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CONTRACTION OF COMPACT SEMISIMPLE LIE GROUPS VIA BEREZIN QUANTIZATION

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Abstract. We establish contractions of the unitary irreducible representations of a compact semisimple Lie group to the unitary irreducible representations of a Heisenberg group by means of Berezin quantization.

1. Introduction

In [28], Inönü and Wigner introduced the notion of contraction of Lie groups on physical grounds: If two physical theories are related by a limiting process, then the associated invariance groups should be also related in a limiting process. For instance, the Galilei group is a contraction, that is, a limiting case in the sense of [28], of the Poincaré group.

When a Lie group G_0 is a contraction of a Lie group G, a natural question is how to obtain the unitary irreducible representations of G_0 as limits of the unitary irreducible representations of G. This delicate problem was first studied systematically by Mickelsson and Niederle [32]. In the paper [32], a proper definition of the contraction of unitary representations of Lie groups was given. The nonzero mass representations of the Euclidean group $\mathbb{R}^{n+1} \rtimes SO(n+1)$ and the positive mass-squared representations of the Poincaré group $\mathbb{R}^{n+1} \rtimes SO_0(n,1)$ were obtained by contraction (i.e., as limits in the sense defined in [32]) of the principal series representations of $SO_0(n+1,1)$. More generally, in [22], Dooley and Rice established a contraction of the principal series representations of a semisimple Lie group to some unitary irreducible representations of its Cartan motion group.

Contraction theory is a way to relate the Harmonic Analysis on two Lie groups. In particular, contractions of Lie group representations allow to recover some classical formulas of the theory of special functions [21], [33]. One

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can also use contractions to transfer results on L^p-multipliers from unitary groups to Heisenberg groups [20], [34].

In [18], Dooley suggested interpreting contractions of representations in the context of the Kirillov–Kostant method of orbits. In this spirit, a contraction of the discrete series representations of $SU(1,1)$ to the positive massive representations of the Poincaré group $\mathbb{R}^2 \rtimes SO(1,1)$ was introduced in [15].

In [17], Cotton and Dooley showed how to obtain contraction results by using the notion of adapted functional calculus which we have introduced in [6], [7] (see also [8]). The basic idea is then to 'read' contraction results on the symbols of the representation operators which are functions on the associated coadjoint orbits. This approach is particularly efficient in the case when these orbits have Kählerian structures [9], [10], [11]. Indeed, in this case, the representation spaces are reproducing kernel Hilbert spaces [14] and the so-called Berezin calculus generally provides an adapted functional calculus on the corresponding coadjoint orbits. Moreover, the representation operators are operators of the functional calculus and one can study the behavior of the corresponding symbols in the contraction process. For example, in [9], we recovered and we improved the contraction results of [33] relative to the contraction of the unitary irreducible representations of $SU(2)$ to the unitary irreducible representations of the 3-dimensional Heisenberg group. Similarly, in [10], we studied a contraction of the discrete series representations of $SU(1,1)$ to the unitary irreducible representations of the 3-dimensional Heisenberg group. More generally, in [11], we established a contraction of the unitary irreducible representations of $SU(n+1)$ to the unitary irreducible representations of the $(2n+1)$-dimensional Heisenberg group. In particular, we approximated the coefficients of the unitary irreducible representations of the $(2n+1)$-dimensional Heisenberg group by the coefficients of the unitary irreducible representations of $SU(n+1)$ in a simpler way that was done in [19].

In the present paper, we extend some results from [9], [11], and [33] to any semisimple compact Lie group G. More precisely, let T be a maximal torus of G. Fix a set Δ^+ of positive roots of G relative to T. Let π_λ be the unitary irreducible representation of G with highest weight λ. Let G_0 be the $(2n+1)$-dimensional Heisenberg group where n is the number of positive roots orthogonal to λ. Then G_0 is a contraction of G in some generalized sense (see [11] and Section 5 below). Let ρ be a nondegenerated unitary irreducible representation of G_0. We establish and study a contraction of the sequence $\pi_{m\lambda}$ to ρ. In particular, we obtain a contraction of the coefficients of $\pi_{m\lambda}$ to the coefficients of ρ in the spirit of [11] by using a realization of π_λ on a (finite-dimensional) Hilbert space of complex polynomials and the explicit formulas for the Berezin symbols of $\pi_\lambda(g)$ for $g \in G$ obtained in [12] (see also [13]).

This paper is organized as follows. In Section 2, we describe the unitary irreducible representations of G_0, and we introduce the Berezin calculus on the associated coadjoint orbits. In Section 3, we realize π_λ on a Hilbert space $\mathcal{H}_\lambda$ of complex polynomials by means of the Gauss decomposition of the complexification G^c of G. We also introduce the corresponding Berezin calculus and we recall from [12] some explicit formulas for the coherent states in $\mathcal{H}_\lambda$, and also for the Berezin symbols of the operators $\pi_\lambda(g)$ for $g\in G$ and $d\pi_\lambda(X)$ for X in the Lie algebra of G. In Section 4, we establish some technical results. In particular, we give a lower bound for the restriction to the diagonal of the reproducing kernel of $\mathcal{H}_\lambda$. In Section 5, we introduce a contraction (in some generalized sense) of G to G_0. In Section 6, we relate the contraction of a sequence of operators acting on the spaces $\mathcal{H}_{m\lambda}$ to the simple convergence of their Berezin symbols. We then obtain in Section 7 our results on the contraction of $\pi_{m\lambda}$ to ρ. In Section 8, we give similar contraction results for the derived representations. Section 9 is devoted to a particular case in which we are able to establish a more precise contraction result. As an example, we study the case when $G=SU(p+q)$ and $\lambda=\lambda_1+\lambda_2+\cdots+\lambda_p$ in the notation of [35], which leads us to consider the Berezin quantization of the symmetric space $SU(p+q)/S(U(p)\times U(q))$.

2. Berezin quantization for the Heisenberg group

In this section, we recall some well-known facts on the Bargmann–Fock realization of the unitary irreducible representations of a $(2n+1)$-dimensional Heisenberg group and the corresponding Berezin calculus [11], [23].

Let G_0 be the Heisenberg group of dimension $2n+1$ and $\mathfrak{g}_0$ its Lie algebra. Let $\{X_1,\dots,X_n,Y_1,\dots,Y_n,\tilde{Z}\}$ be a basis of $\mathfrak{g}_0$ in which the only nontrivial brackets are $[X_k\,,Y_k]=\tilde{Z}$, $1\le k\le n$ and let $\{X_1^*,\dots,X_n^*,Y_1^*,\dots,Y_n^*,\tilde{Z}^*\}$ be the corresponding dual basis of $\mathfrak{g}_0^*$.

For $a=(a_1,a_2,\dots,a_n)\in\mathbb{R}^n$, $b=(b_1,b_2,\dots,b_n)\in\mathbb{R}^n$ and $c\in\mathbb{R}$, we denote by $[a,b,c]$ the element $\exp_H(\sum_{k=1}^n a_kX_k+\sum_{k=1}^n b_kY_k+c\tilde{Z})$ of G_0.

Fix a real number $\gamma>0$, and denote by $\mathcal{O}_\gamma$ the orbit of the element $\xi_\gamma=\gamma\tilde{Z}^*$ of $\mathfrak{g}_0^*$ under the coadjoint action of G_0. By the Stone–von Neumann theorem, there exists a unique (up to unitary equivalence) representation ρ_γ of G_0 whose restriction to the center of G_0 is the character $[0,0,c]\to e^{i\gamma c}$ [23]. The representation ρ_γ is associated to the coadjoint orbit $\mathcal{O}_\gamma$ by the Kirillov–Kostant method of orbits [29]. Here, we introduce the Bargmann–Fock realization of ρ_γ as follows.

Let $\mathcal{H}_\gamma$ be the Hilbert space of holomorphic functions on $\mathbb{C}^n$ such that

$$\|f\|_\gamma^2:=\int_{\mathbb{C}^n}|f(z)|^2\,d\mu_\gamma(z)<+\infty,$$

where $d\mu_\gamma(z)=(2\pi\gamma)^{-n}\exp(-|z|^2/2\gamma)\,dx_1\,dy_1\cdots dx_n\,dy_n$. Here, we use the notation $z=(z_1,z_2,\dots,z_n)$ and $z_k=x_k+iy_k,\quad x_k,y_k\in\mathbb{R}$ for $k=1,2,\dots,n$.

Then ρ_γ is the representation of G_0 on $\mathcal{H}_\gamma$ defined by

$$\rho_\gamma([a,b,c])f(z) = \exp\left(ic\gamma + \frac{1}{4}(b+ai)^t\big(2z+\gamma(-b+ai)\big)\right) f\big(z+\gamma(-b+ai)\big),$$

where z, a and b are considered as column vectors.

For $z\in\mathbb{C}^n$, introduce the coherent states $e_z^\gamma(w) = \exp(z^*w/2\gamma)$. We have the reproducing property

$$f(z) = \langle f, e_z^\gamma\rangle_\gamma \quad (f\in\mathcal{H}_\gamma),$$

where $\langle\cdot,\cdot\rangle_\gamma$ denotes the scalar product on $\mathcal{H}_\gamma$.

Consider now a bounded operator A on $\mathcal{H}_\gamma$. The Berezin (covariant) symbol of A is the function defined on $\mathbb{C}^n$ by

$$s_\gamma(A)(z) = \frac{\langle Ae_z^\gamma, e_z^\gamma\rangle_\gamma}{\langle e_z^\gamma, e_z^\gamma\rangle_\gamma} \tag{2.1}$$

and the double Berezin symbol of A is the function defined by

$$S_\gamma(A)(z,w) = \frac{\langle Ae_w^\gamma, e_z^\gamma\rangle_\gamma}{\langle e_w^\gamma, e_z^\gamma\rangle_\gamma} \tag{2.2}$$

for $z,w\in\mathbb{C}^n$ such that $\langle e_z^\gamma, e_w^\gamma\rangle_\gamma \neq 0$ (see [3], [4] and [5]). The function $S_\gamma(A)$ is holomorphic in the variable z and anti-holomorphic in the variable w. Moreover, we can reconstruct the operator A from its double symbol $S_\gamma(A)$ by the following integral formula (see [14]):

$$Af(z) = \int_{\mathbb{C}^n} f(w) S_\gamma(A)(z,w)\langle e_w^\gamma, e_z^\gamma\rangle_\gamma \, d\mu_\gamma(w). \tag{2.3}$$

From this, we deduce immediately an integral formula for $\langle Af, g\rangle_\gamma$ which we require later

$$\langle Af, g\rangle_\gamma = \int_{\mathbb{C}^n\times\mathbb{C}^n} f(w)\overline{g(z)} S_\gamma(A)(z,w)\langle e_w^\gamma, e_z^\gamma\rangle_\gamma \, d\mu_\gamma(z)\, d\mu_\gamma(w). \tag{2.4}$$

For $g\in G_0$, the Berezin symbol of the operator $\rho_\gamma(g)$ is easily obtained from the reproducing property:

$$S_\gamma(\rho_\gamma(g))(z,w) = \exp\left(i\gamma c - \frac{1}{4}\gamma(|a|^2+|b|^2) + \frac{1}{2}(b+ia)^t z - \frac{1}{2}\overline{w}^t(b-ia)\right). \tag{2.5}$$

Moreover, for $X = \sum_{k=1}^n a_kX_k + \sum_{k=1}^n b_kY_k + c\tilde{Z}\in\mathfrak{g}_0$, we have

$$S_\gamma(d\rho_\gamma(X))(z,w) = \frac{1}{2}(b+ia)^t z + \frac{1}{2}(ia-b)^t\overline{w} + i\gamma c. \tag{2.6}$$

Let us introduce the parametrization ψ_γ of the orbit $\mathcal{O}_\gamma$ defined by

$$\psi_\gamma(z) = (\operatorname{Re} z)X^* + (\operatorname{Im} z)Y^* + \gamma\tilde{Z}^*$$

with obvious notation. Then, for all $z\in\mathbb{C}^n$, $X\in\mathfrak{g}_0$, we have

$$S_\gamma(d\rho_\gamma(X))(z,z) = i\langle\psi_\gamma(z), X\rangle. \tag{2.7}$$

3. Berezin quantization for a compact semisimple Lie group

In this section, we introduce some notation and we recall some results from the paper [12] in which we realized the unitary irreducible representations of a compact semisimple Lie group as representations on a Hilbert space of complex polynomials.

Let G be a connected simply-connected semisimple compact Lie group. Let T be a maximal torus of G and Δ be the root system of G relative to T. Let $\Delta^+ \subset \Delta$ be a system of positive roots. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T, respectively. We denote by $\mathfrak{g}^c$ and $\mathfrak{t}^c$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$. Let G^c and T^c be the connected complex Lie groups whose Lie algebras are $\mathfrak{g}^c$ and $\mathfrak{t}^c$.

Let β be the Killing form on $\mathfrak{g}^c$, that is, $\beta(X,Y)=\mathrm{Tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y)$ for $X,Y\in\mathfrak{g}^c$. For each $\lambda\in(\mathfrak{t}^c)^*$, we denote by H_λ the element of $\mathfrak{t}^c$ satisfying $\beta(H,H_\lambda)=\lambda(H)$ for all $H\in\mathfrak{t}^c$. For $\lambda,\mu\in(\mathfrak{t}^c)^*$, we set $(\lambda,\mu):=\beta(H_\lambda,H_\mu)$.

Now, we fix $\lambda\in(\mathfrak{t}^c)^*$ real-valued on $i\mathfrak{t}$. Then iH_λ lies in $\mathfrak{g}$. Let $T_1\subset G$ be the torus generated by $\exp(iH_\lambda)$. We can apply to T_1 the construction of [36], Section 6.2. Let H be the centralizer of T_1 in G. Clearly, $T\subset H$ and the root system of H relative to T is $\Delta_1=\{\alpha\in\Delta/(\lambda,\alpha)=0\}$. Let $\mathfrak{h}$ be the Lie algebra of H, $\mathfrak{h}^c$ be the complexification of $\mathfrak{h}$ and H^c be the connected subgroup of G^c corresponding to $\mathfrak{h}^c$. Let $\mathfrak{g}^c=\mathfrak{t}^c\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}^c$. Let $\Delta_1^+=\Delta^+\cap\Delta_1$ and $\Phi=\Delta^+\setminus\Delta_1^+$. We set $\mathfrak{n}^+=\sum_{\alpha\in\Phi}\mathfrak{g}_\alpha$ and $\mathfrak{n}^-=\sum_{\alpha\in\Phi}\mathfrak{g}_{-\alpha}$. Then, by [36], 6.2.1, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent Lie algebras satisfying $[\mathfrak{h}^c,\mathfrak{n}^\pm]\subset\mathfrak{n}^\pm$ and we have the decompositions

$$\mathfrak{g}^c=\mathfrak{h}^c\oplus\mathfrak{n}^+\oplus\mathfrak{n}^-,\qquad \mathfrak{h}^c=\mathfrak{t}^c\oplus\sum_{\alpha\in\Delta_1^+}\mathfrak{g}_\alpha\oplus\sum_{\alpha\in\Delta_1^+}\mathfrak{g}_{-\alpha}. \tag{3.1}$$

We denote by N^+ and N^- the analytic subgroups of G^c with Lie algebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$, respectively.

Let us consider the generalized flag manifold $M:=G/H$. A complex structure on M is defined by the diffeomorphism $M=G/H\simeq G^c/H^cN^-$ [36], 6.2.11. We denote by $\tau:G^c\to M\simeq G^c/H^cN^-$ the natural projection. Recall that (1) each g in a dense open subset of G^c has a unique Gauss decomposition $g=n^+hn^-$ where $n^+\in N^+$, $h\in H^c$ and $n^-\in N^-$ and (2) the map $\sigma:Z\to\tau(\exp Z)$ is a holomorphic diffeomorphism from $\mathfrak{n}^+$ onto a dense open subset of M (see [25], Chapter VIII). Then the natural action of G^c on $M\simeq G^c/H^cN^-$ induces an action (defined almost everywhere) of G^c on $\mathfrak{n}^+$. We denote by $g\cdot Z$ the action of $g\in G^c$ on $Z\in\mathfrak{n}^+$. Following [31], we introduce the projections $\kappa:N^+H^cN^-\to H^c$ and $\zeta:N^+H^cN^-\to N^+$. Then, for $g\in G^c$ and $Z\in\mathfrak{n}^+$, we have $g\cdot Z=\log\zeta(g\exp Z)$.

We set $(X+iY)^*=-X+iY$ for $X,Y\in\mathfrak{g}$ and we denote by $g\to g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X+iY\to(X+iY)^*$ to G^c. Also, let θ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$ and let $\tilde{\theta}$ be the automorphism of G^c for which $d\tilde{\theta} = \theta$. Then we have $\theta(X) = -X^*$ for $X \in \mathfrak{g}^c$ and $\tilde{\theta}(g) = (g^*)^{-1}$ for $g \in G^c$.

Let us assume, moreover, that λ is integral and dominant i.e., $\frac{2\lambda(H_\alpha)}{\alpha(H_\alpha)}$ is a nonnegative integer for each $\alpha \in \Delta^+$. Let χ_0 be the unique character on H such that $\lambda = d\chi_0|_{\mathfrak{t}}$ and let χ_λ be the unique extension of χ_0 to H^cN^-. There exists a unique (up to equivalence) unitary irreducible representation π_λ of G with highest weight λ. This representation is usually realized in the space of the holomorphic sections of the holomorphic line bundle $L_\lambda = G^c \times_\chi \mathbb{C}$. Here, we use the realization of π_λ which was obtained in [12] by trivializing L_λ by means of the section $Z \in \mathfrak{n}^+ \to [\exp Z, 1]$. Let χ_Λ be the character of H^c corresponding to $\Lambda = \sum_{\alpha \in \Phi} \alpha$, that is, $\chi_\Lambda(h) = \mathrm{Det}_{\mathfrak{n}^+} \mathrm{Ad}(h)$ for each $h \in H^c$. Then the G-invariant measure on $\mathfrak{n}^+$ is $d\mu(Z) = \chi_\Lambda(k(Z))\, d\mu_L(Z)$ where $k(Z) := \kappa(\exp Z^* \exp Z)$ and $d\mu_L(Z)$ is a Lebesgue measure on $\mathfrak{n}^+$ [12], [31]. The representation space of π_λ is the finite dimensional Hilbert space $\mathcal{H}_\lambda$ consisting of complex polynomials f satisfying

$$\|f\|_\lambda^2 := \int_{\mathfrak{n}^+} |f(Z)|^2 \chi_\lambda(k(Z)) c_\lambda \, d\mu(Z) < +\infty, \tag{3.2}$$

where

$$c_\lambda^{-1} = \int_{\mathfrak{n}^+} \chi_\lambda(k(Z))\, d\mu(Z) < +\infty \tag{3.3}$$

(see [12]). The representation π_λ acts on $\mathcal{H}_\lambda$ as

$$(\pi_\lambda(g)f)(Z) = \chi_\lambda(\exp(-Z) g \exp(g^{-1} \cdot Z)) f(g^{-1} \cdot Z). \tag{3.4}$$

For each $Z \in \mathfrak{n}^+$, there exists a unique element e_Z^λ in $\mathcal{H}_\lambda$ (the coherent state) such that $f(Z) = \langle f, e_Z^\lambda \rangle_\lambda$ for all $f \in \mathcal{H}_\lambda$. Here $\langle \cdot, \cdot \rangle_\lambda$ denotes the scalar product on $\mathcal{H}_\lambda$. We have

$$e_Z^\lambda(W) = \chi_\lambda(\kappa(\exp Z^* \exp W))^{-1} \tag{3.5}$$

for $Z, W \in \mathfrak{n}^+$ [12]. Another expression for the coherent states e_Z^λ in terms of projected determinants can be found in [2].

Remarks 3.1. (1) We have $e_0^\lambda(W) \equiv 1$ and $\|1\|_\lambda = 1$.

(2) It is well known that, for any orthonormal basis (f_p) of $\mathcal{H}_\lambda$, we have $e_Z^\lambda(W) = \sum_{p=1}^{\dim \mathcal{H}_\lambda} f_p(W)\overline{f_p(Z)}$. Taking (f_p) so that $f_1 = 1$, we get

$$\chi_\lambda(k(Z))^{-1} \geq 1$$

for each $Z \in \mathfrak{n}^+$. In particular, this implies that $\mathcal{H}_{m\lambda} \subset \mathcal{H}_{(m+1)\lambda}$ for each $m \in \mathbb{Z}^+$. We shall see in the next section that $\bigcup_{m \in \mathbb{Z}^+} \mathcal{H}_{m\lambda}$ is the space of all complex polynomials on $\mathfrak{n}^+$.

As in Section 2, we can define the Berezin symbol $S_\lambda(A)(Z,W)$ of an operator A on $\mathcal{H}_\lambda$. We also obtain an integral formula for $\langle Af,g\rangle_\lambda$ analogous to that of Section 2:

$$\begin{aligned}(3.6)\quad \langle Af,g\rangle_\lambda = \int_{\mathfrak{n}^+\times\mathfrak{n}^+} & f(W)\overline{g(Z)}S_\lambda(A)(Z,W)\langle e_W^\lambda, e_Z^\lambda\rangle_\lambda \\ & \times (\chi_\lambda.\chi_\Lambda)(k(Z))(\chi_\lambda.\chi_\Lambda)(k(W))c_\lambda^2\, d\mu_L(Z)\, d\mu_L(W).\end{aligned}$$

In [12], we give explicit expressions for the derived representation $d\pi_\lambda$ and for the Berezin symbols of $\pi_\lambda(g)$ and $d\pi_\lambda(X)$. In the following proposition, we recall some results from [12].

Proposition 3.2.

(1) *Let $g\in G$. We have*

$$(3.7)\qquad S_\lambda(\pi_\lambda(g))(Z,W)=\chi_\lambda(\kappa(\exp W^* g^{-1}\exp Z)^{-1}\kappa(\exp W^*\exp Z)).$$

(2) *Let $X\in\mathfrak{g}^c$. We have*

$$(3.8)\qquad S_\lambda(d\pi_\lambda(X))(Z,W)=\lambda(p_{\mathfrak{h}^c}(\mathrm{Ad}(\zeta(\exp W^*\exp Z)^{-1}\exp W^*)X),$$

where $p_{\mathfrak{h}^c}$ denotes the projection of $\mathfrak{g}^c$ onto $\mathfrak{h}^c$ in the decomposition (3.1).

Moreover, we can write

$$(3.9)\qquad S_\lambda(d\pi_\lambda(X))(Z,W)=i\beta(\psi_\lambda(Z,W),X),$$

where

$$(3.10)\qquad \psi_\lambda(Z,W):=\mathrm{Ad}(\tilde\theta(\exp W)\zeta(\exp W^*\exp Z))(-iH_\lambda).$$

In particular, the map $\tilde\psi_\lambda : Z\to\psi_\lambda(Z,Z)$ is a diffeomorphism from $\mathfrak{n}^+$ onto a dense open subset of the orbit $\mathcal{O}_\lambda$ of $-iH_\lambda\in\mathfrak{g}$ for the adjoint action of G.

The Berezin symbol of $\pi_\lambda(g)$ (sometimes called star exponential) plays a prominent role in the construction of the generalized Fourier transform for compact Lie groups [1], [37].

4. Some technical results

In this section, we keep the notation of Section 3. We establish a lower bound for the function $\varphi_\lambda(Z):=\chi_\lambda^{-1}(\kappa(\exp Z^*\exp Z))$ on $\mathfrak{n}^+$.

Let us denote by $|\cdot|$ an arbitrary norm on $\mathfrak{g}^c$. Given a function ψ from a subspace of $\mathfrak{g}^c$ to $\mathfrak{g}^c$ and $p\in\mathbb{Z}^+$, we write $\psi(X)=O(|X|^p)$ if there exist $M>0$ and $\varepsilon>0$ such that $|\psi(X)|\le M|X|^p$ whenever $|X|<\varepsilon$.

We denote by $p_{\mathfrak{n}^+}$ and $p_{\mathfrak{n}^-}$ the projections on $\mathfrak{n}^+$ and $\mathfrak{n}^-$ in the direct decomposition (3.1). The following lemma is a consequence of the Baker–Campbell–Hausdorff formula (see [35] for instance).

LEMMA 4.1.

(1) *For X in a sufficiently small neighborhood of* 0 *in* $\mathfrak{g}^c$, *write*

$$\exp X = \exp Y \exp U \exp V$$

with $Y \in \mathfrak{n}^+$, $U \in \mathfrak{h}^c$, *and* $V \in \mathfrak{n}^-$. *Then we have*

$$U = p_{\mathfrak{h}^c}(X) - \frac{1}{2} p_{\mathfrak{h}^c}([p_{\mathfrak{n}^+}(X), p_{\mathfrak{n}^-}(X)]) + O(|X|^3). \tag{4.1}$$

(2) *For Z and W in a sufficiently small neighborhood of* 0 *in* $\mathfrak{n}^+$, *write*

$$\exp Z^* \exp W = \exp Y \exp U \exp V$$

with $Y \in \mathfrak{n}^+$, $U \in \mathfrak{h}^c$, *and* $V \in \mathfrak{n}^-$. *Then we have*

$$U = p_{\mathfrak{h}^c}([Z^*, W]) + O(\max(|Z|, |W|)^3). \tag{4.2}$$

Proof. (1) We first show that $Y, U, V = O(|X|)$. Set

$$U = u(X) := \log \kappa(\exp X).$$

By differentiating u at $X = 0$, we get $du(0)(X) = p_{\mathfrak{h}^c}(X)$ for each $X \in \mathfrak{g}^c$ (see [12], Proposition 4.1). Then, by writing the first-order Taylor formula for u at 0, we obtain $u(X) = O(|X|)$. Similarly, we find $Y, V = O(|X|)$.

Now, by the Baker–Campbell–Hausdorff formula, we have

$$X = Y + U + V + \frac{1}{2}[Y, U] + \frac{1}{2}[Y, V] + \frac{1}{2}[U, V] + O(\max(|Y|, |U|, |V|)^3)$$

for X in a sufficiently small neighborhood of 0 in $\mathfrak{g}^c$. Since $[Y, U] \in \mathfrak{n}^+$ and $[U, V] \in \mathfrak{n}^-$, this gives

$$p_{\mathfrak{h}^c}(X) = U + \frac{1}{2} p_{\mathfrak{h}^c}([Y, V]) + O(|X|^3), \tag{4.3}$$

$$p_{\mathfrak{n}^+}(X) = Y + \frac{1}{2}[Y, U] + \frac{1}{2} p_{\mathfrak{n}^+}([Y, V]) + O(|X|^3), \tag{4.4}$$

$$p_{\mathfrak{n}^-}(X) = V + \frac{1}{2}[U, V] + \frac{1}{2} p_{\mathfrak{n}^-}([Y, V]) + O(|X|^3). \tag{4.5}$$

Equalities (4.4) and (4.5) imply

$$[p_{\mathfrak{n}^+}(X), p_{\mathfrak{n}^-}(X)] = [Y, V] + O(|X|^3). \tag{4.6}$$

Replacing (4.6) in (4.3), we then obtain the desired equality.

(2) Using the Baker–Campbell–Hausdorff formula again, (4.2) is an immediate consequence of (4.1). $\square$

Let us consider a Chevalley basis $(\tilde{E}_\alpha)_{\alpha \in \Delta} \cup (H_\alpha)_{\alpha \in \Delta_s}$ for $\mathfrak{g}^c$ (see for instance [29], Chapter 5). Here, Δ_s denotes the set of simple roots corresponding to Δ^+. In particular, we have $[\tilde{E}_\alpha, \tilde{E}_{-\alpha}] = H_\alpha$ for $\alpha \in \Delta^+$. Note that $\mathfrak{g}$ is spanned by the elements $\frac{1}{2}(\tilde{E}_\alpha - \tilde{E}_{-\alpha})$, $\frac{1}{2i}(\tilde{E}_\alpha + \tilde{E}_{-\alpha})$ and iH_α for $\alpha \in \Delta^+$ and that we have the property $\tilde{E}_\alpha^* = \tilde{E}_{-\alpha}$ for $\alpha \in \Delta$. Now, we introduce the basis $(E_\alpha)_{\alpha \in \Phi}$ for $\mathfrak{n}^+$ defined by $E_\alpha = \frac{1}{\sqrt{(\lambda, \alpha)}} \tilde{E}_\alpha \quad (\alpha \in \Phi)$ and the corresponding

Euclidean norm defined by $|Z| = (\sum_{\alpha\in\Phi} |z_\alpha|^2)^{1/2}$ for $Z = \sum_{\alpha\in\Phi} z_\alpha E_\alpha$. Note that we have

$$\lambda(p_{\mathfrak{h}^c}([Z, Z^*])) = \lambda\left(\sum_{\alpha\in\Phi} \frac{1}{(\lambda, \alpha)} |z_\alpha|^2 H_\alpha\right) = |Z|^2 \tag{4.7}$$

for each $Z \in \mathfrak{n}^+$. In the following proposition, we give some properties of the function φ_λ.

PROPOSITION 4.2.

(1) *For $Z \in \mathfrak{n}^+$ close to* 0, *we have* $\varphi_\lambda(Z) = 1 + |Z|^2 + O(|Z|^3)$.

(2) *Let $H \in i\mathfrak{t}$ such that $\beta(H) := \max_{\alpha\in\Delta^+} \alpha(H) < 0$. Then, for $Z \in \mathfrak{n}^+$, we have*

$$\varphi_\lambda(\mathrm{Ad}(\exp H)Z) \leq 1 + e^{2\beta(H)}(\varphi_\lambda(Z) - 1). \tag{4.8}$$

(3) *For each $Z \in \mathfrak{n}^+$, we have $\varphi_\lambda(Z) \geq 1$ with equality if and only if $Z = 0$.*

(4) *There exist $r > 0$ and $C > 0$ such that $C|Z|^r \leq \varphi_\lambda(Z)$ if $|Z| \geq 1$.*

Proof. (1) Set $U := \log \kappa(\exp Z^* \exp Z)$. By (2) of Lemma 4.1, we have that

$$U = p_{\mathfrak{h}^c}([Z^*, Z]) + O(|Z|^3) = -\sum_{\alpha\in\Phi} \frac{1}{(\lambda, \alpha)} |z_\alpha|^2 H_\alpha + O(|Z|^3).$$

Then $d\chi_\lambda(U) = \lambda(U) = -|Z|^2 + O(|Z|^3)$. Hence,

$$\varphi_\lambda(Z) = \chi_\lambda^{-1}(\kappa(\exp Z^* \exp Z)) = e^{-\lambda(U)} = 1 + |Z|^2 + O(|Z|^3).$$

(2) We argue as in the proofs of [24], Lemma 88 and [30], Corollary 7.18. We also denote by π_λ the extension of π_λ to G^c and by π_λ^* the contragredient representation to π_λ. Recall that π_λ is a highest weight representation with highest weight λ and unit highest weight vector $f_0 \equiv 1$. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_l, \lambda$ the weights of π_λ. Note that (1) $\pi_\lambda(N^+)$ fixes f_0, (2) for each $h \in H^c$ we have $\pi_\lambda(h)f_0 = \chi_\lambda(h)f_0$ and (3) for $y \in N^-$, $\pi_\lambda(y)f_0$ is of the form $f_0 + \sum_{j=1}^l f_j$ with f_j in the weight space for the weight $\lambda_j < \lambda$.

Now, we obtain an expression for $\varphi_\lambda(Z)$ in terms of π_λ^* as follows. For $Z \in \mathfrak{n}^+$, write $\exp Z^* \exp Z = zhy$ with $z \in N^+$, $h \in H^c$ and $y \in N^-$. Then, we have

$$\begin{aligned}\|\pi_\lambda^*(\exp Z)f_0\|_\lambda^2 &= \langle f_0, \pi_\lambda(\exp Z^* \exp Z)^{-1} f_0\rangle_\lambda \\ &= \langle f_0, \chi_\lambda(h)^{-1}\pi_\lambda(y^{-1})f_0\rangle_\lambda\end{aligned}$$

and taking into account that the weight spaces for distinct weights are orthogonal, we get

$$\|\pi_\lambda^*(\exp Z)f_0\|_\lambda^2 = \overline{\chi_\lambda(h)}^{-1} = \overline{\chi_\lambda^{-1}(\kappa(\exp Z^* \exp Z))} = \varphi_\lambda(Z). \tag{4.9}$$

Note that the weights of π_λ^* are $-\lambda_1, \ldots, -\lambda_l, -\lambda$. Then, for $Z \in \mathfrak{n}^+$, we can write $\pi_\lambda^*(\exp Z)f_0 = f_0 + \sum_{j=1}^l f_j$ with f_j in the weight space for the weight

$-\lambda_j$ for $j=1,2,\ldots,l$. Consider $a=\exp H$ where $H\in i\mathfrak{t}$ satisfies $\beta(H)=\max_{\alpha\in\Delta^+}\alpha(H)<0$. Then we have

$$\pi^*_\lambda(a(\exp Z)a^{-1})f_0=f_0+\sum_{j=1}^{l}e^{(\lambda-\lambda_j)(H)}f_j.$$

Thus,

$$\varphi_\lambda(\mathrm{Ad}(a)Z)=\|\pi^*_\lambda(a(\exp Z)a^{-1})f_0\|^2_\lambda=1+\sum_{j=1}^{l}e^{2(\lambda-\lambda_j)(H)}\|f_j\|^2_\lambda. \tag{4.10}$$

But for each $j=1,2,\ldots,l$, $\lambda-\lambda_j$ is of the form $\sum_{\alpha\in\Delta^+}n_\alpha\alpha$ where the $n_\alpha\in\mathbb{Z}^+$ are not all equal to 0. This implies that $(\lambda-\lambda_j)(H)\leq\beta(H)$ and equality (4.10) gives

$$\varphi_\lambda(\mathrm{Ad}(a)Z)\leq 1+e^{2\beta(H)}\sum_{j=1}^{l}\|f_j\|^2_\lambda\leq 1+e^{2\beta(H)}\big(\varphi_\lambda(Z)-1\big).$$

(3) By 2 of Remark 3.1, we already have $\varphi_\lambda(Z)\geq 1$ for $Z\in\mathfrak{n}^+$. Now consider $Z=\sum_{\alpha\in\Phi}z_\alpha E_\alpha\in\mathfrak{n}^+\setminus(0)$ such that $\varphi_\lambda(Z)=1$. Let $H\in i\mathfrak{t}$ as above. Then by (4.8), we have $\varphi_\lambda(\mathrm{Ad}(\exp tH)Z)\leq 1$ for all $t>0$. Since $\mathrm{Ad}(\exp tH)Z=\sum_{\alpha\in\Phi}z_\alpha e^{t\alpha(H)}E_\alpha$ goes to 0 as $t\to+\infty$, we get a contradiction to 1.

(4) We set $c=\inf_{|Z|=1}\varphi_\lambda(Z)$. By 3, we have $c>1$. Now, let $Z\in\mathfrak{n}^+$ such that $|Z|\geq 1$. Fix $H\in i\mathfrak{t}$ as above. Then there exists $t>0$ such that $|\mathrm{Ad}(\exp tH)Z|^2=\sum_{\alpha\in\Phi}|z_\alpha|^2e^{2t\alpha(H)}=1$. Thus, we have $e^{2\gamma(H)t}|Z|^2\leq 1$ where $\gamma(H):=\min_{\alpha\in\Phi}\alpha(H)<0$. Applying (4.8), we get $(c-1)e^{-2\beta(H)t}\leq\varphi_\lambda(Z)$. Finally, we obtain $(c-1)|Z|^r\leq\varphi_\lambda(Z)$ where $r=2\beta(H)/\gamma(H)>0$. □

Remark 4.3. Let us consider the Iwasawa decomposition $G^c=GAN$ where $A=\exp(i\mathfrak{t})$ and $N=\exp(\sum_{\alpha\in\Delta^+}\mathfrak{g}_{-\alpha})$ (see [25], Chapter VI, Theorem 6.3). Let $Z\in\mathfrak{n}^+$. Write $\exp Z=kan$ with $k\in G$, $a\in A$ and $n\in N$. Since $k^*=k^{-1}$ and $a^*=a$ we have that $\exp Z^*\exp Z=n^*a^2n$ and then $\kappa(\exp Z^*\exp Z)=\kappa(n)^*a^2\kappa(n)$. This gives $\varphi_\lambda(Z)=\chi_\lambda^{-1}(a^2)=\chi_\lambda^{-2}(\exp H(\exp Z))$ where H is the so-called H-function corresponding to the Iwasawa decomposition $G^c=GAN$ [30]. So, we have related the function φ_λ to the H-function for which we find some estimates in the literature, see for instance [16], Lemma 6.1. This gives an alternative proof of Statement 4 of Proposition 4.2.

Proposition 4.4. *There exists a constant $C>0$ and a positive integer m_0 such that for each integer $m\geq m_0$ and each $Z\in\mathfrak{n}^+$, we have $\varphi_{m\lambda}(Z)\geq 1+C|Z|^2$.*

Proof. With the notation of the previous proposition, we fix $m_0 \in \mathbb{Z}$ such that $rm_0 \geq 2$. Then for $|Z| \geq 1$, we have

$$\varphi_{m_0\lambda}(Z) = \varphi_\lambda(Z)^{m_0} \geq c|Z|^{rm_0} \geq c|Z|^2.$$

From this and Proposition 4.2, we easily deduce that

$$C := \inf_{Z \in \mathfrak{n}^+} \frac{\varphi_{m_0\lambda}(Z) - 1}{|Z|^2} > 0.$$

Hence, for $m \geq m_0$ and $Z \in \mathfrak{n}^+$, we have $\varphi_{m\lambda}(Z) \geq \varphi_{m_0\lambda}(Z) \geq 1 + C|Z|^2$. □

A natural question is whether there exists a constant $C > 0$ such that $\varphi_\lambda(Z) \geq 1 + C|Z|^2$ for each $Z \in \mathfrak{n}^+$.

COROLLARY 4.5. *Let $\mathcal{P}$ denote the space of complex polynomials on $\mathfrak{n}^+$. Then $\mathcal{P} = \bigcup_{m \geq 1} \mathcal{H}_{m\lambda}$.*

Proof. Let $f \in \mathcal{P}$. By Proposition 4.4, we can choose m_0 sufficiently large so that $\chi_{m_0\lambda}(k(Z)) \leq (1 + C|Z|^2)^{-1}$ for each $Z \in \mathfrak{n}^+$. Then for each integer $N > 0$, we have $|f(Z)|^2 \chi_{Nm_0\lambda}(k(Z)) \leq (1 + C|Z|^2)^{-N}$ for each $Z \in \mathfrak{n}^+$. Hence, for N sufficiently large, we have $\|f\|_{Nm_0\lambda} < +\infty$ i.e. $f \in \mathcal{H}_{Nm_0\lambda}$. □

5. Contraction of G to the Heisenberg group

We retain the notation from the previous sections. In particular, we fix a real number $\gamma > 0$ as in Section 2 and an integral dominant weight λ as in Section 3. Let us introduce some additional notation. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an enumeration of Φ. For $k = 1, 2, \ldots, n$, we set $X_k = \frac{1}{2i}(E_{\alpha_k} + E_{-\alpha_k})$, $Y_k = \frac{1}{2}(E_{\alpha_k} - E_{-\alpha_k})$ and $\tilde{Z} = \frac{1}{2(\lambda,\lambda)} iH_\lambda$. Let $\mathfrak{g}_0$ be the subspace of $\mathfrak{g}$ generated by X_k, Y_k, $k = 1, 2, \ldots, n$ and $\tilde{Z}$. Let p_0 be the orthogonal projection of $\mathfrak{g}$ on the line generated by $\tilde{Z}$ with respect to the Killing form. For $r > 0$, we denote by C_r the linear isomorphism of $\mathfrak{g}$ defined by $C_r = r^2 p_0 + r(Id - p_0)$. We introduce the Lie bracket on $\mathfrak{g}$ defined by

$$[X, Y]_0 = \lim_{r \to 0} C_r^{-1}([C_r(X), C_r(Y)]).$$

LEMMA 5.1.

(1) *For X and Y in $\mathfrak{g}$, we have $[X, Y]_0 = p_0([(Id - p_0)(X), (Id - p_0)(Y)])$.*

(2) *The Lie algebra $(\mathfrak{g}_0, [\cdot, \cdot]_0)$ is a Heisenberg algebra. In the basis X_k, Y_k, $k = 1, 2, \ldots, n$, $\tilde{Z}$ of $\mathfrak{g}_0$, the only nontrivial brackets are $[X_k, Y_k]_0 = \tilde{Z}$.*

(3) *The Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_0)$ is isomorphic as a Lie algebra to the product of the Heisenberg algebra $(\mathfrak{g}_0, [\cdot, \cdot]_0)$ by an Abelian Lie algebra of dimension* $\dim \mathfrak{g} - \dim \mathfrak{g}_0$.

Proof. Direct computation. □

We denote also by C_r the restriction of C_r to $\mathfrak{g}_0$. Then we have

$$C_r\left(\sum_{k=1}^n a_kX_k+\sum_{k=1}^n b_kY_k+c\tilde{Z}\right)=r\left(\sum_{k=1}^n a_kX_k+\sum_{k=1}^n b_kY_k\right)+r^2c\tilde{Z}$$

and it is immediate from Lemma 5.1 that that $C_r:\mathfrak{g}_0\to\mathfrak{g}$ is a contraction of $\mathfrak{g}$ to $\mathfrak{g}_0$ in the sense of [11], Section 4. The corresponding contraction of G to the Heisenberg group G_0 with Lie algebra $\mathfrak{g}_0$ is the map $c_r:G_0\to G$ defined by $c_r(\exp_{G_0}(X))=\exp_G(C_r(X))$.

In the following proposition, we describe how the parametrizations of the orbits $\mathcal{O}_{m\lambda}$ and $\mathcal{O}_\gamma$ (see Sections 2 and 3) are related in the contraction process. In the rest of the paper, we identify $\mathfrak{n}^+$ with $\mathbb{C}^n$ by means of the linear isomorphism $(z_1,z_2,\ldots,z_n)\to Z=\sum_{k=1}^n z_kE_{\alpha_k}$.

Proposition 5.2. *Let $r(m)$ be a sequence of $]0,1]$ satisfying*

$$\lim_{m\to+\infty} mr(m)^2=2\gamma.$$

Then, for each $X\in\mathfrak{g}_0$ and each $Z\in\mathfrak{n}^+$, we have

$$\lim_{m\to+\infty}\beta\big(\psi_{m\lambda}\big(Z/\sqrt{2\gamma m}\big),C_{r(m)}(X)\big)=\langle\psi_\gamma(Z),X\rangle. \tag{5.1}$$

Proof. Assume that $X=X_k$ or $X=Y_k$. Then $C_{r(m)}(X)=r(m)X$. Taking equality (3.10) into account, we have

$$\begin{aligned}&\beta\big(\psi_{m\lambda}\big(Z/\sqrt{2\gamma m}\big),C_{r(m)}(X)\big)\\ &\quad=-imr(m)\beta\big(\mathrm{Ad}\big(\zeta\big(\exp\big(Z^*/\sqrt{2\gamma m}\big)\exp\big(Z/\sqrt{2\gamma m}\big)\big)\big)H_\lambda,\\ &\qquad\mathrm{Ad}\,\tilde{\theta}\big(\exp\big(Z/\sqrt{2\gamma m}\big)\big)^{-1}X\big).\end{aligned}$$

In order to study the behavior of this expression as $m\to+\infty$, we need a first-order Taylor formula for the function

$$F(t)=\beta(\mathrm{Ad}(\zeta(\exp(tZ^*)\exp(tZ)))H_\lambda,\mathrm{Ad}\,\tilde{\theta}(\exp(tZ))^{-1}X)$$

at $t=0$. We have $F(0)=0$ and $F'(0)=\beta([d\zeta(e)(Z^*+Z),H_\lambda],X)+\beta(H_\lambda,[Z^*,X])$. Since by [12], Proposition 4.1, we have

$$d\zeta(e)(Z^*+Z)=p_{\mathfrak{n}^+}(Z^*+Z)=Z,$$

we then obtain

$$F(t)=\beta(H_\lambda,[Z^*-Z,X])t+o(t). \tag{5.2}$$

If $X=X_k$, then $\beta(H_\lambda,[Z^*-Z,X])=-\frac{1}{2i}(z_k+\overline{z}_k)$ and (5.2) implies

$$\lim_{m\to+\infty}\beta\big(\psi_{m\lambda}\big(Z/\sqrt{2\gamma m}\big),C_{r(m)}(X)\big)=\frac{1}{2}(z_k+\overline{z}_k)=\mathrm{Re}\,z_k.$$

Similarly, if $X=Y_k$ then $\beta(H_\lambda,[Z^*-Z,X])=\frac{1}{2}(z_k-\overline{z}_k)$ and

$$\lim_{m\to+\infty}\beta\big(\psi_{m\lambda}\big(Z/\sqrt{2\gamma m}\big),C_{r(m)}(X)\big)=-\frac{i}{2}(z_k-\overline{z}_k)=\mathrm{Im}\,z_k.$$

Finally, if $X=\tilde{Z}$ then we immediately obtain

$$\lim_{m\to+\infty}\beta\big(\psi_{m\lambda}\big(Z/\sqrt{2\gamma m}\big),C_{r(m)}(X)\big)=\gamma.$$

Comparing with (2.7), we obtain (5.1). □

6. Contraction of operators

In order to simplify the notation, we write π_m, $\mathcal{H}_m$, $\langle\cdot,\cdot\rangle_m$, χ_m, c_m, S_m instead of $\pi_{m\lambda}$, $\mathcal{H}_{m\lambda}$, $\langle\cdot,\cdot\rangle_{m\lambda}$, $\chi_{m\lambda}$, $c_{m\lambda}$, $S_{m\lambda}$, respectively. Moreover, we fix the Lebesgue measure $d\mu_L(Z)$ on $\mathfrak{n}^+$ as follows. Writing $Z=\sum_{k=1}^n z_kE_{\alpha_k}$ and, for $k=1,2,\dots,n$, $z_k=x_k+iy_k$, $x_k,y_k\in\mathbb{R}$, we take $d\mu_L(Z)=dx_1\,dy_1\cdots dx_n\,dy_n$. Recall also that we have set $k(Z):=\kappa(\exp Z^*\exp Z)$ for $Z\in\mathfrak{n}^+$.

Lemma 6.1.

(1) *For $Z\in\mathfrak{n}^+$, we have*

$$\lim_{m\to+\infty}\chi_m\big(k\big(Z/\sqrt{2\gamma m}\big)\big)=e^{-|Z|^2/2\gamma}. \tag{6.1}$$

(2) *For $Z,W\in\mathfrak{n}^+$, we have*

$$\lim_{m\to+\infty}\langle e^m_{W/\sqrt{2\gamma m}},e^m_{Z/\sqrt{2\gamma m}}\rangle_m=\langle e^\gamma_W,e^\gamma_Z\rangle_\gamma. \tag{6.2}$$

(3) *We have*

$$\lim_{m\to+\infty}c_m^{-1}(2\gamma m)^n=\int_{\mathfrak{n}^+}e^{-|Z|^2/2\gamma}\,d\mu_L(Z)=(2\pi\gamma)^n. \tag{6.3}$$

Proof. (1) Fix $Z\in\mathfrak{n}^+$. Note that $\chi_m=\chi_\lambda^m$. By 1 of Proposition 4.2, we have

$$\begin{aligned}\log\chi_m\big(k\big(Z/\sqrt{2\gamma m}\big)\big)&=-m\log\chi_\lambda^{-1}\big(k\big(Z/\sqrt{2\gamma m}\big)\big)\\&=-m\log\varphi_\lambda\big(Z/\sqrt{2\gamma m}\big)\\&=-m\big(|Z|/\sqrt{2\gamma m}\big)^2+O\big(1/\sqrt{m}\big)\\&=-(1/2\gamma)|Z|^2+O\big(1/\sqrt{m}\big).\end{aligned}$$

Statement 1 then follows.

(2) See the proof of Proposition 7.1. Note that the present lemma is not used in the proof of Proposition 7.1.

(3) Recall that

$$c_m^{-1}=\int_{\mathfrak{n}^+}(\chi_m\cdot\chi_\Lambda)(k(Z))\,d\mu_L(Z).$$

Changing variables $Z\to Z/\sqrt{2\gamma m}$ in this integral, we get

$$c_m^{-1}(2\gamma m)^n=\int_{\mathfrak{n}^+}(\chi_m\cdot\chi_\Lambda)\big(k\big(Z/\sqrt{2\gamma m}\big)\big)d\mu_L(Z).$$

By 1, the integrand $I_m(Z) := (\chi_m . \chi_\Lambda)(k(Z/\sqrt{2\gamma m}))$ satisfies

$$\lim_{m\to+\infty} I_m(Z) = e^{-|Z|^2/2\gamma}.$$

In order to obtain 3, it suffices to verify that the Lebesgue dominated convergence theorem can be applied. This can be done as follows. By Proposition 4.4, there exist a constant $C>0$ and an integer $m_0>0$ such that $\chi_{m_0}^{-1}(k(Z)) \geq 1 + C|Z|^2$ for each $Z \in \mathfrak{n}^+$. For $m \in \mathbb{Z}^+$, we set $N(m) = [\frac{m}{m_0}]$. Fix $N_0 \in \mathbb{Z}^+$. There exists $m_1 \in \mathbb{Z}^+$ such that $N(m) \geq N_0$ whenever $m \geq m_1$. Noting that $\chi_\Lambda . \chi_m \leq \chi_m \leq \chi_{m_0 N(m)} \leq \chi_{m_0}^{N(m)}$, for $m \geq m_1$ we have

$$\begin{aligned}
(\chi_\Lambda . \chi_m)\big(k\big(Z/\sqrt{2\gamma m}\big)\big) &\leq \left(1 + N(m)\frac{\chi_{m_0}^{-1}(k(Z/\sqrt{2\gamma m})) - 1}{N(m)}\right)^{-N(m)}\\
&\leq \left(1 + \frac{N(m)(\chi_{m_0}^{-1}(k(Z/\sqrt{2\gamma m})) - 1)}{N_0}\right)^{-N_0}\\
&\leq \left(1 + C\frac{N(m)}{N_0}\frac{|Z|^2}{2\gamma m}\right)^{-N_0}\\
&\leq \left(1 + \frac{C}{4\gamma m_0 N_0}|Z|^2\right)^{-N_0}.
\end{aligned}$$

for each $Z \in \mathfrak{n}^+$. Here we have used the fact that if $N(m) \geq N_0$ then

$$\left(1 + \frac{x}{N_0}\right)^{N_0} \leq \left(1 + \frac{x}{N(m)}\right)^{N(m)}$$

for each $x \in \mathbb{R}^+$. Clearly, the function $Z \to (1 + (C/4\gamma m_0 N_0)|Z|^2)^{-N_0}$ is integrable on $\mathfrak{n}^+$ for $N_0 \geq 1$. This ends the proof. □

PROPOSITION 6.2. *For each integer $m \geq 1$ let A_m be an operator of $\mathcal{H}_m$. Let A be a bounded operator of $\mathcal{H}_\gamma$. Assume that:*

(i) *The sequence $\|A_m\|_{op}$ is bounded;*
(ii) *we have $\lim_{m\to+\infty} S_m(A_m)(Z/\sqrt{2\gamma m}, W/\sqrt{2\gamma m}) = S_\gamma(A)(Z,W)$.*

Then, for any complex polynomials P and Q, we have

$$\lim_{m\to+\infty} \big\langle A_m P\big(\sqrt{2\gamma m}\cdot\big), Q\big(\sqrt{2\gamma m}\cdot\big)\big\rangle_m = \langle AP, Q\rangle_\gamma. \tag{6.4}$$

Proof. First, note that we can assume that P and Q are homogeneous without loss of generality. By Corollary 4.5, for m sufficiently large we have $P \in \mathcal{H}_m$ and $Q \in \mathcal{H}_m$ and, consequently, $P(\sqrt{2\gamma m}\cdot) \in \mathcal{H}_m$ and $Q(\sqrt{2\gamma m}\cdot) \in \mathcal{H}_m$.

By using (3.6), we express $\langle A_m P(\sqrt{2\gamma m}\cdot), Q(\sqrt{2\gamma m}\cdot)\rangle_m$ as an integral. Changing variables $(Z,W) \to (Z/\sqrt{2\gamma m}, W/\sqrt{2\gamma m})$ in this integral, we get

$$\begin{aligned}
&\big\langle A_m P\big(\sqrt{2\gamma m}\cdot\big), Q\big(\sqrt{2\gamma m}\cdot\big)\big\rangle_m \\
&\quad = c_m^2 (2\gamma m)^{-2n} \int_{\mathfrak{n}^+\times\mathfrak{n}^+} I_m(Z,W)\, d\mu_L(Z)\, d\mu_L(W),
\end{aligned} \tag{6.5}$$

where the integrand $I_m(Z,W)$ is given by

$$\begin{aligned} I_m(Z,W) = P(W)\overline{Q(Z)}S_m(A_m)\big(Z/\sqrt{2\gamma m}, W/\sqrt{2\gamma m}\big) \\ \times \langle e^m_{W/\sqrt{2\gamma m}}, e^m_{Z/\sqrt{2\gamma m}}\rangle_m (\chi_m \cdot \chi_\lambda)\big(k\big(Z/\sqrt{2\gamma m}\big)\big) \\ \times (\chi_m \cdot \chi_\lambda)\big(k\big(W/\sqrt{2\gamma m}\big)\big). \end{aligned}$$

Note that, by 3 of Lemma 6.1, we have $\lim_{m\to+\infty} c_m^2 (2\gamma m)^{-2n} = (2\pi\gamma)^{-2n}$. Moreover, by 1 of Lemma 6.1, we have

$$\begin{aligned} &\lim_{m\to+\infty} I_m(Z,W) \\ &\quad = P(W)\overline{Q(Z)}S_\gamma(A)(Z,W)\langle e^\gamma_W, e^\gamma_Z\rangle_\gamma e^{-|Z|^2/2\gamma} e^{-|W|^2/2\gamma}. \end{aligned}$$

Now, as in the proof of Lemma 6.1, we want to verify that the Lebesgue dominated convergence theorem can be applied. To this aim, we write by using the Cauchy–Schwarz inequality

$$|\langle A_m e^m_{W/\sqrt{2\gamma m}}, e^m_{Z/\sqrt{2\gamma m}}\rangle_m| \leq \|A_m\|_{op} \|e^m_{Z/\sqrt{2\gamma m}}\|_m \|e^m_{W/\sqrt{2\gamma m}}\|_m.$$

This implies that

$$|I_m(Z,W)| \leq |P(W)||Q(Z)|\|A_m\|_{op} \chi_\lambda^{m/2}\big(k\big(Z/\sqrt{2\gamma m}\big)\big)\chi_\lambda^{m/2}\big(k\big(W/\sqrt{2\gamma m}\big)\big).$$

As seen in the proof of Lemma 6.1, for each integer $N_0 \geq 0$ there exist a constant $C_1 > 0$ and an integer m_1 such that for each $m \geq m_1$ we have

$$\chi_\lambda^{m/2}\big(k\big(Z/\sqrt{2\gamma m}\big)\big) \leq (1 + C_1|Z|^2)^{-N_0/2}.$$

Hence, we obtain

$$|I_m(Z,W)| \leq C_2 |P(W)||Q(Z)|(1+C_1|Z|^2)^{-N_0/2}(1+C_1|W|^2)^{-N_0/2}, \tag{6.6}$$

where C_2 is a constant. Now, we can choose N_0 sufficiently large so that the right-hand side of (6.6) is an integrable function on $\mathfrak{n}^+ \times \mathfrak{n}^+$. This ends the proof. □

Corollary 6.3. *Let P and Q two homogeneous complex polynomials different to* 0. *Then we have*

$$\lim_{m\to+\infty} \langle A_m(\|P\|_m^{-1}P), \|Q\|_m^{-1}Q\rangle_m = \langle A(\|P\|_\gamma^{-1}P), \|Q\|_\gamma^{-1}Q\rangle_\gamma. \tag{6.7}$$

Proof. Since P and Q are homogeneous, one has

$$\begin{aligned} &\langle A_m(\|P\|_m^{-1}P), \|Q\|_m^{-1}Q\rangle_m \\ &\quad = \big\|P\big(\sqrt{2\gamma m}\cdot\big)\big\|_m^{-1} \big\|Q\big(\sqrt{2\gamma m}\cdot\big)\big\|_m^{-1} \big\langle A_m P\big(\sqrt{2\gamma m}\cdot\big), Q\big(\sqrt{2\gamma m}\cdot\big)\big\rangle_m. \end{aligned}$$

But applying Proposition 6.2 to the particular case $A_m = Id$, $A = Id$, we get

$$\lim_{m\to+\infty} \langle P(\sqrt{2\gamma m}\cdot), Q(\sqrt{2\gamma m}\cdot)\rangle_m = \langle P, Q\rangle_\gamma.$$

Also, we have $\lim_{m\to+\infty} \|P(\sqrt{2\gamma m}\cdot)\|_m = \|P\|_\gamma$. The desired result then follows from Proposition 6.2. □

For each integer $m>0$, we denote by B_m the linear isomorphism of $\mathcal{P}$ defined by $(B_mP)(Z)=P(Z/\sqrt{2\gamma m})$.

PROPOSITION 6.4. *Under the assumptions of Proposition 6.2, we have*

$$\lim_{m\to+\infty}(B_mA_mB_m^{-1})P(Z)=AP(Z) \tag{6.8}$$

for each complex polynomial P and each $Z\in\mathfrak{n}^+$.

Proof. For each $Z\in\mathfrak{n}^+$, we express

$$(B_mA_mB_m^{-1})P(Z)=A_m(B_m^{-1}P)\big(Z/\sqrt{2\gamma m}\big)$$

as an integral by using a formula for A_m analogous to (2.3), and we proceed as in the proof of Proposition 6.2. $\square$

7. Contraction of representations

We retain the notation from the previous sections. In this section, we use the results of Section 6 in order to establish our main results on the contraction of the sequence π_m to the representation ρ_γ of G_0. As in Section 6, we consider a sequence $r(m)$ such that $\lim_{m\to+\infty}mr(m)^2=2\gamma$.

PROPOSITION 7.1. *For $g_0\in G_0$ and $Z,W\in\mathfrak{n}^+$, we have*

$$\lim_{m\to+\infty}S_m(\pi_m(c_{r(m)}(g_0)))\big(Z/\sqrt{2\gamma m},W/\sqrt{2\gamma m}\big)=S_\gamma(\rho_\gamma(g_0))(Z,W). \tag{7.1}$$

Proof. Let $Z,W\in\mathfrak{n}^+$ and $g_0=\exp(\sum_{k=1}^n a_kX_k+\sum_{k=1}^n b_kY_k+c\tilde{Z})\in G_0$. Let

$$\begin{aligned}U_m:=\log\kappa\Bigg(&\exp\big(Z^*/\sqrt{2\gamma m}\big)\\ &\times\exp\Bigg(-r(m)\Bigg(\sum_{k=1}^n a_kX_k+\sum_{k=1}^n b_kY_k\Bigg)-r(m)^2c\tilde{Z}\Bigg)\exp\big(W/\sqrt{2\gamma m}\big)\Bigg).\end{aligned}$$

We have to study the behavior of the sequence $u_m:=\chi_\lambda^{-m}(\exp U_m)$ as $m\to+\infty$. We proceed as in the proof of (1) of Lemma 6.1. From Lemma 4.1, we deduce that

$$\begin{aligned}U_m=&-r(m)^2c\tilde{Z}-\frac{r(m)^2}{2\gamma}p_{\mathfrak{h}^c}([Z^*,\Sigma])-\frac{r(m)^2}{2\gamma}p_{\mathfrak{h}^c}([\Sigma,W])\\ &+\frac{r(m)^2}{4\gamma^2}p_{\mathfrak{h}^c}([Z^*,W])-\frac{1}{2}r(m)^2p_{\mathfrak{h}^c}\big([p_{\mathfrak{n}^+}(\Sigma),p_{\mathfrak{n}^-}(\Sigma)]\big)+O(r(m)^3),\end{aligned}$$

where

$$\Sigma:=\sum_{k=1}^n(a_kX_k+b_kY_k)=\sum_{k=1}^n\Bigg(\frac{1}{2i}(a_k+ib_k)E_{\alpha_k}+\frac{1}{2i}(a_k-ib_k)E_{-\alpha_k}\Bigg).$$

Noting that we have

$$p_{\mathfrak{h}^c}([Z^*,\Sigma]) = -\sum_{k=1}^{n} \frac{1}{2i}\overline{z}_k(a_k+ib_k)\frac{1}{(\lambda,\alpha_k)}H_{\alpha_k} \quad \mathrm{mod.}\,\mathfrak{n}_1^+ + \mathfrak{n}_1^-,$$

$$p_{\mathfrak{h}^c}([\Sigma,W]) = -\sum_{k=1}^{n} \frac{1}{2i}w_k(a_k-ib_k)\frac{1}{(\lambda,\alpha_k)}H_{\alpha_k} \quad \mathrm{mod.}\,\mathfrak{n}_1^+ + \mathfrak{n}_1^-,$$

$$p_{\mathfrak{h}^c}([Z^*,W]) = -\sum_{k=1}^{n} \overline{z}_k w_k\frac{1}{(\lambda,\alpha_k)}H_{\alpha_k} \quad \mathrm{mod.}\,\mathfrak{n}_1^+ + \mathfrak{n}_1^-$$

we find

$$\begin{aligned}\lim_{m\to+\infty}\log u_m &= \lim_{m\to+\infty}(-m\lambda(U_m)) \\ &= i\gamma c + \sum_{k=1}^{n}\frac{1}{2}\overline{z}_k(ia_k-b_k) \\ &\quad + \sum_{k=1}^{n}\frac{1}{2}w_k(ia_k+b_k) + \frac{1}{2\gamma}\sum_{k=1}^{n}\overline{z}_k w_k - \frac{\gamma}{4}\sum_{k=1}^{n}(a_k^2+b_k^2).\end{aligned}$$

The result then follows from equality (2.5). □

Applying the results of Section 6 to the operators $A_m = \pi_m(c_{r(m)}(g_0))$ and $A = \rho_\gamma(g_0)$ for $g_0\in G_0$, we obtain immediately the following proposition.

PROPOSITION 7.2.

(1) *For $P,Q\in\mathcal{P}$ and $g_0\in G_0$, we have*

$$\lim_{m\to+\infty}\big\langle \pi_m\big(c_{r(m)}(g_0)\big)P\big(\sqrt{2\gamma m}\cdot\big), Q\big(\sqrt{2\gamma m}\cdot\big)\big\rangle_m = \langle\rho_\gamma(g_0)P,Q\rangle_\gamma. \tag{7.2}$$

(2) *For P,Q homogeneous polynomials different to 0 and $g_0\in G_0$, we have*

$$\begin{aligned}\lim_{m\to+\infty}&\big\langle \pi_m\big(c_{r(m)}(g_0)\big)(\|P\|_m^{-1}P), \|Q\|_m^{-1}Q\big\rangle_m \\ &= \langle\rho_\gamma(g_0)(\|P\|_\gamma^{-1}P), \|Q\|_\gamma^{-1}Q\rangle_\gamma.\end{aligned} \tag{7.3}$$

(3) *For $P\in\mathcal{P}$, $Z\in\mathfrak{n}^+$ and $g_0\in G_0$, we have*

$$\lim_{m\to+\infty}\big(B_m\pi_m\big(c_{r(m)}(g_0)\big)B_m^{-1}\big)P(Z) = \rho_\gamma(g_0)P(Z). \tag{7.4}$$

8. Contraction of derived representations

In this section, we establish contraction results for the derived representations $d\pi_m$ and $d\pi_\gamma$ analogous to those of Section 7.

PROPOSITION 8.1. *For $X\in\mathfrak{g}_0$ and $Z,W\in\mathfrak{n}^+$, we have*

$$\begin{aligned}\lim_{m\to+\infty}&S_m\big(d\pi_m\big(C_{r(m)}(X)\big)\big)\big(Z/\sqrt{2\gamma m}, W/\sqrt{2\gamma m}\big) \\ &= S_\gamma(d\rho_\gamma(X))(Z,W).\end{aligned} \tag{8.1}$$

Proof. Taking (2.7) and (3.9) into account, the result follows from Proposition 5.2. ☐

PROPOSITION 8.2.

(1) *For $P, Q \in \mathcal{P}$ and $X \in \mathfrak{g}_0$, we have*

$$\lim_{m\to+\infty} \langle d\pi_m\big(C_{r(m)}(X)\big)P\big(\sqrt{2\gamma m}\cdot\big), Q\big(\sqrt{2\gamma m}\cdot\big)\rangle_m = \langle d\rho_\gamma(X)P, Q\rangle_\gamma. \tag{8.2}$$

(2) *For $P \in \mathcal{P}$, $Z \in \mathfrak{n}^+$ and $X \in \mathfrak{g}_0$, we have*

$$\lim_{m\to+\infty} \big(B_m d\pi_m\big(C_{r(m)}(X)\big)B_m^{-1}\big)P(Z) = d\rho_\gamma(X)P(Z). \tag{8.3}$$

Proof. (1) The proof is similar to that of Proposition 6.2. However, here we do not know an a priori estimate for $\|d\pi_m(C_{r(m)}(X))\|_{op}$. In order to dominate the integrand

$$\begin{aligned} J_m(Z,W) := P(W)\overline{Q(Z)}&\langle d\pi_m\big(C_{r(m)}(X)\big)e^m_{W/\sqrt{2\gamma m}}, e^m_{Z/\sqrt{2\gamma m}}\rangle_m \\ &\times(\chi_m\cdot\chi_\Lambda)\big(k\big(Z/\sqrt{2\gamma m}\big)\big)(\chi_m\cdot\chi_\Lambda)\big(k\big(W/\sqrt{2\gamma m}\big)\big), \end{aligned}$$

we can find an estimate for $|\langle d\pi_m(C_{r(m)}(X))e^m_{W/\sqrt{2\gamma m}}, e^m_{Z/\sqrt{2\gamma m}}\rangle_m|$ as follows. For each $X \in \mathfrak{g}$, we consider the polynomial

$$q_X(Z,W) = \sum_{jl} q_{jl}(X)Z^jW^l$$

defined by $q_X(Z^*,W) = \langle d\pi_\lambda(X)e^\lambda_Z, e^\lambda_W\rangle_\lambda$. Then by (3.5) and (3.8), we have

$$\langle d\pi_m(X)e^m_Z, e^m_W\rangle_m = mq_X(Z^*,W)\chi_\lambda^{-m+1}(\kappa(\exp Z^*\exp W)). \tag{8.4}$$

Assume now that $X = X_k$ or $X = Y_k$. Then $C_{r(m)}(X) = r(m)X$ and the constant term of $q_{C_{r(m)}(X)} = r(m)q_X$ is equal to 0 since $q_X(0,0) = \langle d\pi_\lambda(X)e^\lambda_0, e^\lambda_0\rangle_\lambda = \lambda(p_{\mathfrak{h}^c}(X)) = 0$. Thus, we have

$$\begin{aligned} &\big|mq_{C_{r(m)}(X)}\big(Z/\sqrt{2\gamma m}, W/\sqrt{2\gamma m}\big)\big| \\ &\quad\le \frac{mr(m)}{\sqrt{2\gamma m}} \sum_{|j|+|l|\ge 1} |q_{jl}(X)||Z^j||W^l|. \end{aligned} \tag{8.5}$$

On the other hand, we have

$$\begin{aligned} &\big|\chi_\lambda^{-m+1}\big(\kappa\big(\exp\big(Z^*/\sqrt{2\gamma m}\big)\exp\big(W/\sqrt{2\gamma m}\big)\big)\big)\big| \\ &\quad= |\langle e^{m-1}_{Z/\sqrt{2\gamma m}}, e^{m-1}_{W/\sqrt{2\gamma m}}\rangle_{m-1}| \\ &\quad\le \|e^{m-1}_{Z/\sqrt{2\gamma m}}\|_{m-1}\|e^{m-1}_{W/\sqrt{2\gamma m}}\|_{m-1} \\ &\quad\le \chi_\lambda^{-\frac{m-1}{2}}\big(k\big(Z/\sqrt{2\gamma m}\big)\big)\chi_\lambda^{-\frac{m-1}{2}}\big(k\big(W/\sqrt{2\gamma m}\big)\big). \end{aligned}$$

Combining this inequality with (8.5), we get

$$|J_m(Z,W)| \leq |P(W)Q(Z)|\Bigg(\sum_{|j|+|l|\geq 1} |q_{jl}(X)||Z^j||W^l|\Bigg) \\ \times \chi_\lambda^{\frac{m+1}{2}}\big(k\big(Z/\sqrt{2\gamma m}\big)\big)\chi_\lambda^{\frac{m+1}{2}}\big(k\big(W/\sqrt{2\gamma m}\big)\big)$$

and we conclude as in the proof of Proposition 6.2.

Assume now that $X=\tilde{Z}$. Then $C_{r(m)}(X)=r(m)^2X$ and we have

$$\big|mq_{C_{r(m)}(X)}\big(Z/\sqrt{2\gamma m}, W/\sqrt{2\gamma m}\big)\big| \leq mr(m)^2\sum_{jl}|q_{jl}(X)||Z^j||W^l| \\ \leq 3\gamma\sum_{jl}|q_{jl}(X)||Z^j||W^l|$$

for m sufficiently large. From this, we deduce a bound for $|J_m(Z,W)|$ and we conclude as in the proof of Proposition 6.2.

(2) The proof is similar to that of 1. $\square$

9. A particular case

In this section, we consider the case when the action of H on $\mathcal{P}$ is multiplicity free. In this case, we shall establish a more precise contraction result.

We denote by σ the action of H on $\mathcal{P}$ given by $\sigma(h)P(Z)=P(h^{-1}\cdot Z)$ for each $h\in H$, $P\in\mathcal{P}$ and $Z\in\mathfrak{n}^+$. Noting that $h\cdot Z=\mathrm{Ad}(h)Z$ for each $h\in H$ and $Z\in\mathfrak{n}^+$, we easily verify that the Euclidean norm on $\mathfrak{n}^+$ introduced in Section 4 is H-invariant and then the inner product $\langle\cdot,\cdot\rangle_\gamma$ on $\mathcal{P}$ is $\sigma(H)$-invariant.

Now, we assume that σ is a multiplicity free action. That is, each irreducible (unitary) representation of H appears at most once in σ. Let $\mathcal{P}=\sum_{j\in\mathbb{Z}^+}\mathcal{P}_j$ be the decomposition of $\mathcal{P}$ into H-irreducible components. Note that (1) if $j\neq j'$ then $\langle\mathcal{P}_j,\mathcal{P}_{j'}\rangle_\gamma=(0)$ and (2) since for each integer $m\geq 0$ the space of homogeneous polynomials on $\mathfrak{n}^+$ of degree m is $\sigma(H)$-invariant, each $\mathcal{P}_j$ is a space of homogeneous polynomials.

Let $m\geq 1$ be an integer. Since we have $\pi_m(h)P(Z)=\chi_m(h)P(h^{-1}\cdot Z)$ for each $h\in H$, $P\in\mathcal{P}$ and $Z\in\mathfrak{n}^+$, the space $\mathcal{H}_m$ is $\sigma(H)$-invariant. Then there exists a finite subset J_m of $\mathbb{Z}^+$ such that $\mathcal{H}_m=\sum_{j\in J_m}\mathcal{P}_j$. For each $j\neq j'$ in J_m, $\mathcal{P}_j$ and $\mathcal{P}_{j'}$ are inequivalent irreducible submodules of $\mathcal{P}$ and then we also have $\langle\mathcal{P}_j,\mathcal{P}_{j'}\rangle_m=(0)$. Moreover, for each $j\in J_m$ the restrictions of $\langle\cdot,\cdot\rangle_\gamma$ and of $\langle\cdot,\cdot\rangle_m$ to the irreducible submodule $\mathcal{P}_j$ of $\mathcal{P}$ are $\sigma(H)$-invariant and then there exists a constant $c_{m,j}>0$ such that $\langle P,Q\rangle_\gamma=c_{m,j}\langle P,Q\rangle_m$ for all $P,Q\in\mathcal{P}_j$.

For each integer $m\geq 1$, we denote by D_m the unitary operator from $(\mathcal{H}_m,\langle\cdot,\cdot\rangle_\gamma)$ onto $(\mathcal{H}_m,\langle\cdot,\cdot\rangle_m)$ defined by $D_mP=\sqrt{c_{m,j}}P$ if $P\in\mathcal{P}_j\subset\mathcal{H}_m$, $j\in J_m$. For each j, we fix an orthonormal basis $(f^\gamma_{j,l})_{1\leq l\leq\dim\mathcal{P}_j}$ for $(\mathcal{P}_j,\langle\cdot,\cdot\rangle_\gamma)$.

If $j \in J_m$ for some m then $f^m_{j,l} := D_m f^\gamma_{j,l}$, $1 \le l \le \dim \mathcal{P}_j$ is an orthonormal basis for $(\mathcal{P}_j, \langle \cdot,\cdot\rangle_m)$. Hence $f^m_{j,l}, j \in J_m, 1 \le l \le \dim \mathcal{P}_j$ is an orthonormal basis for $\mathcal{H}_m$.

PROPOSITION 9.1. *Let* (A_m) *and* A *as in Proposition* 6.2.

(1) *For each* $j, j' \in \mathbb{Z}^+$ *and each* l, l' *such that* $1 \le l \le \dim \mathcal{P}_j$ *and* $1 \le l' \le \dim \mathcal{P}_{j'}$, *we have*

$$\lim_{m\to+\infty} \langle A_m f^m_{j,l}, f^m_{j',l'}\rangle_m = \langle A f^\gamma_{j,l}, f^\gamma_{j',l'}\rangle_\gamma.$$

(2) *If moreover* A_m $(m \ge 1)$ *and* A *are unitary operators then we have*

$$\lim_{m\to+\infty} \|(D_m^{-1} A_m D_m)P - AP\|_\gamma = 0$$

for each $P \in \mathcal{P}$. *We also have*

$$\lim_{m\to+\infty} (D_m^{-1} A_m D_m)P(Z) = AP(Z)$$

for each $P \in \mathcal{P}$ *and* $Z \in \mathfrak{n}^+$.

Proof. First note that the expressions that are taken to the limit in 1 and 2 are well-defined for m sufficiently large, according to Corollary 4.5.

(1) Since the polynomials $f^\gamma_{j,l}$ are homogeneous, the result is a consequence of Corollary 4.5.

(2) By linearity, we can assume that $P = f^\gamma_{j,l}$. For each (j', l'), we have

$$\langle (D_m^{-1} A_m D_m) f^\gamma_{j,l}, f^\gamma_{j',l'}\rangle_\gamma = \langle A_m f^m_{j,l}, f^m_{j',l'}\rangle_m \to \langle A f^\gamma_{j,l}, f^\gamma_{j',l'}\rangle_\gamma$$

as $m \to +\infty$. This implies that $((D_m^{-1} A_m D_m) f^\gamma_{j,l})_m$ converges weakly to $A f^\gamma_{j,l}$ in $\mathcal{H}_\gamma$. Since $\|(D_m^{-1} A_m D_m) f^\gamma_{j,l}\|_\gamma = 1 = \|A f^\gamma_{j,l}\|_\gamma$, we can conclude that $((D_m^{-1} A_m D_m) f^\gamma_{j,l})_m$ converges strongly to $A f^\gamma_{j,l}$ in $\mathcal{H}_\gamma$. This also implies that $((D_m^{-1} A_m D_m) f^\gamma_{j,l})_m$ converges pointwise to $A f^\gamma_{j,l}$ on $\mathfrak{n}^+$. □

Applying Proposition 9.1 to the operators $A_m = \pi_m(c_{r(m)}(g_0))$ and $A = \rho_\gamma(g_0)$ for $g_0 \in G_0$, we immediately obtain the following result.

COROLLARY 9.2.

(1) *For each* $g_0 \in G_0$, $j, j' \in \mathbb{Z}^+$, $1 \le l \le \dim \mathcal{P}_j$ *and* $1 \le l' \le \dim \mathcal{P}_{j'}$ *we have*

$$\lim_{m\to+\infty} \big\langle \pi_m\big(c_{r(m)}(g_0)\big) f^m_{j,l}, f^m_{j',l'}\big\rangle_m = \langle \rho_\gamma(g_0) f^\gamma_{j,l}, f^\gamma_{j',l'}\rangle_\gamma.$$

(2) *For each* $g_0 \in G_0$ *and* $P \in \mathcal{P}$, *we have*

$$\lim_{m\to+\infty} \big\| \big(D_m^{-1} \pi_m\big(c_{r(m)}(g_0)\big) D_m\big)P - \rho_\gamma(g_0)P\big\|_\gamma = 0$$

and, for each $g_0 \in G_0$, $P \in \mathcal{P}$ *and* $Z \in \mathfrak{n}^+$, *we have*

$$\lim_{m\to+\infty} \big(D_m^{-1} \pi_m\big(c_{r(m)}(g_0)\big) D_m\big)P(Z) = \rho_\gamma(g_0)P(Z).$$

EXAMPLE. We take $G = SU(p+q)$. Then $G^c = SL(p+q,\mathbb{C})$. We use the notation of [35]. Let T be the maximal torus of G consisting of the matrices

$$\mathrm{Diag}(e^{ia_1}, e^{ia_2}, \dots, e^{ia_{p+q}}), \quad a_1, a_2, \dots, a_{p+q} \in \mathbb{R}, \prod_{k=1}^{p+q} e^{ia_k} = 1.$$

The complexification T^c of T has Lie algebra

$$\mathfrak{t}^c = \left\{ X = \mathrm{Diag}(x_1, x_2, \dots, x_{p+q}) : x_k \in \mathbb{C}, \quad \sum_{k=1}^{p+q} x_k = 0 \right\}.$$

The set Δ of roots of $\mathfrak{t}^c$ on $\mathfrak{g}^c$ is $\lambda_i - \lambda_j$ for $1 \le i \ne j \le p+q$ where $\lambda_i(X) = x_i$ for $X \in \mathfrak{t}^c$ as above. We take the set of positive roots Δ^+ to be $\lambda_i - \lambda_j$ for $1 \le i < j \le p+q$.

Also, we take $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_p$. The positive root $\lambda_i - \lambda_j$ is then orthogonal to λ if and only if $1 \le i < j \le p$ or $p+1 \le i < j \le p+q$. Consequently, H consists of the matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad A \in U(p), D \in U(q), \ \mathrm{Det}(A) \cdot \mathrm{Det}(D) = 1,$$

that is, we have $H = S(U(p) \times U(q))$ and the character χ_λ is given by

$$\chi_\lambda \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \mathrm{Det}(A).$$

Moreover, we have

$$N^+ = \left\{ \begin{pmatrix} I_p & Z \\ 0 & I_q \end{pmatrix} : Z \in M_{pq}(\mathbb{C}) \right\}, \qquad N^- = \left\{ \begin{pmatrix} I_p & 0 \\ Y & I_q \end{pmatrix} : Y \in M_{qp}(\mathbb{C}) \right\}.$$

In particular, we can identify $\mathfrak{n}^+$ to $M_{pq}(\mathbb{C})$. We also have

$$H^c = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \ A \in M_p(\mathbb{C}), D \in M_q(\mathbb{C}), \ \mathrm{Det}(A) \cdot \mathrm{Det}(D) = 1 \right\}.$$

We easily compute the $N^+H^cN^-$-decomposition of a matrix $g \in G^c$ (see [12]). From this, we deduce that the action of G^c on $\mathfrak{n}^+$ is given by

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

In this case, the Hilbert space $\mathcal{H}_{m\lambda}$ associated with $\chi_{m\lambda}$ consists of complex polynomials f such that

$$\|f\|_{m\lambda}^2 = \int_{\mathfrak{n}^+} |f(Z)|^2 \big(\mathrm{Det}(I_q + Z^*Z)\big)^{-p-q-m} c_{m\lambda} \, d\mu_L(Z) < +\infty.$$

Here $d\mu_L(Z)$ denotes the Lebesgue measure on $M_{pq}(\mathbb{C})$ defined by $d\mu_L(Z) = \prod_{kl} dx_{kl}\, dy_{kl}$ where $Z = (x_{kl} + iy_{kl})_{kl}$, $x_{kl}, y_{kl} \in \mathbb{R}$ for $1 \le k \le p$ and $1 \le l \le q$

and the constant $c_{m\lambda}$ can be calculated by adapting Hua's method for computing some integrals over matrix balls (see [27], Theorem 2.2.1 and [12], Section 7):

$$c_{m\lambda}^{-1} = \pi^{pq} \prod_{k=0}^{q-1} \frac{\Gamma(q+m-k)}{\Gamma(p+q+m-k)}.$$

Also, the coherent states for the Hilbert space $\mathcal{H}_{m\lambda}$ are given by

$$e_Z^{m\lambda}(W) = \chi_{m\lambda}(\kappa(\exp Z^* \exp W)^{-1}) = \big(\mathrm{Det}(I_q + Z^* W)\big)^m$$

and the the representation $\pi_{m\lambda}$ is given by

$$\pi_{m\lambda}(g)f(Z) = \big(\mathrm{Det}(CZ+D)\big)^m f\big((AZ+B)(CZ+D)^{-1}\big)$$

if $g^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Note that we have in this case

$$\varphi_\lambda(Z) = \mathrm{Det}(I_q + Z^* Z) \geq 1 + \mathrm{Tr}(Z^* Z) = 1 + \sum_{kl} |z_{kl}|^2$$

in accordance with Proposition 4.4.

The action of H on $\mathfrak{n}^+$ is given by

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \cdot Z = AZD^{-1}.$$

The corresponding action σ of H on $\mathcal{P}$ is multiplicity free (see for instance [26]). Then Proposition 7.2 and Corollary 9.2 can be applied here in order to obtain contraction results for the sequence $(\pi_{m\lambda})$ to the unitary irreducible representation ρ_γ of the $(2pq+1)$-dimensional Heisenberg group.

References

[1] D. Arnal, M. Cahen and S. Gutt, *Representations of compact Lie groups and quantization by deformation*, Acad. R. Belg. Bull. Cl. Sc. 3e série LXXIV **45** (1988), 123–141. MR 1027456

[2] D. Bar-Moshe and M. S. Marinov, *Realization of compact Lie algebras in Kähler manifolds*, J. Phys. A: Math. Gen. **27** (1994), 6287–6298. MR 1306179

[3] F. A. Berezin, *Covariant and contravariant symbols of operators*, Math. USSR Izv. **6** (1972), 1117–1151. MR 0350504

[4] F. A. Berezin, *Quantization*, Math. USSR Izv. **8** (1974), 1109–1165. MR 0395610

[5] F. A. Berezin, *Quantization in complex symmetric domains*, Math. USSR Izv. **9** (1975), 341–379.

[6] B. Cahen, *Deformation program for principal series representations*, Lett. Math. Phys. **36** (1996), 65–75. MR 1371298

[7] B. Cahen, *Quantification d'une orbite massive d'un groupe de Poincaré généralisé*, C.R. Acad. Sci. Paris t. **325** (1997), 803–806. MR 1483721

[8] B. Cahen, *Quantification d'orbites coadjointes et théorie des contractions*, J. Lie Theory **11** (2001), 257–272. MR 1851792

[9] B. Cahen, *Contraction de SU(2) vers le groupe de Heisenberg et calcul de Berezin*, Beiträge Algebra Geom. **44** (2003), 581–603. MR 2017860

[10] B. Cahen, *Contraction de $SU(1,1)$ vers le groupe de Heisenberg*, Mathematical works, Part XV, Université du Luxembourg, Luxembourg, Séminaire de Mathématique, 2004 pp. 19–43. MR 2143420

[11] B. Cahen, *Contractions of $SU(1,n)$ and $SU(n+1)$ via Berezin quantization*, J. Anal. Math. **97** (2005), 83–102. MR 2274974

[12] B. Cahen, *Berezin quantization on generalized flag manifolds*, Preprint Univ. Metz (2008), to appear in Math. Scand.

[13] B. Cahen, *Multiplicities of compact Lie group representations via Berezin quantization*, Math. Vesnik. **60** (2008), 295–309. MR 2465811

[14] M. Cahen, S. Gutt and J. Rawnsley, *Quantization on Kahler manifolds I: Geometric interpretation of Berezin quantization*, J. Geom. Phys. **7** (1990), 45–62. MR 1094730

[15] C. Cishahayo and S. de Bièvre, *On the contraction of the discrete series of $SU(1,1)$*, Ann. Inst. Fourier **43** (1993), 551–567. MR 1220284

[16] L. Cohn, *Analytic Theory of the Harish–Chandra C-function*, Lecture Notes in Math., vol. 429, Springer, 1974.

[17] P. Cotton and A.H. Dooley, *Contraction of an adapted functional calculus*, J. Lie Theory **7** (1997), 147–164. MR 1473162

[18] A. H. Dooley, *Contractions of Lie groups and applications to analysis*, Topics in modern harmonic analysis, Proc. Semin., Torino and Milano 1982, vol. I, Ist. di Alta Mat, Rome, 1983, pp. 483–515. MR 0748872

[19] A. H. Dooley and S. K. Gupta, *The Contraction of S^{2p-1} to H^{p-1}*, Monatsh. Math. **128** (1999), 237–253. MR 1719423

[20] A. H. Dooley and S. K. Gupta, *Transferring Fourier multipliers from S^{2p-1} to H^{p-1}*, Illinois J. Math. **46** (2002), 657–677. MR 1951233

[21] A. H. Dooley and J. W. Rice, *Contractions of rotation groups and their representations*, Math. Proc. Camb. Phil. Soc. **94** (1983), 509–517. MR 0720801

[22] A. H. Dooley and J. W. Rice, *On contractions of semisimple Lie groups*, Trans. Amer. Math. Soc. **289** (1985), 185–202. MR 0779059

[23] B. Folland, *Harmonic analysis in phase space*, Princeton Univ. Press, 1989. MR 0983366

[24] Harish-Chandra, *Discrete series for semisimple Lie groups II. Explicit determination of the characters*, Acta Math. **116** (1966), 1–111. MR 0219666

[25] S. Helgason, *Differential geometry*, Lie groups and symmetric spaces, Graduate Studies in Mathematics, vol. 34, Amer. Math. Soc., Providence, RI, 2001. MR 1834454

[26] R. Howe and T. Umeda, *The Capelli identity, the double commutant theorem and multiplicity-free actions*, Math. Ann. **290** (1991), 565–619. MR 1116239

[27] L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Translations of Mathematical Monographs, vol. 6, Amer. Math. Soc., Providence, RI, 1963. MR 0171936

[28] E. Inönü, E. P. Wigner, *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. USA **39** (1953), 510–524. MR 0055352

[29] A. A. Kirillov, *Lectures on the Orbit Method*, Graduate Studies in Mathematics, vol. 64, Amer. Math. Soc., Providence, RI, 2004. MR 2069175

[30] A. W. Knapp, *Representation theory of semi simple groups. An overview based on examples*, Princeton Math. Series, vol. 36, 1986. MR 0855239

[31] K.-H. Neeb, *Holomorphy and convexity in Lie theory*, de Gruyter Expositions in Mathematics, vol. 28, Walter de Gruyter, Berlin, 2000. MR 1740617

[32] J. Mickelsson and J. Niederle, *Contractions of representations of de Sitter groups*, Commun. Math. Phys. **27** (1972), 167–180. MR 0309502

[33] F. Ricci, *A Contraction of $SU(2)$ to the Heisenberg group*, Monatsh. Math. **101** (1986), 211–225. MR 0847376

[34] F. Ricci and R. L. Rubin, *Transferring Fourier multipliers from $SU(2)$ to the Heisenberg group*, Amer. J. Math. **108** (1986), 571–588. MR 0844631

[35] V. S. Varadarajan, *Lie groups*, Lie algebras and their representations, Graduate Texts in Mathematics, vol. 102, Springer, New York, 1986. MR 0746308

[36] N. R. Wallach, *Harmonic analysis on homogeneous spaces*, Pure and Applied Mathematics, vol. 19, Marcel Dekker, New York, 1973. MR 0498996

[37] N. J. Wildberger, *On the Fourier transform of a compact semi simple Lie group*, J. Austral. Math. Soc. A **56** (1994), 64–116. MR 1250994

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