n)/l]}\right\}\right]\right| \\
&\qquad \le E\left[\left|\exp\left\{ir\left(Y_0^{l[m(n)/l]} - Y_0^{m(n)}\right)\right\} - 1\right|\right] \\
&\qquad \le E\left[\left|Y_0^{l[m(n)/l]} - Y_0^{m(n)}\right|\right] \\
&\qquad \le E\left[\left(Y_0^{l[m(n)/l]} - Y_0^{m(n)}\right)^2\right]^{1/2} \\
&\qquad \to 0, \quad n\to\infty.
\end{aligned} \tag{18}$$

Furthermore, with $l = [m(n)/m]$, using the fact, which follows from stationarity, that $Y_0^{ml} \stackrel{\mathcal{L}}{=} l^{-d/2}\sum_{j\in B_0^l} Y_j^{m,n}$, by Theorem 2.3

$$\begin{aligned}
&\left|E\left[\exp\left\{irY_0^{ml}\right\}\right] - E\left[\exp\left\{irl^{-d/2}Y_0^{m,n}\right\}\right]^{l^d}\right| \\
&\qquad \le \frac{1}{2}\sum_{j,k\in B_0^l, j\ne k} r^2 l^{-d}\operatorname{Cov}\left(Y_j^{m,n}, Y_k^{m,n}\right) \\
&\qquad = \frac{r^2}{2}\left(\operatorname{Cov}\left(Y_0^{ml}, Y_0^{ml}\right) - l^{-d}\sum_{j\in B_0^l}\operatorname{Cov}\left(Y_j^{m,n}, Y_j^{m,n}\right)\right) \\
&\qquad = \frac{r^2}{2}\left(\operatorname{Var}\left(Y_0^{ml}\right) - \operatorname{Var}\left(Y_0^{m,n}\right)\right) \\
&\qquad \to \frac{r^2}{2}(A - A_m), \quad l\to\infty.
\end{aligned} \tag{19}$$

For the term, $E[\exp\{irl^{-d/2}Y_0^{m,n}\}]^{l^d}$, we use the inequality

$$\left|e^{ix} - 1 - ix + \frac{x^2}{2}\right| \le c(\epsilon)|x|^{2+\epsilon}, \quad x\in\mathbf{R}.$$

This implies,

$$\begin{aligned}
&E\big[\exp\big\{irl^{-d/2}Y_0^{m,n}\big\}\big]\\
&\quad=\left(1-\frac{r^2}{2}l^{-d}A_m+O(1)|r|^{2+\epsilon}l^{-d(1+\frac{\epsilon}{2})}E\big[\big(Y_0^{m,n}\big)^{2+\epsilon}\big]\right)\\
&\quad=\left(1-\frac{r^2}{2}l^{-d}A_m+O(1)l^{-d(1+\frac{\epsilon}{2})}E_Q\big[\big(X_0^{m}\big)^{2+\epsilon}\big]\right)\\
&\quad=\left(1-\frac{r^2}{2}l^{-d}A_m+O(1)l^{-d(1+\frac{\epsilon}{2})}e^{(\lambda(2+\epsilon)-(1+\frac{\epsilon}{2})\lambda(2)+o(1))n}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&E\big[\exp\big\{irl^{-d/2}Y_0^{m}\big\}\big]^{l^d}\\
&\quad=\left(1-\frac{r^2}{2}l^{-d}A_m+O(1)l^{-d(1+\frac{\epsilon}{2})}e^{(\lambda(2+\epsilon)-(1+\frac{\epsilon}{2})\lambda(2)+o(1))n}\right)^{l^d}\\
&\quad\sim\left(1-l^{-d}\left(\frac{r^2A_m}{2}+O(1)l^{-d\frac{\epsilon}{2}}e^{(\lambda(2+\epsilon)-(1+\frac{\epsilon}{2})\lambda(2)+o(1))n}\right)\right)^{l^d}\\
&\quad=\left(1-l^{-d}\left(\frac{r^2A_m}{2}+O(1)e^{(-d\gamma\frac{\epsilon}{2}+\lambda(2+\epsilon)-(1+\frac{\epsilon}{2})\lambda(2)+o(1))n}\right)\right)^{l^d}.
\end{aligned}$$

Now set $\gamma=\frac{1}{d}(2\lambda'(2)-\lambda(2))+\alpha$ where $\alpha>0$. Then,

$$\begin{aligned}
&-d\gamma\frac{\epsilon}{2}+\lambda(2+\epsilon)-\left(1+\frac{\epsilon}{2}\right)\lambda(2)+o(1)\\
&\quad=-\big(2\lambda'(2)-\lambda(2)\big)\frac{\epsilon}{2}+\lambda(2+\epsilon)\\
&\qquad-\left(1+\frac{\epsilon}{2}\right)\lambda(2)-d\alpha\frac{\epsilon}{2}+o(1)\\
&\quad=-\epsilon\left(\lambda'(2)-\frac{\lambda(2+\epsilon)-\lambda(2)}{\epsilon}\right)-d\alpha\frac{\epsilon}{2}+o(1)\\
&\quad=O(1)\epsilon^2-d\alpha\frac{\epsilon}{2}+o(1)\\
&\quad<0
\end{aligned}\tag{20}$$

for n sufficiently large. Therefore,

$$\lim_{n\to\infty}E\big[e^{irl^{-d/2}Y_0^{m,n}}\big]^{l^d}=e^{-A_m\frac{r^2}{2}}.\tag{21}$$

Putting together (18), (19) and (21) we have for any m,

$$\begin{aligned}
&\limsup_{n\to\infty}\left|E\big[\exp\big\{irY_0^{m(n)}\big\}\big]-\exp\left\{-A\frac{r^2}{2}\right\}\right|\\
&\quad\le\frac{r^2}{2}(A-A_m)+\big(e^{-A_m\frac{r^2}{2}}-e^{-A\frac{r^2}{2}}\big)
\end{aligned}$$

Since $A_m \to A$ as $m \to \infty$

$$\lim_{n\to\infty} E\big[\exp\big\{irY_0^{m(n)}\big\}\big] = e^{-A\frac{r^2}{2}}$$

and the theorem is proved. □

We end by remarking that the estimate in Theorem 3.1 holds under reasonable conditions on Γ if one uses the field $\{\widetilde{W}_x : x \in \mathbf{Z^d}\}$ instead of the δ-correlated field $\{W_x : x \in \mathbf{Z^d}\}$. Thus under reasonable conditions, say Γ has compact support, Theorem 4.1 holds for the solution field of (4).

References

[1] G. Ben Arous, S. Molchanov and A. F. Ramírez, *Transition from the annealed to the quenched asymptotics for a random walk on random obstacles*, Ann. Probab. **33** (2005), 2149–2187. MR 2184094

[2] R. A. Carmona and S. A. Molchanov, *Parabolic Anderson problem and intermittency*, Mem. Amer. Math. Soc., vol. **108**, Amer. Math. Soc., Providence, RI, 1994, pp. viii+125. MR 1185878

[3] M. Cranston and S. Molchanov, *On phase transitions and limit theorems for homopolymers*, Probability and mathematical physics, CRM Proc. Lecture Notes, vol. **42**, Amer. Math. Soc., Providence, RI, 2007, pp. 97–112. MR 2352264

[4] M. Cranston and S. Molchanov, *Quenched to annealed transition in the parabolic Anderson problem*, Probab. Theory Related Fields **138** (2007), 177–193. MR 2288068

[5] M. Cranston, T. Mountford and T. Shiga, *Lyapunov exponents for the parabolic Anderson model*, Acta Math. Univ. Comenian. **71** (2002), 163–188. MR 1980378

[6] J. D. Esary, F. Proschan and D. W. Walkup, *Association of random variables, with applications*, Ann. Math. Statist. **38** (1967), 1466–1474. MR 0217826

[7] C. M. Newman, *Normal fluctuations and the FKG inequalities*, Comm. Math. Phys. **74** (1980), 119–128. MR 0576267

[8] L. D. Pitt, *Positively correlated normal variables are associated*, Ann. Probab. **10** (1982), 496–499. MR 0665603

M. Cranston, Department of Mathematics, University of California, Irvine, Irvine, CA 92612, USA

E-mail address: mcransto@math.uci.edu

G. Mueller, Department of Mathematics, University of California, Irvine, Irvine, CA 92612, USA

E-mail address: gsmuelle@math.uci.edu