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LOCALLY COMPLETE INTERSECTION STANLEY–REISNER IDEALS

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Abstract. In this paper, we prove that the Stanley–Reisner ideal of any connected simplicial complex of dimension ≥ 2 that is locally complete intersection is a complete intersection ideal.

As an application, we show that the Stanley–Reisner ideal whose powers are Buchsbaum is a complete intersection ideal.

Introduction

By a simplicial complex Δ on a vertex set $V = [n] = \{1, 2, \ldots, n\}$, we mean that Δ is a nonvoid family of subsets of V such that (i) $\{v\} \in \Delta$ for every $v \in V$, and (ii) $F \in \Delta$, $G \subseteq F$ imply $G \in \Delta$. Let $S = K[X_1, \ldots, X_n]$ be a polynomial ring over a field K. The *Stanley–Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by all squarefree monomials $X_{i_1} \cdots X_{i_p}$ such that $1 \leq i_1 < \cdots < i_p \leq n$ and $\{i_1, \ldots, i_p\} \notin \Delta$. The *Stanley–Reisner ring* of Δ over K is the K-algebra $K[\Delta] = S/I_\Delta$. Any squarefree monomial ideal I with $I \subseteq (X_1, \ldots, X_n)^2$ is a Stanley–Reisner ideal I_Δ for some simplicial complex Δ on $V = [n]$.

An element $F \in \Delta$ is called a *face* of Δ. A maximal face of Δ with respect to inclusion is called a *facet* of Δ. The *dimension* of Δ, denoted by $\dim \Delta$, is the maximum of the dimensions $\dim F = \sharp(F) - 1$, where F runs through all faces F of Δ and $\sharp(F)$ denotes the cardinality of F. Note that the Krull dimension of $K[\Delta]$ is equal to $\dim \Delta + 1$. A simplicial complex is called *pure* if all facets have the same dimension. See [1], [6] for more information on Stanley–Reisner rings.

A homogeneous ideal I in $S = K[X_1, \ldots, X_n]$ is said to be a *locally complete intersection* ideal if I_P is a complete intersection ideal (that is, generated by a regular sequence) for any prime $P \in \operatorname{Proj}(S/I)$. A simplicial complex Δ on V is said to be a *locally complete intersection* complex if $I_{\operatorname{link}_\Delta(\{v\})}$ is a

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complete intersection ideal for every $v \in V$. Then Δ is a locally complete intersection complex if and only if I_Δ is a locally complete intersection ideal. Note that a locally complete intersection ideal I is called a *generalized complete intersection* ideal in the sense of Goto–Takayama (see [3]) if $I = I_\Delta$ is the Stanley–Reisner ideal for some pure simplicial complex Δ.

In Section 1, we consider the structure of simplicial complexes which are locally complete intersection. This is the main purpose of the paper. One can easily see that if a Stanley–Reisner ideal I is a complete intersection ideal, then it can be written as

$$I = (X_{11} \cdots X_{1q_1}, \dots, X_{c1} \cdots X_{cq_c}),$$

where $c \geq 0$ and q_i is a positive integer with $q_i \geq 2$ for $i = 1, \dots, c$ and all X_{ij} are distinct variables.

A complete intersection simplicial complex Δ is connected if $\dim \Delta \geq 1$, and it is a locally complete intersection complex. When $\dim \Delta \geq 2$, the converse is also true, which is a main result in this paper.

THEOREM 1 (See also Theorems 1.5, 1.15). *Let Δ be a connected simplicial complex with* $\dim \Delta \geq 2$ (*resp.* $\dim \Delta = 1$). *If it is a locally complete intersection complex, then it is a complete intersection complex* (*resp. an n-gon for $n \geq 3$ or an n-pointed path for some $n \geq 2$*).

Let Δ be a connected simplicial complex on V with $\dim \Delta \geq 2$. Our main theorem says that if $\mathrm{link}_\Delta(\{x\})$ is a complete intersection complex for every vertex $x \in V$ then so is Δ. If we also assume Serre's condition (S_2), then we can obtain a stronger result. That is, when $K[\Delta]$ satisfies (S_2), Δ is a complete intersection complex if and only if $\mathrm{link}_\Delta(F)$ is a complete intersection complex for any face $F \in \Delta$ with $\dim \mathrm{link}_\Delta F = 1$; see Corollary 1.10 for more details.

In Section 2, we discuss Buchsbaumness for powers of Stanley–Reisner ideals. Let us explain our motivation briefly. Let A be a Cohen–Macaulay local ring. If I is a complete intersection ideal of A, then A/I^ℓ is Cohen–Macaulay for every $\ell \geq 1$ because $I^\ell/I^{\ell+1}$ is a free A/I-module. In [2], Cowsik and Nori proved the converse. That is, if I is a generically complete intersection ideal (i.e., I_P is a complete intersection ideal for all minimal prime divisors P of I) and A/I^ℓ is Cohen–Macaulay for all (sufficiently large) $\ell \geq 1$, then I is a complete intersection ideal. Note that one can apply this result to Stanley–Reisner ideals: I_Δ is a complete intersection ideal if and only if $S/I_\Delta^{\ell+1}$ is Cohen–Macaulay for every $\ell \geq 1$.

A standard graded ring $A = S/I$ with homogeneous maximal ideal $\mathfrak{m}$ is said to be *Buchsbaum* (resp. (FLC)) if the canonical map

$$H^i(\mathfrak{m}, A) \to H^i_\mathfrak{m}(A) = \varinjlim \mathrm{Ext}^i_S(S/\mathfrak{m}^\ell, A)$$

is surjective (resp. if $H^i_\mathfrak{m}(A)$ has finite length) for all $i < \dim A$, where $H^i(\mathfrak{m}, A)$ (resp. $H^i_\mathfrak{m}(A)$) denotes the ith Koszul cohomology module (resp.

ith local cohomology module); see [8, Chapter I, Theorem 2.15]. Then we have the following implications:

$$\begin{array}{ccccc} \text{Complete intersection} & \Longrightarrow & \text{Locally complete intersection} & & \\ \Downarrow & & \Downarrow \text{if pure} & & \\ \text{Cohen–Macaulay} & \Longrightarrow & \text{Buchsbaum} & \Longrightarrow & \text{(FLC).} \end{array}$$

Goto and Takayama [3] proved that I_Δ is a pure locally complete intersection ideal if and only if $S/I_\Delta^{\ell+1}$ is (FLC) for every $\ell \geq 1$ as an analogue of Cowsik–Nori theorem.

Let S be a polynomial ring and I a squarefree monomial ideal of S. Then S/I is Buchsbaum if and only if it is (FLC); see e.g., [6, p. 73, Theorem 8.1]. But a similar statement is no longer true for nonsquarefree monomial ideals. The following is a natural question.

Question 2. When is S/I_Δ^ℓ Buchsbaum for every $\ell \geq 1$?

As an application of our main theorem and the lower bound formula on the multiplicity of Buchsbaum homogeneous K-algebras in [4], we can prove the following theorem.

Theorem 3. *Put* $S = K[X_1, \dots, X_n]$. *Let* Δ *be a simplicial complex on* $V = [n]$. *Then the following conditions are equivalent*:

(1) I_Δ *is generated by a regular sequence*;
(2) S/I_Δ^ℓ *is Cohen–Macaulay for all* $\ell \geq 1$;
(3) S/I_Δ^ℓ *is Buchsbaum for all* $\ell \geq 1$;
(3)′ $\sharp\{\ell \in \mathbb{Z}_{\geq 1} : S/I_\Delta^\ell$ *is Buchsbaum*$\} = \infty$.

We do not know whether a similar statement is true for general homogeneous ideals.

1. Connected complexes which are locally complete intersection

Throughout this paper, let Δ be a simplicial complex on V. For a face F of Δ and $W \subseteq V$, we put

$$\operatorname{link}_\Delta(F) = \{G \in \Delta : G \cup F \in \Delta, F \cap G = \emptyset\},$$
$$\Delta_W = \{G \in \Delta : G \subseteq W\}.$$

These complexes are the *link* of F, and, the *restriction* to W of Δ, respectively.

Let $\mathcal{H}$ be a subset of 2^V. The minimum simplicial complex $\Gamma \subseteq 2^V$ which contains $\mathcal{H}$ as a subset, denoted by $\langle \mathcal{H} \rangle$, is said to be the *simplicial complex spanned by* $\mathcal{H}$ *on* V.

Suppose that $V = V_1 \cup \cdots \cup V_r$ is a disjoint union. Let Δ_i be a simplicial complex on V_i for each $i = 1, \dots, r$. Then $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$ is a simplicial complex on V. We call Δ "a *disjoint union* of Δ_i's" by abuse of language although $\Delta_i \cap \Delta_j = \{\emptyset\}$ for $i \neq j$.

A simplicial complex Δ is a *complete intersection* complex if the Stanley–Reisner ideal I_Δ is generated by a regular sequence. Now, let us define the notion of locally complete intersection for complexes.

DEFINITION 1.1. A simplicial complex Δ on V is said to be a *locally complete intersection* complex if $I_{\mathrm{link}_\Delta(\{v\})}$ is a complete intersection ideal for all vertex $v \in V$.

A simplicial complex Δ is a locally complete intersection complex if and only if its Stanley–Reisner ideal I_Δ is a locally complete intersection ideal.

LEMMA 1.2. *For a Stanley–Reisner ideal $I = I_\Delta$, the following conditions are equivalent*:

(1) *Δ is a locally complete intersection complex*;
(2) *$K[\Delta]_{X_i}$ is a complete intersection ring for all $i \in V$*;
(3) *I_P is a complete intersection ideal for all prime $P \in \mathrm{Proj}(S/I_\Delta)$.*

Proof. The equivalence of (1) and (2) immediately follows from the fact that

$$K[\mathrm{link}_\Delta(\{i\})][X_i, X_i^{-1}] \cong K[\Delta]_{X_i}.$$

(2) $\Longrightarrow$ (3) is clear. In order to show the converse, we suppose that $K[\Delta]_{X_1}$ is not a complete intersection ring. Without loss of generality, we may assume that

$$\{X_i : 2 \le i \le m\} = \{X_i : i \in \mathrm{link}_\Delta(\{1\})\}.$$

Since $X_1 X_j \in I_\Delta$ for $m+1 \le j \le n$, one has that $X_j \in I_\Delta S_{X_1}$. If we put $P = (X_2, \ldots, X_m)$, then we can easily see that $I_\Delta S_P$ is not a complete intersection ideal by assumption. Hence, we obtain (3) $\Longrightarrow$ (2). □

COROLLARY 1.3. *If Δ is a connected locally complete intersection complex, then it is pure.*

Proof. Suppose that Δ is not pure. Since Δ is connected, there exist a vertex $i \in V$ and facets F_1, F_2 such that $i \in F_1 \cap F_2$ and $\sharp(F_1) < \sharp(F_2)$. This implies that $\mathrm{link}_\Delta(\{i\})$ is not pure. This contradicts the assumption that $\mathrm{link}_\Delta(\{i\})$ is Cohen–Macaulay. Hence, Δ must be pure. □

REMARK 1.4. A pure locally complete intersection complex is called a generalized complete intersection complex in [3].

The main purpose of this section is to prove the following theorem.

THEOREM 1.5. *Let Δ be a connected simplicial complex on V with $\dim \Delta \ge 2$. If Δ is a locally complete intersection complex, then it is a complete intersection complex.*

Let Δ be a connected complex of dimension $d-1$. Suppose that Δ is a locally complete intersection complex, but not a complete intersection complex. Note that Δ is pure and thus a generalized complete intersection complex. Let $G(I_\Delta)=\{m_1,\dots,m_\mu\}$ denote the minimal set of monomial generators of I_Δ. Then $\mu\geq 2$ and $\deg m_i\geq 2$ for every $i=1,2,\dots,\mu$, and that there exist i, j $(1\leq i<j\leq n)$ such that $\gcd(m_i,m_j)\neq 1$.

LEMMA 1.6. *In the above notation, we may assume that* $\deg m_i = \deg m_j = 2$.

Proof. Take m_j, m_k $(j\neq k)$ such that $\gcd(m_j,m_k)\neq 1$. If $\deg m_j = \deg m_k = 2$, then there is nothing to prove.

Now suppose that $\deg m_k\geq 3$. By [3, Lemmas 3.4, 3.5], we may assume that $\deg m_j=2$ and $\gcd(m_j,m_k)=X_p$. Write $m_k=X_pX_{i_1}\cdots X_{i_r}$ and $m_j=X_pX_q$. Then [3, Lemma 3.6] implies that $X_{i_1}X_q\in G(I_\Delta)$. Set $m_i=X_{i_1}X_q\in I_\Delta$. Then $\deg m_i=\deg m_j=2$ and $\gcd(m_i,m_j)=X_q\neq 1$, as required. □

The following lemma is simple but important. We use the following convention in this section: the vertices x, y, z etc. correspond to the indeterminates X, Y, Z etc., respectively.

LEMMA 1.7. *Let x_1,x_2,y be distinct vertices such that X_1Y, $X_2Y\in I_\Delta$. For any $z\in V\setminus\{x_1,x_2,y\}$, at lease one of monomials X_1Z, X_2Z and YZ belongs to I_Δ.*

Proof. Note that $K[\mathrm{link}_\Delta(\{z\})$ is obtained from $K[\Delta]$ by setting $Z=1$. Then the assertion follows from the fact that $K[\mathrm{link}_\Delta(\{z\})]$ is a complete intersection ring. □

In what follows, we prove Theorem 1.5. In order to do that, let Δ be a connected simplicial complex of dimension $d-1\geq 1$. Moreover, assume that Δ is a locally complete intersection complex and that there exist vertices x_1,x_2,y such that X_1Y, $X_2Y\in I_\Delta$ (we assign a variable X_i for a vertex x_i). Then we must show that $\dim\Delta(=d-1)=1$. Let us begin with proving the following key lemma.

LEMMA 1.8. *Under the above notation, there exist some integers k, $\ell\geq 2$ such that*

(1) $V=\{x_1,\dots,x_k,y_1,\dots,y_\ell\}$;
(2) $X_1Y_1,\dots,X_kY_1\in I_\Delta$;
(3) $\sharp\{i:1\leq i\leq k, X_iY_j\notin I_\Delta\}\leq 1$ *holds for each* $j=2,\dots,\ell$.

Proof. By assumption, there exist vertices $x_1,x_2,y_1\in V$ such that X_1Y_1, $X_2Y_1\in I_\Delta$. Thus, one can write $V=\{x_1,\dots,x_k,y_1,\dots,y_\ell\}$ such that

$$X_1Y_1,\, X_2Y_1,\, \dots,\, X_kY_1\in I_\Delta,$$
$$Y_1Y_2,\ Y_1Y_3,\ \dots,\ Y_1Y_\ell\notin I_\Delta.$$

If $\ell = 1$, then $\Delta = \Delta_{\{y_1\}} \cup \Delta_{\{x_1,\dots,x_k\}}$ is a disjoint union since $\{y_1, x_i\} \notin \Delta$ for all i. This contradicts the connectedness of Δ. Hence, $\ell \geq 2$. Thus, it is enough to show (3) in this notation.

Now, suppose that there exists an integer j with $2 \leq j \leq \ell$ such that

$$\sharp\{i : 1 \leq i \leq k,\, X_iY_j \notin I_\Delta\} \geq 2.$$

When $k = 2$, we have X_1Y_j, $X_2Y_j \notin I_\Delta$. On the other hand, as X_1Y_1, $X_2Y_1 \in I_\Delta$ and $Y_j \neq X_1, X_2, Y_1$, we obtain that at least one of X_1Y_j, X_2Y_j, Y_1Y_j belongs to I_Δ. It is impossible. So we may assume that $k \geq 3$ and $X_{k-1}Y_j$, $X_kY_j \notin I_\Delta$. Then $\{x_{k-1}\}$, $\{x_k\}$ and $\{y_1\}$ belong to $\mathrm{link}_\Delta(\{y_j\})$, and $X_{k-1}Y_1$, X_kY_1 form part of the minimal system of generators of $I_{\mathrm{link}_\Delta(\{y_j\})}$. This contradicts the assumption that $\mathrm{link}_\Delta(\{y_j\})$ is a complete intersection complex. □

In what follows, we fix the notation as in Lemma 1.8. First, we suppose that there exists an i_0 with $1 \leq i_0 \leq k$ such that

$$\sharp\{j : 1 \leq j \leq \ell,\, X_{i_0}Y_j \notin I_\Delta\} = 1.$$

In this case, we may assume that $X_1Y_2 \notin I_\Delta$ and $X_1Y_j \in I_\Delta$ for all $3 \leq j \leq \ell$ without loss of generality. Note that $X_2Y_2, \dots, X_kY_2 \in I_\Delta$ by Lemma 1.8. We claim that $\{x_1, y_2\}$ is a facet of Δ. As $X_iY_2 \in I_\Delta$ for each $i = 2, \dots, k$, it follows that $\{x_1, y_2, x_i\} \notin \Delta$. Similarly, $\{x_1, y_2, y_j\} \notin \Delta$ since $X_1Y_j \in I_\Delta$ for $j = 1$ or $3 \leq j \leq \ell$. Hence $\{x_1, y_2\}$ is a facet of Δ, and $\dim \Delta = 1$ because Δ is pure.

By the observation as above, we may assume that for every i with $1 \leq i \leq k$,

$$\sharp\{j : 1 \leq j \leq \ell, X_iY_j \notin I_\Delta\} \geq 2$$

or $X_iY_j \in I_\Delta$ holds for all $j = 1, \dots, \ell$.

Now, suppose that there exist j_1, j_2 with $1 \leq j_1 < j_2 \leq \ell$ such that $X_iY_{j_1}$, $X_iY_{j_2} \notin I_\Delta$. Then $X_rY_{j_1}$, $X_rY_{j_2} \in I_\Delta$ for all $r \neq i$ by Lemma 1.8. It follows that $X_rX_i \in I_\Delta$ from Lemma 1.7. Then we can relabel x_i (say $y_{\ell+1}$). Repeating this procedure, we can get one of the following cases:

Case 1: $V = \{x_1, \dots, x_r, y_1, \dots, y_s\}$ such that $X_iY_j \in I_\Delta$ for all i, j with $1 \leq i \leq r$, $1 \leq j \leq s$.

Case 2: $V = \{x_1, x_2, y_1, \dots, y_m, z_1, \dots, z_p, w_1, \dots, w_q\}$ such that

$$\begin{cases} X_1Y_j \in I_\Delta, & X_2Y_j \in I_\Delta \quad (j = 1, \dots, m), \\ X_1Z_j \notin I_\Delta, & X_2Z_j \in I_\Delta \quad (j = 1, \dots, p), \\ X_1W_j \in I_\Delta, & X_2W_j \notin I_\Delta \quad (j = 1, \dots, q), \end{cases}$$

holds for some $m \geq 1$, $p, q \geq 2$.

If Case 1 occurs, then $\Delta = \Delta_{\{x_1,\dots,x_r\}} \cup \Delta_{\{y_1,\dots,y_s\}}$ is a disjoint union. This contradicts the assumption. Thus, Case 2 must occur. If $\{x_1, x_2\} \in \Delta$, then it is a facet and so $\dim \Delta = 1$. Hence, we may assume that $\{x_1, x_2\} \notin \Delta$. However, since Δ is connected, there exists a path between x_1 and x_2.

Cases (2-a): the case where $\{z_1, w_k\} \in \Delta$ for some k with $1 \le k \le q$.

We may assume that $\{z_1, w_1\} \in \Delta$. Now suppose that $\dim \Delta \ge 2$. Then since $\{z_1, w_1\}$ is *not* a facet, there exists a vertex $u \in V \setminus \{x_1, x_2\}$ such that $\{z_1, w_1, u\} \in \Delta$. If $u = z_j$ $(2 \le j \le p)$ (resp. $u = y_i$ $(1 \le i \le m)$), then $G(I_{\mathrm{link}_\Delta(\{w_1\})})$ contains X_2Z_1 and X_2Z_j (resp. X_2Y_i); see Figure 1. It is impossible since $\mathrm{link}_\Delta(\{w_1\})$ is a complete intersection complex. When $u = w_k$, we can obtain a contradiction by a similar argument as above. Therefore, $\dim \Delta = 1$.

Cases (2-b): the case where $\{z_j, w_k\} \notin \Delta$ for all j, k.

Then we may assume that (i) $\{z_1, y_1\} \in \Delta$ and (ii) $\{y_1, y_2\} \in \Delta$ or $\{y_1, w_1\} \in \Delta$. Now suppose that $\dim \Delta \ge 2$. Then since $\{z_1, y_1\}$ is *not* a facet, we have

$$\{z_1, y_1, y_i\} \in \Delta, \qquad \{z_1, y_1, w_k\} \in \Delta \quad \text{or} \quad \{z_1, y_1, z_j\} \in \Delta.$$

When $\{z_1, y_1, y_i\} \in \Delta$, we obtain that $X_1Y_1, X_1Y_i \in G(I_{\mathrm{link}_\Delta(\{z_1\})})$. This is a contradiction. When $\{z_1, y_1, w_k\} \in \Delta$, we can obtain a contradiction by a similar argument as in Case (2-a). Thus, it is enough to consider the case $\{z_1, y_1, z_j\} \in \Delta$.

First, we suppose that $\{y_1, y_2\} \in \Delta$ (see Figure 2).

Then $\mathrm{link}_\Delta(\{y_1\})$ contains $\{z_1, z_j\}$ and $\{y_2\}$. Since $\mathrm{link}_\Delta(\{y_1\})$ is also connected, we can find vertices z_α, y_β such that $\{z_\alpha, y_\beta\} \in \mathrm{link}_\Delta(\{y_1\})$. In particular, $\{z_\alpha, y_\beta, y_1\} \in \Delta$. This yields a contradiction because X_1Y_1, X_1Y_β are contained in $G(I_{\mathrm{link}_\Delta(\{z_\alpha\})})$.

Next, suppose that $\{y_1, w_1\} \in \Delta$ (see Figure 3).

Then $\mathrm{link}_\Delta(\{y_1\})$ contains $\{z_1, z_j\}$ and $\{w_1\}$. Since $\mathrm{link}_\Delta(\{y_1\})$ is also connected, we can also find vertices z_α, y_β such that $\{z_\alpha, y_\beta\} \in \mathrm{link}_\Delta(\{y_1\})$ (notice that $\{z_j, w_k\} \notin \Delta$). In particular, $\{z_\alpha, y_1, y_\beta\} \in \Delta$. This yields a contradiction because X_1Y_1, X_1Y_β are contained in $G(I_{\mathrm{link}_\Delta(\{z_\alpha\})})$.

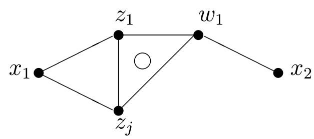


FIGURE 1. The case $\{z_1, z_j, w_1\} \in \Delta$ in Case (2-a).

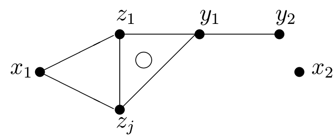


FIGURE 2. The case $\{z_1, y_1, z_j\}, \{y_1, y_2\} \in \Delta$ in Case (2-b).

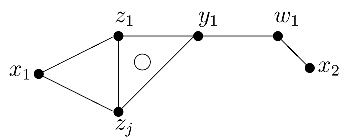


FIGURE 3. The case $\{z_1, y_1, z_j\}, \{y_1, w_1\} \in \Delta$ in Case (2-b).

Therefore, we have $\dim \Delta = 1$. So, we have finished the proof of Theorem 1.5.

An arbitrary Noetherian ring R is said to satisfy Serre's condition (S_2) if $\operatorname{depth} R_P \geq \min\{\dim R_P, 2\}$ for every prime P of R. A Stanley–Reisner ring $K[\Delta]$ satisfies (S_2) if and only if Δ is pure and $\operatorname{link}_\Delta(F)$ is connected for every face F with $\dim \operatorname{link}_\Delta(F) \geq 1$; see e.g., [10, p. 454]. In particular, if $K[\Delta]$ satisfies (S_2), then Δ is pure and connected if $\dim \Delta \geq 1$.

Let Δ be a connected simplicial complex on V with $\dim \Delta \geq 2$. Our main theorem says that if $\operatorname{link}_\Delta(\{x\})$ is a complete intersection complex for every $x \in V$ then so is Δ itself. Thus, it is natural to ask the following question:

QUESTION 1.9. Does there exist a proper subset $W \subsetneq V$ for which "$\operatorname{link}_\Delta(\{x\})$ is a complete intersection complex for all $x \in W$" implies that Δ is a complete intersection complex?

The following corollary gives an answer to the above question in the (S_2) case.

COROLLARY 1.10. *Let Δ be a simplicial complex with $\dim \Delta \geq 2$. Assume that $K[\Delta]$ satisfies (S_2). Then the following conditions are equivalent*:

(1) *$K[\Delta]$ is a complete intersection ring*;
(2) *For any face F with $\dim \operatorname{link}_\Delta(F) = 1$, $\operatorname{link}_\Delta(F)$ is a complete intersection complex*;
(3) *There exists $W \subseteq V$ such that $\dim \Delta_{V \setminus W} \leq \dim \Delta - 3$ which satisfies the following condition*:

"*$\operatorname{link}_\Delta(\{x\})$ is a complete intersection complex for all $x \in W$.*"

Proof. Note that Δ is pure. Put $d = \dim \Delta + 1$.

(1) $\Longrightarrow$ (3): It is enough to put $W = V$.

(3) $\Longrightarrow$ (2): Let $W \subseteq V$ be a subset that satisfies the condition (3). Let F be a face with $\dim \operatorname{link}_\Delta(F) = 1$. Since Δ is pure, $\sharp(F) = d - 1 - \dim \operatorname{link}_\Delta(F) = d - 2$. As $\dim \Delta_{V \setminus W} \leq d - 4$, F is not contained in $V \setminus W$. Thus, there exists a vertex $i \in F$ such that $i \in W$. Then since $\operatorname{link}_\Delta(\{i\})$ is a complete intersection complex by assumption, $\operatorname{link}_\Delta(F)$ is also a complete intersection complex, as required.

(2) $\Longrightarrow$ (1): We use an induction on $d \geq 3$. First, suppose that $d = 3$. Then for each $i \in V$, since $\dim \operatorname{link}_\Delta(\{i\}) = 1$, $\operatorname{link}_\Delta(\{i\})$ is a complete intersection

complex by the assumption (2). Hence, $K[\Delta]$ is a complete intersection ring by Theorem 1.5.

Next, suppose that $d \geq 4$. Let $i \in V$. Since $K[\Delta]$ satisfies (S_2), we have that $\Gamma = \mathrm{link}_\Delta(\{i\})$ is connected and $\dim \Gamma = d-2 \geq 2$. Moreover, for any face G in Γ with $\dim \mathrm{link}_\Gamma(G) = 1$, $\mathrm{link}_\Gamma(G) = \mathrm{link}_\Delta(G \cup \{i\})$ is a complete intersection complex by assumption. Hence, by the induction hypothesis, $K[\mathrm{link}_\Delta(\{i\})]$ is a complete intersection ring. Therefore, $K[\Delta]$ is a complete intersection ring by Theorem 1.5 again. □

To complete the proof of Theorem 1, we must consider the case $\dim \Delta = 1$. In this case, there exist connected locally complete intersection complexes which are not complete intersection.

Let Δ be a one-dimensional simplicial complex on $V = [n]$. Δ is said to be the *n-gon* for $n \geq 3$ (resp. the *n-pointed path* for $n \geq 2$) if Δ is pure and its facets consist of $\{i, i+1\}$ $(i = 1, 2 \ldots, n-1)$ and $\{n, 1\}$ (resp. its facets consists of $\{i, i+1\}$ $(i = 1, 2 \ldots, n-1)$) after suitable change of variables.

PROPOSITION 1.11. *Let Δ be a 1-dimensional connected complex. Then the following conditions are equivalent*:

(1) *Δ is a locally complete intersection complex*;
(2) *Δ is locally Gorenstein* (*i.e.*, *$K[\mathrm{link}_\Delta(\{i\})]$ is Gorenstein for every $i \in V$*);
(3) *Δ is isomorphic to either one of the following*:
  (a) *the n-gon for $n \geq 3$*;
  (b) *the n-pointed path for $n \geq 2$*.

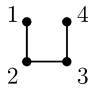
Proof. Note that (1) $\Longrightarrow$ (2) is clear.

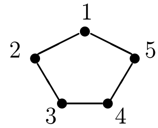
Suppose that Δ is a locally Gorenstein. Then since $\mathrm{link}_\Delta(\{i\})$ is a zero-dimensional Gorenstein complex, it consists of at most two points. Such a complex is isomorphic to either one of the n-gon $(n \geq 3)$ or the n-pointed path $(n \geq 2)$.

Conversely, if Δ is isomorphic to either n-gon or n-pointed path, then $\mathrm{link}_\Delta(\{i\})$ is a complete intersection complex. Hence, Δ is locally complete intersection. □

REMARK 1.12. Let Δ be a connected simplicial complex on $V = [n]$ of $\dim \Delta = 1$. Then Δ is a locally complete intersection complex but not a complete intersection complex if and only if it is isomorphic to the n-gon for some $n \geq 5$ or the n-pointed path for some $n \geq 4$.

EXAMPLE 1.13. *Let K be a field. The Stanley–Reisner ring of the 4-pointed path Δ_1 is $K[X_1, X_2, X_3, X_4]/(X_1X_3, X_1X_4, X_2X_4)$. The Stanley–Reisner ring of the 5-gon Δ_2 is $K[X_1, X_2, X_3, X_4, X_5]/(X_1X_3, X_1X_4, X_2X_4, X_2X_5, X_3X_5)$.*

$\Delta_1 =$ 

$\Delta_2 =$ 

REMARK 1.14. When $\dim \Delta \geq 2$, there are many examples of locally Gorenstein complexes which are not locally complete intersection complexes.

In the last of this section, we give a structure theorem for locally complete intersection complexes.

THEOREM 1.15. *Let Δ be a simplicial complex on V such that $V \neq \emptyset$. Then Δ is a locally complete intersection complex if and only if it is a finitely many disjoint union of the following connected complexes*:

(a) *a complete intersection complex Γ with $\dim \Gamma \geq 2$;*
(b) *m-gon $(m \geq 3)$;*
(c) *m'-pointed path $(m' \geq 2)$;*
(d) *a point.*

When this is the case, $K[\Delta]$ is Cohen–Macaulay (resp. Buchsbaum) if and only if $\dim \Delta = 0$ or Δ is connected (resp. pure).

To prove the theorem, it suffices to show the following lemma.

LEMMA 1.16. *Assume that $V = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$. Let Δ_i be a simplicial complex on V_i for $i = 1, 2$. If Δ_1 and Δ_2 are both locally complete intersection complexes, then so is $\Delta_1 \cup \Delta_2$.*

Proof. Put $\Delta = \Delta_1 \cup \Delta_2$ and $V_1 = [m]$ and $V_2 = [n]$. If we write

$$K[\Delta_1] = K[X_1, \ldots, X_m]/I_{\Delta_1} \quad \text{and} \quad K[\Delta_2] = K[Y_1, \ldots, Y_n]/I_{\Delta_2},$$

then

$$K[\Delta] \cong K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/(I_{\Delta_1}, I_{\Delta_2}, \{X_iY_j\}_{1 \leq i \leq m, 1 \leq j \leq n}).$$

Hence, $K[\Delta]_{X_i} \cong K[\Delta_1]_{X_i}$ and $K[\Delta]_{Y_j} \cong K[\Delta_2]_{Y_j}$ are complete intersection rings. Thus, Δ is also a locally complete intersection complex by Lemma 1.2. □

REMARK 1.17. In the above lemma, we suppose that both Δ_1 and Δ_2 are generalized complete intersection complexes. Then $\Delta_1 \cup \Delta_2$ is a generalized complete intersection complexes if and only if $\dim \Delta_1 = \dim \Delta_2$.

EXAMPLE 1.18. *Let Δ be the disjoint union of the standard $(m-1)$-simplex and the standard $(n-1)$-simplex. Then Δ is a locally complete intersection complex by Lemma 1.16. Moreover, $K[\Delta]$ is isomorphic to*

$$K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/(X_iY_j : 1 \leq i \leq m, 1 \leq j \leq n)$$

and it is a generalized complete intersection complex if and only if $m = n$.

2. Buchsbaumness of powers for Stanley–Reisner ideals

The Stanley–Reisner ring $K[\Delta]$ has (FLC) if and only if Δ is pure and $K[\mathrm{link}_\Delta(\{v\})]$ is Cohen–Macaulay for every $v \in V$. Then $H^i_{\mathfrak{m}}(K[\Delta]) = [H^i_{\mathfrak{m}}(K[\Delta])]_0$ for all $i < \dim K[\Delta]$ and so that $K[\Delta]$ is Buchsbaum. See [6, p. 73, Theorem 8.1].

Let $\ell \geq 2$ be an integer. Suppose that S/I_Δ^ℓ is Buchsbaum. In [5], Herzog, Takayama, and the first author showed that this condition implies that S/I_Δ is Buchsbaum. The converse is not true. What can we say about the structure of Δ? This gives a motivation of our study in this section.

The main result in this section is the following theorem, which is an analogue of the Cowsik–Nori theorem in [2], and the Goto–Takayama theorem in [3].

Theorem 2.1. *Put $S = K[X_1, \ldots, X_n]$. Let I_Δ denote the Stanley–Reisner ideal of a simplicial complex Δ on $V = [n]$. Then the following conditions are equivalent*:

(1) *I_Δ is generated by a regular sequence*;
(2) *S/I_Δ^ℓ is Cohen–Macaulay for all $\ell \geq 1$*;
(3) *S/I_Δ^ℓ is Buchsbaum for all $\ell \geq 1$*;
(3)′ *$\sharp\{\ell \in \mathbb{Z}_{\geq 1} : S/I_\Delta^\ell$ is Buchsbaum$\} = \infty$.*

Note that (1) $\Longleftrightarrow$ (2) is a special case of the Cowsik–Nori theorem and (2) $\Longrightarrow$ (3) $\Longrightarrow$ (3)′ is trivial. Thus, our contribution is (3)′ $\Longrightarrow$ (1).

In what follows, we put $d = \dim S/I_\Delta$, $c = \mathrm{height}\, I_\Delta (= \mathrm{codim}\, I_\Delta) = n - d$. Put $q = \mathrm{indeg}\, I_\Delta \geq 2$, the *initial degree* of I, that is, q is the least degree of the minimal generators of I, in other words, $q = \min\{\sharp(F) : F \in 2^V \setminus \Delta\}$. Put $e = e(S/I_\Delta)$, the *multiplicity* of I_Δ, which is equal to the number of facets of dimension $d-1$. Note that for any homogeneous ideal I of S, the following formula for multiplicities is known:

$$e(S/I) = \sum_{P \in \mathrm{Assh}_S(S/I)} e(S/P) \cdot \lambda_{S_P}(S_P/IS_P),$$

where $\mathrm{Assh}_S(S/I) = \{P \in \mathrm{Min}_S(S/I) : \dim S/P = \dim S/I\}$ and $\lambda_R(M)$ denotes the length of an R-module M over an Artinian local ring R.

In order to prove the theorem, it suffices to show that if S/I_Δ^ℓ is Buchsbaum for infinitely many $\ell \geq 1$, then Δ is a complete intersection complex.

First, we give a formula for multiplicities of S/I_Δ^ℓ for every $\ell \geq 1$.

Lemma 2.2. *Under the above notation, we have*

$$e(S/I_\Delta^\ell) = e \cdot \binom{c + \ell - 1}{c}.$$

Proof. Let $P \in \mathrm{Assh}_S(S/I_\Delta^\ell)$. Then P is a minimal prime over I_Δ such that S/P is isomorphic to a polynomial ring in d variables and S_P is a regular local ring of dimension c. Thus, we get

$$e(S/I_\Delta^\ell) = \sum_{P \in \operatorname{Assh} S/I_\Delta} e(S/P) \cdot \lambda_{S_P}(S_P/I_\Delta^\ell S_P) = e \cdot \binom{c+\ell-1}{c},$$

as required. □

We recall the following theorem, which gives a lower bound on multiplicities for homogeneous Buchsbaum algebras:

LEMMA 2.3 ([4, Theorem 3.2]). *Assume that S/I is a homogeneous Buchsbaum K-algebra. Put $c = \operatorname{codim} I \geq 2$, $q = \operatorname{indeg} I \geq 2$ and $d = \dim S/I \geq 1$. Then*

$$e(S/I) \geq \binom{c+q-2}{c} + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot \dim_K H_{\mathfrak{m}}^i(S/I).$$

Applying this formula to S/I_Δ^ℓ, yields the following corollary.

COROLLARY 2.4. *If S/I_Δ^ℓ is Buchsbaum, then*

$$e(S/I_\Delta^\ell) \geq \binom{c+q\ell-2}{c}.$$

In particular, we have

$$e(S/I_\Delta) \geq \frac{\binom{c+q\ell-2}{c}}{\binom{c+\ell-1}{c}} = \frac{(q\ell+c-2)\cdots(q\ell+1)q\ell(q\ell-1)}{(\ell+c-1)\cdots(\ell+1)\ell}.$$

In the above corollary, if we fix c, q and let ℓ tend to ∞, then the limit of the right hand side in the last inequality tends to q^c. Therefore, if S/I_Δ^ℓ is Buchsbaum for infinitely many $\ell \geq 1$, then $e(S/I_\Delta) \geq q^c$. For instance, if $I_\Delta = (m_1, \ldots, m_c)$ is a complete intersection ideal, then this inequality holds because

$$e(S/I_\Delta) = \deg m_1 \cdots \deg m_c \geq q^c.$$

However, if I is a locally complete intersection ideal but not a complete intersection ideal, then this is not true. This is a key point in the proof of Theorem 2.1. Namely we have the following proposition.

PROPOSITION 2.5. *Assume that Δ is pure and a locally complete intersection complex but not a complete intersection complex. Then*

$$e(K[\Delta]) < 2^c.$$

Proof. First, we consider the case $d = 1$. Then Δ consists of n points, and so that $c = n-1$, $e = n$. As Δ is not a complete intersection complex, we have $n \geq 3$. Then $e = n < 2^c = 2^{n-1}$ is clear.

Next, we consider the case $d = 2$. By assumption, Δ is isomorphic to the following complexes:

(a) the n-gon for $n \geq 5$;

(b) the n-pointed path for $n \geq 4$;
(c) the disjoint union of k connected complexes $\Delta_1, \dots, \Delta_k$ for some $k \geq 2$, where each Δ_i is isomorphic to the m-gon for some $m \geq 3$ or the m-pointed path for $m \geq 2$.

In particular, we have $e \leq n$ and $c = n-2$. If $n \geq 5$, then $e \leq n < 2^{n-2} = 2^c$ is clear. So we may assume that $3 \leq n \leq 4$. Then Δ is isomorphic to either the 4-pointed path or two disjoint union of the 2-pointed paths. In any case, we have $e \leq 3 < 4 = 2^c$.

Finally, we consider the case $d \geq 3$. Theorem 1.5 implies that Δ is disconnected, and so that $c \geq d$. Then we consider the following three cases:

(a) the case $c = d$;
(b) the case $c = d+1$;
(c) the case $c \geq d+2$.

When $c = d$, Δ is a disjoint union of two $(d-1)$-simplices. Then $e = 2 < 2^3 \leq 2^c$, as required. When $c = d+1$, Δ has just two connected components. One of components is a $(d-1)$-simplex and the other one is a pure $(d-1)$-subcomplex of the boundary complex of a d-simplex. In particular, $e \leq d+2 < 2^c = 2^{d+1}$.

So we may assume that $c \geq d+2$. Then Δ is a disjoint union of complete intersection complexes of dimension $d-1$ (say, $\Delta_1, \dots, \Delta_k$) by Theorem 1.15, where $k \leq \frac{n}{d} = 1 + \frac{c}{d}$. Moreover, since $c \geq d+2$, we obtain that $c(d-1) \geq (d+2)(d-1) > d^2$, and thus $d + \frac{c}{d} < c$. Hence,

$$e(K[\Delta]) = \sum_{i=1}^{k} e(K[\Delta_i]) \leq 2^d \cdot k \leq 2^d \cdot \left(1 + \frac{c}{d}\right) \leq 2^d \cdot 2^{\frac{c}{d}} = 2^{d+\frac{c}{d}} < 2^c,$$

where the first inequality follows from the lemma below. □

LEMMA 2.6. *Assume that Δ is a complete intersection complex of dimension $d-1$. Then $e(K[\Delta]) \leq 2^d$.*

Proof. Write $I_\Delta = (m_1, \dots, m_c)$, where $\deg m_i = h_i$ $(i = 1, \dots, c)$. Then

$$e(K[\Delta]) = h_1 \cdots h_c \leq 2^{h_1 - 1} \cdots 2^{h_c - 1} = 2^{h_1 + \cdots + h_c - c} \leq 2^{n-c} = 2^d,$$

as required. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. It suffices to show that I_Δ is a complete intersection ideal whenever S/I_Δ^ℓ is Buchsbaum for infinitely many $\ell \geq 1$.

By assumption and the above observation, $e(K[\Delta]) \geq 2^c$. On the other hand, S/I_Δ is Buchsbaum and thus pure by [5, Theorem 2.6]. We also have that Δ is a locally complete intersection complex by the Goto–Takayama theorem.

Suppose that Δ is not a complete intersection complex. Then by Proposition 2.5, we have that $e(K[\Delta]) < 2^c$. This is a contradiction. Hence, Δ must be a complete intersection complex. □

EXAMPLE 2.7. *Let $\Delta = \Delta_n$ be the n-gon for $n \geq 5$ (or the n-pointed path for $n \geq 4$). Then S/I_Δ^ℓ is not Buchsbaum for $\ell \geq 6$.*

Proof. We consider the case of n-gons only. Set $I = I_\Delta = (X_1X_3, X_1X_4, \dots, X_{n-2}X_n)$. Then $e = e(S/I) = n$, $c = \operatorname{codim} I = n-2$ and $q = \operatorname{indeg} I = 2$.

Suppose that S/I_Δ^ℓ is Buchsbaum. By Corollary 2.4,

$$n = e(S/I) \geq \frac{(2\ell+n-4)\cdots(2\ell+1)2\ell(2\ell-1)}{(\ell+n-3)\cdots(\ell+1)\ell}.$$

Fix $n \geq 5$ and put $f(\ell)$ to be the right-hand side of the above inequality. Then one can easily see that $f(\ell)$ is an increasing function of ℓ. Thus if $\ell \geq 6$, then

$$1 \geq \frac{(n+8)\cdots 12\cdot 11}{(n+3)\cdots 7\cdot 6} \times \frac{1}{n} = \frac{(n+8)(n+7)(n+6)(n+5)(n+4)}{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot n}.$$

Put $g(n)$ to be the right-hand side of the above inequality. Then since

$$g(n+1)/g(n) = \frac{n^2+9n}{n^2+5n+4} \geq 1 \quad \text{and} \quad g(5) = 1.02\cdots > 1$$

we get a contradiction. □

It is difficult to determine the Buchsbaumness for S/I^ℓ.

EXAMPLE 2.8. *Let $S = K[X_1, X_2, X_3, X_4, X_5]$ be a polynomial ring. Let $I = (X_1X_3, X_1X_4, X_2X_4, X_2X_5, X_3X_5)$ be the Stanley–Reisner ideal (of height 3) of the 5-gon. Then S/I^2 is Cohen–Macaulay with* $\dim S/I^2 = 2$. *Indeed, Macaulay 2 yields the following minimal free resolution of S/I^2:*

$$0 \to S^{10}(-6) \to S^{24}(-5) \to S^{15}(-4) \to S \to S/I^2 \to 0.$$

On the other hand, $\operatorname{depth} S/I^3 = 0$ *since $X_1X_2X_3X_4X_5 \in I^3 : \mathfrak{m} \setminus I^3$. We do not know whether S/I^3 is Buchsbaum or not.*

In the following, we give an example of the simplicial complex Δ for which S/I_Δ^2 is Buchsbaum but *not* Cohen–Macaulay (and this implies that Δ is not a complete intersection complex). In order to do that, we use an extension of Hochster's formula describing the local cohomology of a monomial ideal; see [9]. Fix $\ell \geq 1$ and set $G(I_\Delta^\ell) = \{m_1, \dots, m_\mu\}$. Write $m = X_1^{\nu_1(m)} \cdots X_n^{\nu_n(m)}$ for any monomial m in $S = K[X_1, \dots, X_n]$. For a vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we put

$$G_a = \{i \in V : a_i < 0\}.$$

Then we define the simplicial complex $\Delta_{\mathbf{a}}(I_\Delta^\ell) \subseteq \Delta$ by

$$\Delta_{\mathbf{a}}(I_\Delta^\ell) = \{L \setminus G_a : G_a \subseteq L \in \Delta, L \text{ satisfies the condition } (*)\},$$

where

(∗) for all $m \in G(I_\Delta^\ell)$, there exists an $i \in V \setminus L$ such that $\nu_i(m) > a_i (\geq 0)$.

For a graded S-module M, $F(A, \mathbf{t}) = \sum_{\mathbf{a} \in \mathbb{Z}^n} \dim_K A_\mathbf{a} \mathbf{t}^\mathbf{a}$ is called the Hilbert–Poincaré series of M. Then Hochster–Takayama formula (see [9]) says that

$$F\big(H_\mathfrak{m}^i(S/I_\Delta^\ell), \mathbf{t}\big) = \sum_{F \in \Delta} \sum_{\substack{\mathbf{a} \in \mathbb{Z}^n \\ G_a = F, a_i \leq \ell - 1}} \dim_K \widetilde{H}_{i - \sharp(F) - 1}(\Delta_\mathbf{a}(I_\Delta^\ell); K) \mathbf{t}^\mathbf{a},$$

where $\widetilde{H}_i(\Delta; K)$ denotes the ith simplicial reduced homology of Δ with values in K. In particular, we have

$$F\big(H_\mathfrak{m}^1(S/I_\Delta^\ell), \mathbf{t}\big) = \sum_{\mathbf{a} \in \mathcal{A}} \dim_K \widetilde{H}_0(\Delta_\mathbf{a}(I_\Delta^\ell); K) \mathbf{t}^\mathbf{a} + \sum_{i=1}^{n} \sum_{\mathbf{a} \in \mathcal{A}_i} \mathbf{t}^\mathbf{a},$$

where

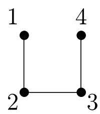
$$\mathcal{A} = \{\mathbf{a} \in \mathbb{Z}^n : 0 \leq a_1, \ldots, a_n \leq \ell - 1, \Delta_\mathbf{a}(I_\Delta^\ell) \text{ is disconnected}\};$$
$$\mathcal{A}_i = \big\{\mathbf{a} \in \mathbb{Z}^n : 0 \leq a_1, \ldots, \widehat{a_i} \ldots, a_n \leq \ell - 1, \Delta_\mathbf{a}(I_\Delta^\ell) = \{\emptyset\}\big\}$$

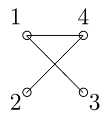
for each $i = 1, \ldots, n$.

Example 2.9. *Let $S = K[X_1, X_2, X_3, X_4]$ be a polynomial ring over a field K. Let $I = (X_1X_3, X_1X_4, X_2X_4)$ be the Stanley–Reisner ideal of the 4-pointed path Δ.*

Then S/I^2 is Buchsbaum but not Cohen–Macaulay. In fact, $\dim S/I^2 = 2$, $\operatorname{depth} S/I^2 = 1$ *and* $\dim_K H_\mathfrak{m}^1(S/I^2) = 1$.

Proof. The ideal I can be considered as the edge ideal of some bipartite graph G. Thus we have $I^2 = I^{(2)}$, the second symbolic power of I, by [7, Section 5], and so $H_\mathfrak{m}^0(S/I^2) = 0$.

$\Delta =$ 

$G =$ 

Hence, it suffices to show that $\mathfrak{m}H_\mathfrak{m}^1(S/I^2) = 0$ and $H_\mathfrak{m}^1(S/I^2) \neq 0$. We first show the following claim. Put $\Delta_\mathbf{a} = \Delta_\mathbf{a}(I^2)$ for simplicity.

Claim 1: $\mathcal{A} = \{(1,0,0,1)\}$ and $\Delta_{(1,0,0,1)}$ is spanned by $\{\{(1,2)\}, \{3,4\}\}$. (This implies that $Kt_1t_4 \subseteq H_\mathfrak{m}^1(S/I^2)$.)

First of all, we define monomials $m_1, \ldots, m_6$ as in Table 1:
Namely,

$$G(I^2) = \{X_1^2X_3^2, X_1^2X_3X_4, X_1^2X_4^2, X_1X_2X_3X_4, X_1X_2X_4^2, X_2^2X_4^2\}.$$

Fix $\mathbf{a} = (a_1, a_2, a_3, a_4) \in (\mathbb{Z} \cap \{0,1\})^4$. As $\nu_3(m_4) = \nu_4(m_4) = 1$, it follows that $\{1,2\} \in \Delta_\mathbf{a}$ if and only if $a_3 = 0$ or $a_4 = 0$. Similarly, $\{3,4\} \in \Delta_\mathbf{a}$ if and only

TABLE 1.

	m_1	m_2	m_3	m_4	m_5	m_6
$\nu_1(m)$	2	2	2	1	1	0
$\nu_2(m)$	0	0	0	1	1	2
$\nu_3(m)$	2	1	0	1	0	0
$\nu_4(m)$	0	1	2	1	2	2

if $a_1 = 0$ or $a_2 = 0$. If $\sharp\{i : 1 \le i \le 4, a_i = 1\} \ge 3$, then $\Delta_{\mathbf{a}} = \emptyset$. So, we may assume that $\sharp\{i : 1 \le i \le 4, a_i = 1\} \le 2$ and $a_1 \ge a_4$.

If $\{2,3\} \notin \Delta_{\mathbf{a}}$, then $a_1 = a_4 = 1$. That is $\mathbf{a} = (1,0,0,1)$. Indeed, $\Delta_{(1,0,0,1)} = \langle\{1,2\},\{3,4\}\rangle$ is disconnected. Otherwise, $\{2,3\} \in \Delta_{\mathbf{a}}(I^2)$. Then $(a_1, a_4) = (0,0)$ or $(1,0)$. In these cases, we have

$$\Delta_{(0,*,*,0)} = \Delta_{(1,0,0,0)} = \Delta_{(1,0,1,0)} = \Delta, \qquad \Delta_{(1,1,0,0)} = \langle\{1,2\},\{2,3\}\rangle.$$

In particular, $\Delta_{\mathbf{a}}$ is connected in any case. Therefore, we proved Claim 1.

Next, we show the following claim.

Claim 2: $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = \emptyset$.

To see $\mathcal{A}_1 = \emptyset$, let $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$ such that $a_1 < 0$, $0 \le a_2, a_3, a_4 \le 1$. Note that

$$\Delta_{\mathbf{a}}(I^2) = \big\{L \setminus \{1\} : \{1\} \subseteq L \in \Delta, L \text{ satisfies } (*)\big\}$$

and that $\{1\} \subseteq L \in \Delta$ if and only if $L = \{1\}$ or $\{1,2\}$. By a similar argument as in the proof of the claim 1, we obtain that

$$\{2\} = \{1,2\} \setminus \{1\} \in \Delta_{\mathbf{a}}(I^2) \quad \Longleftrightarrow \quad a_3 = 0 \quad \text{or} \quad a_4 = 0.$$

Then $\Delta_{\mathbf{a}}(I^2) = \{\emptyset, \{2\}\} \ne \{\emptyset\}$.

Now suppose that $a_3 = a_4 = 1$. Then $\emptyset \notin \Delta_{\mathbf{a}}(I^2)$ because $m_2 = X_1^2 X_3 X_4 \in G(I^2)$. This yields that $\Delta_{\mathbf{a}}(I^2) \ne \{\emptyset\}$. Therefore, $\mathcal{A}_1 = \emptyset$. Similarly, one has $\mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4 = \emptyset$.

The above two claims imply that $H^1_{\mathfrak{m}}(S/I^2) \cong K t_1 t_4$, as required. □

QUESTION 2.10. Can you replace Buchsbaumness with quasi-Buchsbaumness in Theorem 2.1?

QUESTION 2.11. Let I be a generically complete intersection homogeneous ideal of a polynomial ring S. If S/I^ℓ is Buchsbaum for all $\ell \ge 1$, then is I a complete intersection ideal?

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