

## ON PROJECTIVE VARIETIES OF DIMENSION $n + k$ COVERED BY $k$ -SPACES

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**ABSTRACT.** We study families of linear spaces in projective space whose union is a proper subvariety  $X$  of the expected dimension. We establish relations between configurations of focal points and the existence or non-existence of a fixed tangent space to  $X$  along a general element of the family. We apply our results to the classification of ruled 3-dimensional varieties.

### 0. Introduction

Since the publication of [GH] there has been a renewal of interest in the study of differential geometric properties of algebraic varieties. This study has its origins in classical works, such as several papers by C. Segre (in particular, [S1] and [S2]). There, topics such as the second fundamental form of projective varieties, varieties with degenerate Gauss mapping, and in general varieties ruled by linear subspaces, are introduced and discussed. Recently, contributions to these topics have been made by Akivis, Goldberg, Landsberg, and Rogora (see [AG], [L], [AGL], [R]). These papers highlight the importance of the study of the *focal scheme*.

Foci are a classical tool for studying families of linear spaces (see [S2]), which in modern algebraic geometry has been reformulated by Ciliberto and Sernesi [CS] by means of the focal diagram and has been applied to the study of congruences of lines (see [ABT], [Arr], [D]).

In this paper we will deal with families of linear spaces that generate proper subvarieties of the expected dimension in the projective space. For instance, consider a family  $B$  of  $k$ -spaces in the projective space  $\mathbb{P}^N$ , the variety  $X$  ruled by  $B$ , and assume that  $\dim B = n$  and  $\dim X = n + k < N$ . Then we will study the relationship between the existence and the properties of the

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focal scheme on a general space of  $B$ , and the existence of spaces of dimension  $\leq n+k$  tangent to  $X$  along a general space of  $B$ . A complete description of this relationship for a family of lines is given in the following theorem:

**THEOREM 0.1.** *Let  $B \subset \mathbb{G}(1, N)$  be a family of lines in  $\mathbb{P}^N$  of dimension  $n$ ,  $n \leq N-2$ . Suppose that the union of the lines belonging to  $B$  is an algebraic variety  $X$  of dimension  $n+1$ . Then, for all  $k$  in the range  $0 \leq k \leq n$ , the following conditions are equivalent:*

- (i) *The focal locus on the general element  $r \in B$  has length  $k$ .*
- (ii)  *$X$  has a fixed tangent  $\mathbb{P}^{k+1}$  along every general  $r \in B$ .*

There is an analogue of Theorem 0.1 for varieties with degenerate Gauss mapping:

**THEOREM 0.2.** *Let  $B$  be a family of linear subspaces of  $\mathbb{P}^N$  of dimension  $k$ , and denote by  $n$  the dimension of  $B$ . Suppose that the union of the  $k$ -planes of the family  $B$  is an algebraic variety  $X \subset \mathbb{P}^N$  of dimension  $n+k < N$ . Then the following conditions are equivalent:*

- (i) *The tangent space to  $X$  is constant along general elements of  $B$ .*
- (ii) *For all  $\Lambda$  belonging to an open set of  $B$  the focal subvariety of  $B$  is a hypersurface of  $\Lambda$  of degree  $n$ ; otherwise all points of  $\Lambda$  are focal.*

We will apply Theorem 0.1 and Theorem 0.2 to the study of ruled varieties of dimension 3. Our results comprise and complete results proved in earlier papers, such as [GH], [R], [AGL]. It should be noted, however, that the result in [GH] about varieties with degenerate Gauss mapping is not precisely stated, and that [R] considers only necessary conditions and not sufficient ones.

In the following two theorems we give the classification of threefolds with a tangent 2-plane constant along lines, and that of threefolds with degenerate Gauss mapping.

**THEOREM 0.3.** *Let  $B$  be a surface in the Grassmannian  $\mathbb{G}(1, N)$ , with  $N \geq 4$ . Suppose that the union of the lines belonging to  $B$  is an algebraic variety  $X$  of dimension 3, and that the Gauss image of  $X$  has dimension 3. Then, along a general line of  $B$  there is a fixed tangent 2-plane not contained in  $X$  if and only if  $X$  is one of the following:*

- (1) *a union of lines, all tangent to a surface  $S \subset \mathbb{P}^N$ , whose direction at the tangency point is not in general a conjugate direction for the second fundamental form of  $S$ ;*
- (2) *the union of a one-dimensional family of 2-dimensional cones, whose vertices sweep a curve.*

**THEOREM 0.4.** *Let  $X$  be a variety of dimension 3 with degenerate Gauss mapping. Then one of the following cases holds:*

- (1) *The Gauss image of  $X$  has dimension 2, and  $X$  is one of the following:*
- (a) *a union of lines bitangent to a surface;*
  - (b) *a union of lines tangent to two surfaces;*
  - (c) *a union of lines tangent to a surface, and meeting a fixed curve;*
  - (d) *the union of asymptotic tangent lines of a surface;*
  - (e) *the join of two curves;*
  - (f) *the variety of secant lines of a curve;*
  - (g) *a band (see Definition 2.4);*
  - (h) *the cone over a surface, with a point as vertex.*
- (2) *The Gauss image of  $X$  has dimension 1, and  $X$  is built up by a composite construction of cones and varieties ruled by osculating spaces over some curve.*

*All of these cases are possible, and each of them always represents a class of varieties with degenerate Gauss mapping.*

The plan of the paper is as follows. In Section 1 we introduce the notion of foci for a family of linear spaces and we give an interpretation of foci in terms of tangent spaces to the Grassmannian.

In Section 2 we prove the two classification theorems for ruled varieties of dimension 3. We prove, moreover, that all surfaces  $S$  appearing in cases (a)–(d) of Theorem 0.3 are not general, but must satisfy the condition that the osculating space to  $S$  at a general point has dimension at most 4.

In Section 3 we prove Theorem 0.1 and establish the properties of the focal locus in the case of varieties ruled by lines. In the final section we consider varieties ruled by subspaces of higher dimension. We give an example showing that Theorem 0.1 cannot be extended to a family of subspaces of dimension  $\geq 2$ . However, Theorem 0.2 shows that a description is still possible for varieties with degenerate Gauss mapping. This result has already been established by Akivis and Goldberg [AG] using differential geometry techniques; we give now an algebraic proof.

**Notation.** We will study projective algebraic varieties over the complex field or, more generally, over an algebraically closed field  $\mathbb{K}$  with  $\text{char } \mathbb{K} = 0$ .

$V$  will denote a linear space of dimension  $N + 1$  over  $\mathbb{K}$ , and  $\mathbb{P}^N = \mathbb{P}(V)$  the projectivization of  $V$ . Analogously,  $\mathbb{A}^{N+1} = \mathbb{A}(V)$  will denote the affine space associated to  $V$ .

If  $\Lambda \subset \mathbb{P}^N$  is a projective linear subspace,  $\hat{\Lambda} \subset V$  will denote the linear subspace associated to  $\Lambda$  such that  $\Lambda = \mathbb{P}(\hat{\Lambda})$ .  $[v] \in \mathbb{P}(V)$  will denote the point of  $\mathbb{P}^N$  corresponding to the equivalence class of  $v \in V \setminus \{0\}$ .

$T_x X$  will denote the Zariski tangent space to the variety  $X$  at its point  $x$ , while we will denote by  $\mathbb{T}_x X \subset \mathbb{P}^N$  the embedded tangent space to  $X$  at  $x$ .

$G(h, V)$  will denote the Grassmannian variety of linear subspaces of dimension  $h$  in  $V$ .  $\mathbb{G}(k, N)$  will denote the Grassmannian of projective subspaces

of dimension  $k$  of  $\mathbb{P}^N$ . We will use the same symbol to denote the points of the Grassmannian and the corresponding linear subspaces.

**1. The focal diagram**

Let  $B \subset \mathbb{G}(k, N)$  be a family of dimension  $n$  of  $k$ -spaces in  $\mathbb{P}^N$ . Denote by  $B'$  a desingularization of  $B$  and by  $\mathcal{I}$  the incidence correspondence of  $B'$ , with the natural projections

$$\begin{array}{ccccc} B' & \xleftarrow{p_1} & B' \times \mathbb{P}^N & \xrightarrow{p_2} & \mathbb{P}^N \\ & & \cup & & \\ B & \xleftarrow{g} & \mathcal{I} & \xrightarrow{f} & \mathbb{P}^N \end{array}$$

In what follows, we will restrict ourselves to families  $B$  such that the image of  $f$  (i.e., the union of the lines belonging to  $B$ ) is a variety  $X$  of dimension  $n + k$ . This is the same as assuming that the general fibre of  $f : \mathcal{I} \rightarrow X$  is finite.

DEFINITION 1.1. A point  $x \in X$  is said to be a *fundamental point* of the family  $B$  if the fibre  $f^{-1}(x)$  has positive dimension. This condition defines a closed subset of  $X$  called the *fundamental locus*  $\Phi$  of  $B$ .

With this setup, we can construct a commutative diagram of exact sequences, called the *focal diagram* of  $B$ :

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & (p_1^*(\mathcal{T}_{B'}))|_{\mathcal{I}} & \xrightarrow{\chi} & \mathcal{N}_{\mathcal{I}|B' \times \mathbb{P}^N} & \\ & & & \downarrow & & \parallel & \\ 0 & \longrightarrow & \mathcal{T}_{\mathcal{I}} & \longrightarrow & \mathcal{T}_{B' \times \mathbb{P}^N}|_{\mathcal{I}} & \longrightarrow & \mathcal{N}_{\mathcal{I}|B' \times \mathbb{P}^N} \longrightarrow 0 \\ & & \downarrow df & & \downarrow & & \\ & & f^*(\mathcal{T}_{\mathbb{P}^N}) & \xlongequal{\quad} & (p_2^*(\mathcal{T}_{\mathbb{P}^N}))|_{\mathcal{I}} & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

The focal diagram is built up by crossing the exact sequence defining the normal sheaf to  $\mathcal{I}$  inside  $B' \times \mathbb{P}^N$  with the sequence (restricted to  $\mathcal{I}$ ) expressing the tangent sheaf of the product variety  $B' \times \mathbb{P}^N$  as a product of tangent sheaves.

DEFINITION 1.2. The map denoted by  $\chi$  in the focal diagram is called the *characteristic map* of the family  $B$ . For every  $\Lambda \in B_{ns}$  the restriction of  $\chi$

to  $g^{-1}(\Lambda)$  is called the *characteristic map of  $B$  relative to  $\Lambda$* ; it lies in the following diagram:

$$\begin{array}{ccc} \chi(\Lambda) : T_\Lambda B' \otimes \mathcal{O}_\Lambda & \longrightarrow & \mathcal{N}_{\Lambda|\mathbb{P}^N} \\ \wr \parallel & & \wr \parallel \\ \mathcal{O}_\Lambda^n & \longrightarrow & \mathcal{O}_\Lambda^{N-k}(1) \end{array}$$

DEFINITION 1.3. The condition

$$\text{rank } \chi(\Lambda, x) < \min\{\text{rank}((p_1^*(\mathcal{T}_{B'})|_{\mathcal{I}}), \text{rank}(\mathcal{N}_{\mathcal{I}|B' \times \mathbb{P}^N})\}$$

defines a closed subscheme  $V(\chi) \subset \mathcal{I}$  which will be called the *subscheme of first order foci* (or, simply, the *focal subscheme*) of the family  $B$ . Analogously,  $F = f(V(\chi))$  is called the *locus of first order foci*, or the *focal locus* of  $B$  in  $\mathbb{P}^N$ .

By the commutativity of the focal diagram, the focal locus has a second interpretation. Indeed, the kernel of  $\chi$  and the kernel of  $df$  must coincide (as subsheaves of  $\mathcal{T}_{B \times \mathbb{P}^N|_{\mathcal{I}}}$ ). Then the focal locus is the ramification locus of  $f$ . As a consequence, the fundamental locus is contained in the focal locus. These considerations can be rephrased as follows.

PROPOSITION 1.4. *The following conditions are equivalent:*

- (1) *The rank of  $\chi$  is maximal.*
- (2) *The rank of  $df$  is maximal.*
- (3)  *$V(\chi)$  is a closed proper subscheme of  $\mathcal{I}$ .*

We have assumed that the union of the  $k$ -spaces belonging to  $B$  is a variety  $X$  of dimension  $n + k$ . By Proposition 1.4 this implies that a general point on a general space of  $B$  is not a focus. Nevertheless, some particular spaces of  $B$  can be contained in the focal locus: they are called *focal spaces*.

The characteristic map is closely connected with the structure of the tangent space to the Grassmannian variety as a space of homomorphisms (see [H]). Let  $B$  be a subvariety of  $\mathbb{G}(k, N)$ . We can identify this Grassmannian with the Grassmannian of linear subspaces of dimension  $k+1$  of  $V$ ,  $G(k+1, V)$ . Then, by associating to each  $\Lambda \in B$  the affine cone  $\hat{\Lambda} \subset \mathbb{A}(V) = \mathbb{A}^{N+1}$ , we can construct a new incidence correspondence  $\mathcal{I}' \subset B' \times \mathbb{A}^{N+1}$  and projections  $f', g'$  as follows:

$$\begin{array}{ccccc} B' & \xleftarrow{q_1} & B' \times \mathbb{A}^{N+1} & \xrightarrow{q_2} & \mathbb{A}^{N+1} \\ & & \cup & & \\ B & \xleftarrow{g'} & \mathcal{I}' & \xrightarrow{f'} & \mathbb{A}^{N+1} \end{array}$$

Considering  $B$  as a family of subspaces in  $\mathbb{A}^{N+1}$  yields an affine version of the focal diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & (q_1^*(\mathcal{T}_{B'}))|_{\mathcal{I}'} & \xrightarrow{\chi'} & \mathcal{N}_{\mathcal{I}'|_{B' \times \mathbb{A}^{N+1}}} & \\
 & & & \downarrow & & \parallel & \\
 0 & \longrightarrow & \mathcal{T}_{\mathcal{I}'} & \longrightarrow & \mathcal{T}_{B' \times \mathbb{A}^{N+1}|_{\mathcal{I}'}} & \longrightarrow & \mathcal{N}_{\mathcal{I}'|_{B' \times \mathbb{A}^{N+1}}} \longrightarrow 0 \\
 & & \downarrow d_f' & & \downarrow & & \\
 & & f'^*(\mathcal{T}_{\mathbb{A}^{N+1}}) & \xlongequal{\quad} & (q_2^*(\mathcal{T}_{\mathbb{A}^{N+1}}))|_{\mathcal{I}'} & & \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

As in the projective case, we can define the characteristic map  $\chi'$  relative to  $\Lambda$ , a non-singular element of  $B$ , by

$$\chi'(\Lambda) : T_\Lambda B \otimes \mathcal{O}_{\hat{\Lambda}} \longrightarrow \mathcal{N}_{\hat{\Lambda}|\mathbb{A}^{N+1}}.$$

If we compare the definition of the focal diagram and the characterization of  $T_\Lambda B$  as a space of homomorphisms, we easily get the following proposition.

**PROPOSITION 1.5.** *Let  $\Lambda$  be a non-singular point of  $B \subset \mathbb{G}(k, N) = G(k + 1, V)$ . Let us consider  $T_\Lambda B$  as a linear subspace of  $T_\Lambda \mathbb{G}(k, N) \cong \text{Hom}(\hat{\Lambda}, V/\hat{\Lambda})$ . Then the characteristic map  $\chi'$  relative to  $\Lambda$ , considered as a morphism of vector bundles, for all  $v \in \hat{\Lambda}$  associates to  $\eta \in T_\Lambda B$  the normal vector  $\eta(v)$ .*

The projectivization of the usual characteristic map  $\chi$  coincides with that of the affine version  $\chi'$ , so we have:

**COROLLARY 1.6.** *Let  $\Lambda$  be a non-singular point of  $B \subset \mathbb{G}(k, N)$ . Let us consider  $T_\Lambda B$  as a linear subspace of  $T_\Lambda \mathbb{G}(k, N) \cong \text{Hom}(\hat{\Lambda}, V/\hat{\Lambda})$ . Then the projectivization of the characteristic map  $\chi$  relative to  $\Lambda$ , considered as a morphism of vector bundles, for all  $p \in \Lambda$  associates to  $[\eta] \in \mathbb{P}(T_\Lambda B)$  the point  $[\eta(v)] \in \mathbb{P}(V/\hat{\Lambda})$ , where  $v \in V$  is such that  $[v] = p$ .*

This corollary yields an interpretation of focal points which is particularly clear in the case of a family of lines.

**REMARK 1.** Consider a variety  $B \subset \mathbb{G}(1, N)$  and a general line  $r \in B$ . Then the foci on  $r$  are the points  $p = [v]$  such that  $v \in \ker \eta$  for a non-trivial

$\eta \in T_r B$ . Since under our hypotheses the rank of the general  $\eta \in T_r B$  is 2, the existence of focal points depends on the existence of rank 1 homomorphisms in  $T_r B$ . Focal points with multiplicity represent a special case. A point  $p = [v] \in r$  is a focal point of multiplicity  $\geq 2$  if and only if there exist two linearly independent tangent vectors  $\eta_1, \eta_2 \in T_r B$  satisfying

$$\begin{aligned} \eta_1(v) &= 0, & \text{Im}(\eta_1) &\neq 0, \\ \eta_2(v) &\in \text{Im}(\eta_1), & \text{Im}(\eta_2) &\neq \text{Im}(\eta_1), \end{aligned}$$

since the condition on the multiplicity is that the composition of the characteristic map relative to  $r$  with the natural map  $V/\hat{r} \rightarrow (V/\hat{r})/\text{Im}(\eta_1)$  does not have maximal rank. Iteration of this construction yields a characterization for focal points of higher multiplicity.

By means of the Plücker embedding, we can consider the embedded tangent space to  $B \subset \mathbb{G}(1, N)$  at a point  $\Lambda$ . In the case of lines, there is a connection between the existence of focal points on  $r \in B$  and the existence of a line in  $T_r B \cap \mathbb{G}(1, N)$ .

**PROPOSITION 1.7.** *Let  $B \subset \mathbb{G}(1, N)$  be a family of lines. Let  $r$  be a general element of  $B$  and suppose  $r$  is not focal. Then there is a bijection between the focal points on  $r$  and the lines in the intersection of the Grassmannian  $\mathbb{G}(1, N)$  with  $T_r B$  (embedded in  $\mathbb{P}(\wedge^2 V)$ ).*

*Proof.* We know that  $[v] \in r$  is a focal point if and only if  $v \in \ker \eta$ , where  $\eta \in T_r B$  has rank 1. With a simple computation, it is possible to prove that if a homomorphism  $\eta \in T_r \mathbb{G}(1, N)$  has rank 1 then the pencil of lines passing through  $\mathbb{P}(\ker \eta)$  and lying in  $\text{Im}(\eta) \oplus r$  is a line contained in the intersection of  $T_r B$  with the Grassmannian. The converse is also true: if there is a line in the intersection, then we can find a homomorphism  $\eta$  of rank 1 and hence a focal point.  $\square$

### 2. Varieties of dimension 3

We now apply the study of the focal locus to the specific problem of classifying ruled varieties of dimension 3 with degenerate tangential properties. More precisely, we consider:

- (1) varieties ruled by lines with a constant tangent 2-plane along any line of the ruling;
- (2) varieties ruled by lines with a constant tangent space of dimension 3 along every line;
- (3) varieties ruled by planes with a constant tangent space of dimension 3 along every plane.

Note that the last two cases yield the classification of varieties of dimension 3 with degenerate Gauss mapping. For the proofs we will use some results to be proved in Sections 3 and 4.

The classical references for our approach to classification are the works of C. Segre. In particular, a classical proof of the classification of case (1) can be found in [S2]. The classification of varieties of dimension 3 with degenerate Gauss mapping has recently been given in the papers [R] and [AGL]. In both papers, the classification is based on the study of the focal scheme of the family of fibres of the Gauss map, but there is no distinction between strict focal locus and (total) focal locus (see Definition 3.1). In [R] the classification is outlined without a study of the second fundamental form of focal surfaces. Therefore, there is no description of how to construct a variety with degenerate Gauss mapping. In [AGL] one of the cases (that of bands) is not completely solved.

In what follows, the concept of *conjugate directions* for the second fundamental form will naturally arise. We will denote by

$$II_y : T_y Y \otimes T_y Y \longrightarrow N_y Y$$

the *second fundamental form* of a variety  $Y$  at a non-singular point  $y$  (see [H] for the definition). It is a symmetric bilinear form, and hence can be interpreted as a linear system of quadrics  $|II_y|$  in  $\mathbb{P}(T_y Y)$ . The dimension of the linear system is connected with the dimension of the second osculating space to  $Y$  in  $y$ ,  $T_y^{(2)} Y$ , by the relation

$$\dim |II_y| = \dim T_y^{(2)} Y - \dim T_y Y - 1.$$

DEFINITION 2.1. Let  $Y \subset \mathbb{P}^N$  be a variety, and let  $y$  be a non-singular point of  $Y$ . Then two tangent vectors  $v, w \in T_y Y$  are said to represent *conjugate directions* at  $y$  if  $II_y(v, w) = 0$ . This means that the points  $[v]$  and  $[w]$  are conjugate with respect to all quadrics in  $|II_y|$ . If there is a self-conjugate tangent vector, its direction is called an *asymptotic direction* at  $y$ .

The existence of conjugate directions at every non-singular point is not true in general, but depends on the dimension of the second osculating space to the variety at the general point. We are interested in the study of conjugate directions for surfaces. For general surfaces at general points the dimension of the second osculating space is 5. In this case, at a general point there are no conjugate directions. Conjugate directions exist only if the dimension of the second osculating space is  $\leq 4$ . If the dimension is 3, every direction possesses a conjugate direction. It is well known that in  $\mathbb{P}^N$ ,  $N \geq 4$ , this property holds only for developable surfaces, i.e., cones and varieties swept out by tangent lines to a curve.

DEFINITION 2.2 ([S1]). A surface is called a  $\Phi$  surface if and only if the dimension of its second osculating space at the general point is 4.

PROPOSITION 2.3. *For a surface  $S \subset \mathbb{P}^N$ ,  $N \geq 5$ , the following properties are equivalent:*

- (i)  *$S$  is a  $\Phi$  surface.*
- (ii) *At a general point of  $S$  there is exactly one couple of conjugate directions, or one asymptotic direction.*
- (iii) *The union of the tangent planes to  $S$  is a variety of dimension 4 with tangent space fixed along those planes.*

*Proof.* The equivalence of the first two properties is a consequence of the fact that a linear system of quadrics in  $\mathbb{P}^1$  admits exactly one couple of conjugate points if and only if its dimension is 1. Let now  $S \subset \mathbb{P}^N$  be a surface, and let us denote by  $V$  the closure in  $\mathbb{P}^N$  of the union of the tangent planes to  $S$  at its non-singular points. Then the dimension of  $V$  is 4 if and only if  $S$  is not a developable surface or a plane. Let us consider the osculating space  $\mathbb{T}_p^2 S$  to  $S$  at a general point  $p$ , embedded in  $\mathbb{P}^N$ . Using a local parametric representation of  $S$ , it is easy to show the following equality:

$$\mathbb{T}_p^2 S = \overline{\bigcup_{q \in \mathbb{T}_p S \cap V_{ns}} \mathbb{T}_q V}.$$

This implies that the tangent space to  $V$  is constant along planes if and only if the dimension of the osculating space to  $S$  equals the dimension of  $V$ . Hence the equivalence of (i) and (iii) is established.  $\square$

REMARK 2. The general situation for the union of tangent planes to a surface is that the fibres of the Gauss map are 1-dimensional.

When we have a  $\Phi$  surface  $S$ , we can always construct an irreducible family  $\Sigma$  of dimension 2, whose elements are lines tangent to  $S$ , such that for each general point  $p \in S$  there is exactly one line of  $\Sigma$  tangent to  $S$  at  $p$ , and, moreover, its tangent direction at  $p$  is conjugate to some (other) tangent direction. In this case, we will say that the lines of  $\Sigma$  *admit a conjugate direction* on  $S$ . If at the general point of  $S$  the two conjugate directions coincide, i.e., there is an asymptotic direction, then the lines of  $\Sigma$  are called *asymptotic lines* on  $S$ .

Let us first consider case (1). Then we have a 3-dimensional variety  $X$  which is covered by the lines belonging to a surface  $B$  in the Grassmannian  $\mathbb{G}(1, N)$ , such that  $X$  has a constant tangent plane along a general line of  $B$ . A classification of these families is provided by Theorem 0.3 (see also [M]).

*Proof of Theorem 0.3.* We can apply Theorem 0.1 to the family  $B$ . The existence of the tangent plane implies then that on the general line belonging to  $B$  there exists one focal point (with multiplicity 1). This focus cannot be a fixed point  $p$ , for then there would be a 2-dimensional subfamily of lines of  $B$  passing through  $p$ , and  $p$  would be a focal point of multiplicity 2, which is

not allowed. Thus, the closure of the union of the focal points on such lines (the *strict* focal locus, in the terminology to be introduced in Section 3) can be either a surface  $S$  or a curve  $C$ . In the former case, the first part of the claim follows from Theorem 3.2. The exception considered in our statement is necessary in order to exclude varieties with degenerate Gauss mapping, as we will see later. In the latter case  $C$  lies in the fundamental locus of  $B$ , which yields the second part of the claim. By a direct calculation, we can check that the union of tangent lines to a surface  $S$  has a constant tangent plane along a general line  $r$ . This plane is the tangent plane to  $S$  at the point of tangency of  $r$ . Analogously, the fixed tangent plane along the lines of a cone is contained in the tangent space to the union of cones.  $\square$

REMARK 3. In the hypotheses of Theorem 0.3 we have excluded the (trivial) case of varieties  $X$  ruled by planes. In this case the Fano variety of lines has dimension  $> 2$ , but it is always possible to find a subvariety  $B$  of this variety, with  $\dim B = 2$ , such that the lines of  $B$  cover  $X$ . There are two ways of constructing  $B$ . We can choose a unisecant curve  $C$  to the family of planes and consider for every plane the pencil of lines, with the corresponding point of  $C$  as center. The points of  $C$  are fundamental points of  $B$ , and in general they are not singular points of  $X$ . Note that a general ruled surface in  $\mathbb{G}(1, N)$  gives an example of this situation. Alternatively, inside every plane we can fix a curve (varying algebraically with the plane) and consider the family of its tangent lines. Also, in this case the focal points are not in general singular for  $X$ . In fact, in both cases the focal points have no real geometric meaning for  $X$ .

We next prove Theorem 0.4, giving the classification of varieties of dimension 3 with degenerate Gauss mapping.

*Proof of Theorem 0.4.* The second part is well known and classical. We give here a simple proof based on the analysis of foci. Let us suppose that  $X \subset \mathbb{P}^N$  is a 3-dimensional variety with Gauss map whose fibres have dimension 2. Let us consider the family  $B \subset \mathbb{G}(2, N)$  of the fibres of the Gauss map of  $X$ . Then, by Theorem 0.2, there is a focal line on every general plane of  $B$ . If there is a fundamental line  $L$ ,  $X$  must be a cone over a curve, with vertex  $L$ . Otherwise, the focal locus is a ruled surface  $S$ , and by Theorem 3.2 every plane of  $B$  is tangent to  $S$  along a line of its ruling. Hence  $S$  is a surface with degenerate Gauss mapping, so  $S$  is a cone or the tangent developable to a curve. In the first case,  $X$  is a cone, with a point as vertex, over the tangent developable to a curve. In the second case,  $X$  is the union of osculating planes to a curve.

We next prove the first part of the theorem. For more details we refer to [T].

Let  $B \subset \mathbb{G}(1, N)$  be the family of fibres of the Gauss map of  $X$ . By Theorem 0.1, on a general line of  $B$  there are two foci (counting multiplicity). We now consider the number of distinct focal points on a general line belonging to  $B$ , the number (1 or 2) and the dimension of the irreducible components of the strict focal locus (see Definition 3.1), i.e., the variety obtained as closure of the union of focal points on non-focal lines of  $B$ . This is a general procedure, which will be extended to varieties of higher dimension in Section 3.

If the focal points on a general line of  $B$  are distinct, Theorem 3.2 gives us the classification of all possible cases, as shown in Table 1.

foci on a general line	strict focal locus	description
two distinct points	each point sweeps a surface	union of lines bitangent to a surface
		union of lines tangent to two surfaces
	a point sweeps a surface, the other point sweeps a curve	union of lines tangent to a surface and meeting a curve
	each point sweeps a curve	secant variety of a curve
		join of two curves

TABLE 1. Two distinct foci

If on a general line of  $B$  there is one focal point with multiplicity 2, we need additional information. This information is obtained by considering the interpretation, given in Section 1, of the characteristic map of  $B$  relative to a general  $r \in B$  as describing the subspace  $T_r B \subset T_r \mathbb{G}(1, N) \cong \text{Hom}(\hat{r}, V/\hat{r})$ .

Using this, we now prove that if the strict focal locus is a surface  $S$ , then a general line of  $B$  represents an asymptotic direction of  $S$ , i.e., a self-conjugate direction with respect to the second fundamental form of  $S$ . Let  $r$  be a general fibre of the Gauss map and let  $F = [v]$  be the (double) focus on  $r$ . Then by Remark 1 there exist two linearly independent tangent vectors  $\eta_1, \eta_2$  in  $T_r B$  such that  $\eta_1(r) = 0$  in  $V/\hat{r}$  and  $\eta_2(v) \in \text{Im}(\eta_1)$ . Let  $\{b_1(t)\}$  be an arc of a smooth curve in  $B$ , parametrized by an open disc containing the origin, with  $b_1(0) = r$  and  $b'_1(0) = \eta_1$ , and let  $\{c_1(t)\}$  be a lifting of  $\{b_1(t)\}$  through  $F$ , i.e., any regular curve in  $X$  such that  $c_1(t) \in b_1(t)$  for all  $t$  and  $c_1(0) = F$ . Then  $r$  is the tangent line to the curve  $\{c_1(t)\}$  at  $F$ . In particular, the curve  $C (\subset S)$  generated by the unique focus of  $b_1(t)$  as  $t$  varies in the disc is such a lifting. If  $\{g_1(t)\}$  is another lifting of  $\{b_1(t)\}$ , but with  $g_1(0) \neq F$ , then the tangent vector  $g'_1(0)$  is not parallel to  $r$ , and so, together with  $r$ , generates

the tangent plane at  $g_1(0)$  to the ruled surface  $Y$ , the union of the lines  $b_1(t)$ . Since  $\dim \text{Im}(\eta_1) = 1$ , this plane is constant along  $r$ , so it coincides with the osculating plane to the curve  $C$  at  $F$ . Let now  $\{b_2(t)\}$  be a regular curve in  $B$  such that  $b_2(0) = r$  and  $b_2'(0) = \eta_2$ . If  $\{d_2(t)\}$  is a lifting of  $\{b_2(t)\}$  through  $F$ , then its tangent line at  $F$  is contained in the osculating plane to  $C$  at  $F$ . In particular, we can choose as lifting the curve  $D$  of the foci of the lines  $b_2(t)$ . Because of the generality assumptions, the tangent plane to  $S$  at  $F$  is generated by the tangent lines to  $C$  and  $D$ , so it is the osculating plane to  $C$ . We have thus proved that through a general point  $F$  of  $S$  there is a curve  $C$  whose osculating plane at  $F$  coincides with the tangent plane to  $S$ . The tangent line to  $C$ , which is a general line of  $B$ , is therefore an asymptotic tangent line of  $S$ . This proves our claim.

If the strict focal locus is a curve  $C$ , then we will show that  $X$  is not just a union of cones, as in the case in which the focal point has multiplicity 1, but a union of planes tangent to  $C$ . We proceed as in the previous case. Let  $r \in B$  be a general line and let  $F = [v]$  be its focus. Since  $F$  is a fundamental point for the family  $B$ , there is a curve  $Z$  in the Grassmannian passing through  $r$ , which represents the lines of  $B$  through  $F$ . It is easy to show that every lifting of  $Z$  through  $F$  has  $r$  as tangent line at  $F$ , so  $\eta_1$ , the tangent vector to  $Z$  at  $r$ , is such that  $\eta_1(v) = 0$ . But  $F$  is a focus with multiplicity two and  $\dim T_r B = 2$ , so it follows that every regular curve contained in  $B$ , passing through  $r$ , but with tangent vector  $\eta_2$  different from  $\eta_1$ , is such that  $\eta_2(v)$  belongs to the image of  $\eta_1$ . The focal curve  $C$  can be interpreted as a lifting of such a curve. Let  $w$  be its tangent vector at  $F$ . Then the plane generated by  $r$  and  $w$  contains also the tangent line to any lifting of  $Z$  at its intersection point with  $r$ . Let  $\varphi(t)$  be a local parametrization of such a lifting, with  $\varphi(0) = P \in r$ . Then  $\varphi'(0)$  lies in the plane generated by  $w$  and the direction of  $r$ . By repeated derivations, we see that all derivatives  $\varphi^{(k)}(0)$ , and hence the whole curve, are in this plane. Therefore every lifting of  $Z$  is a plane curve, which proves that the lines of  $B$  passing through  $F$  form a pencil, contained in the plane generated by  $r$  and by the tangent line to  $C$  at  $F$ .

Thus  $B$  is a ruled surface. In this case,  $X$  is called a 3-dimensional band. The precise definition is as follows (see [AG]):

DEFINITION 2.4. A variety  $X \subset \mathbb{P}^N$  is said to be a *3-dimensional band* if there exist two distinct curves  $C, D \subset X$ , not belonging both to the same space  $\mathbb{P}^3$ , and a birational equivalence  $\psi : C \rightarrow D$ , such that  $X$  is the closure of the union of the planes lying in the image of the morphism

$$\begin{aligned} f : C_0 &\longrightarrow \mathbb{G}(2, N) \\ p &\longrightarrow \langle \mathbb{T}_p C, \psi(p) \rangle, \end{aligned}$$

where  $C_0$  is a non-singular open subset of  $C$  contained in the domain of definition of  $\psi$ .

Table 2 describes all varieties ruled by a 2-dimensional family of lines with a focal point of multiplicity 2 on the general line. □

foci on a general line	strict focal locus	description
a point with multiplicity 2	the focal point sweeps a surface	union of asymptotic lines
	the focal point sweeps a curve	band
	the focal point is fixed	cone over a surface

TABLE 2. One double focus

By a direct calculation, it is possible to show that every variety obtained in Theorem 0.3 is a variety with degenerate Gauss mapping. The interesting point is that, whereas any curve can be obtained as the focal curve of a 3-dimensional variety with degenerate Gauss mapping, the focal surfaces must verify some special conditions. For instance, it is not true in general that for a surface there always exists a family of dimension 2 of bitangent lines.

**THEOREM 2.5.** *Let  $X \subset \mathbb{P}^N$  be a variety of dimension 3 with Gauss image of dimension 2, satisfying one of the conditions (a)–(d) in Theorem 0.4. Suppose that the strict focal locus of the family  $B$  of the fibres of the Gauss map of  $X$  has an irreducible component  $S$  of dimension 2. Then  $S$  is either a developable surface or a  $\Phi$  surface. Moreover, the lines of  $B$  are tangent to  $S$  and they are either asymptotic tangent lines or they admit a conjugate direction.*

*Proof.* If  $X$  is as in (d), then the assertion is clearly true. Hence we can assume that on a general fibre of the Gauss map there are two distinct foci. Let  $F_1 \in S$  be general. Then  $F_1$  is a focus on a non-focal line  $r$ , which contains also a second focus  $F_2$ . Thus there exist two tangent vectors  $\eta_1, \eta_2 \in T_r B$  such that, for all regular curves  $\{b_i(t)\} \subset B$ ,  $i = 1, 2$ , with  $b_i(0) = r$  and  $b'_i(0) = \eta_i$ , every lifting through  $F_i$  has  $r$  as tangent line at  $F_i$ . As a lifting of  $\{b_1(t)\}$ , we can choose a curve  $C_1 \subset S$ , with local parametrization  $\{c_1(t)\}$ , such that  $c_1(t)$  is a focus of the line  $b_1(t)$  for all  $t$ . Note that  $\text{Im}(\eta_1)$ , which is 1-dimensional, is generated by the tangent vector  $x'(0)$  for all choices of a lifting  $x(t)$  of  $b_1(t)$  with  $x(0) \neq F_1$ . Hence, as in the proof of Theorem 0.4, it follows that  $x'(0)$  belongs to the osculating plane of the curve  $C_1$ .

Next, assume that  $X$  satisfies conditions (a) or (b). Then the above construction can be repeated for the second focus  $F_2$  on  $r$  relative to the focal

surface  $S'$  to which it belongs, which coincides with  $S$  in case (a) or is the second component of the strict focal locus of  $X$  in case (b). This gives a second curve  $C_2 \subset S'$  passing through  $F_2$  and with  $\mathbb{T}_{F_1}C_1 = r = \mathbb{T}_{F_2}C_2$ . Let now  $D_2$  be the curve generated by the second focus of the lines  $b_1(t)$ . Similarly let  $D_1$  be the curve generated by the second focus of the lines  $b_2(t)$ . Note that  $C_1 \neq D_1$  and  $C_2 \neq D_2$ . We can choose a local parametrization for  $S$  of the form  $\psi(t, s)$ , where  $\psi(0, 0) = F_1$ , and  $\psi(t, 0)$  and  $\psi(0, s)$  are local parametrizations of  $C_1$  and  $D_1$ , respectively. By considering the other focus, we get a parametrization  $\varphi(t, s)$  of the second surface  $S'$  near  $F_2$  such that  $\varphi(t, 0)$  and  $\varphi(0, s)$  are local parametrizations of  $D_2$  and  $C_2$ , respectively. By comparing the tangent vectors, we get  $\psi_t = \varphi_s$ ,  $\varphi_t \in \langle \psi_t, \psi_{tt} \rangle$ ,  $\psi_s \in \langle \varphi_s, \varphi_{ss} \rangle$ , and also  $\psi_{tt} \in \langle \varphi_t, \varphi_s \rangle$ ,  $\varphi_{ss} \in \langle \psi_t, \psi_s \rangle$ , so that  $\psi_{ts} \in \langle \psi_t, \psi_s \rangle$ . Hence the pair of vectors  $(\psi_t, \psi_s)$  annihilates the second fundamental form of  $S$ , and these vectors represent conjugate directions.

If we are in case (c),  $F_2$  is a fundamental point for the family  $B$ , so there are infinitely many lines of  $B$  through  $F_2$ . Each of them contains also a second focus, describing a curve  $E$ . In this case we can find a local parametrization of  $S$ ,  $\psi(t, s)$ , centered at  $F_1$  and such that  $\psi(t, 0)$  describes  $C_1$  and  $\psi(0, s)$  describes  $E$ . Note that  $\psi_t(0, s)$  is the direction of the line of the ruling passing through  $\psi(0, s)$ , and  $\psi_{ts}$  is tangent at  $F_1$  to the cone of vertex  $F_2$  on the curve  $E$ , and therefore is contained in the tangent plane to this cone along  $r$ . But this plane is generated by  $\psi_t$  and  $\psi_s$ , so it coincides with the tangent plane to  $S$  at the point  $F_1$ . We conclude then as in the previous case.  $\square$

We close this section with a remark on the second fundamental form. It is known (see [GH]) that the second fundamental form of the varieties with degenerate Gauss mapping has non-empty singular locus. In particular, this singular locus is a point in the case of varieties of dimension 3 with Gauss image of dimension 2. Assume that  $X$  is such a variety, which is not a hypersurface. Then there is a connection between the properties of the second fundamental form and the configuration of focal points on the general fibre of the Gauss map of  $X$ . Indeed, if  $X$  is a variety with distinct focal points of multiplicity 1, then the dimension of the second osculating space is 5 and the second fundamental form is a pencil of conics with a point both as base and as singular locus. If  $X$  has one focal point of multiplicity 2 on the general line and is not a cone, then the dimension of the second osculating space is also 5, but the pencil of conics of the second fundamental form has a line as base locus. In the case of cones over a surface, the dimension of the second osculating space is 6 (in general), so the second fundamental form is a net of conics and the base locus can only be a point, coinciding with the singular point.

### 3. Varieties covered by lines

Let  $B \subset \mathbb{G}(1, N)$  be a family of lines in  $\mathbb{P}^N$  of dimension  $n \leq N - 2$ . Suppose that the union of the lines belonging to  $B$  is an algebraic variety  $X$  of dimension  $n + 1$ . When this condition holds, we do not expect in general to find any focal points. In particular, a general line of  $B$  cannot be focal. This allows us to consider the length of the focal locus on the general  $r \in B$ . It turns out that this length has a geometric interpretation in terms of fixed tangent spaces along  $r$ . Theorem 0.1 states that the length of the focal locus on  $r \in B$  is  $k$  if and only if  $X$  possesses a fixed tangent space of dimension  $k + 1$  along a general line  $r$ . We now prove this result.

*Proof of Theorem 0.1.* We can suppose without loss of generality that  $X$  is a hypersurface, i.e.,  $N = n + 2$ . Indeed, if  $X$  is not a hypersurface, we can project it to  $\mathbb{P}^{n+2}$ , and a general projection will not affect either its tangential properties or its focal properties. Let  $r$  be a general point of  $B$ . Suppose that on  $r$  there are exactly  $k$  focal points, counting multiplicity. These are the points where the characteristic map relative to  $r$ ,

$$\begin{array}{ccc} \chi(r) : T_r B \otimes \mathcal{O}_r & \longrightarrow & \mathcal{N}_{r|\mathbb{P}^N} \\ \wr \parallel & & \wr \parallel \\ \mathcal{O}_r^n & & \mathcal{O}_r(1)^{n+1} \end{array}$$

does not have maximal rank. If we choose projective coordinates  $x_0, x_1$  on  $r$ , by means of the natural identification given above, we can represent  $\chi(r)$  by an  $(n + 1) \times n$  matrix

$$A = \begin{pmatrix} l_{1,1} & \dots & l_{1,n} \\ \vdots & \dots & \vdots \\ l_{n+1,1} & \dots & l_{n+1,n} \end{pmatrix},$$

whose entries  $l_{i,j}$  are linear forms in  $x_0, x_1$ . Let us consider the minors (with sign) of  $A$  of maximal order,

$$\varphi_i = (-1)^{i+1} \det \begin{pmatrix} l_{1,1} & \dots & l_{1,n} \\ \vdots & \dots & \vdots \\ \widehat{l_{i,1}} & \dots & \widehat{l_{i,n}} \\ \vdots & \dots & \vdots \\ l_{n+1,1} & \dots & l_{n+1,n} \end{pmatrix}, \quad i = 1, \dots, n + 1.$$

The existence of  $k$  focal points implies that  $\varphi_1, \dots, \varphi_{n+1}$  have a common factor  $F$  of degree  $k$ , so we have  $\varphi_i = F\psi_i$ , where  $\psi_1, \dots, \psi_{n+1}$  are suitable polynomials of degree  $n - k$  in  $x_0, x_1$ . We are interested in finding vectors that are tangent to  $X$  at every point of  $r$ . Thus we seek normal vectors of



Fix two equations of the system above by choosing two indices  $1 \leq i_1 < i_2 \leq n$ . We multiply the first equation by  $l_{n-k+1, i_2}$ , and the second by  $l_{n-k+1, i_1}$  and subtract to get a homogeneous relation among  $\varphi_1, \dots, \varphi_{n-k}$  with coefficients of degree 2. Considering every possible choice of  $i_1$  and  $i_2$ , we obtain the homogeneous system

$$\begin{cases} A_{i_1, i_2}^{1, n-k} \varphi_1 + \dots + A_{i_1, i_2}^{n-k, n-k+1} \varphi_{n-k} = 0, \\ 1 \leq i_1 < i_2 \leq n. \end{cases}$$

Analogously, we can find homogeneous relations with coefficients of every degree between 2 and  $n - k$ , involving successively fewer minors. For the highest degree we have a system of  $\binom{n}{k}$  equations in 2 minors. For  $\varphi_1, \varphi_2$ , for instance, we have

$$\begin{cases} A_{i_1, \dots, i_{n-k}}^{1, 3, 4, \dots, n-k+1} \varphi_1 + A_{i_1, \dots, i_{n-k}}^{2, 3, \dots, n-k+1} \varphi_2 = 0, \\ 0 \leq i_1 < i_2 < \dots < i_{n-k} \leq n. \end{cases}$$

As the relations belonging to this system cannot all be trivial, we conclude that  $\varphi_1$  and  $\varphi_2$  must have a common factor of degree  $\geq k$ . Moreover, it is possible to prove that  $\varphi_1, \dots, \varphi_{n-k}$  have a common factor of degree  $k$ . In fact, consider the following system of relations (of degree  $n - k - 1$ ) among 3 minors, say  $\varphi_1, \varphi_2, \varphi_3$ :

$$\begin{cases} A_{i_1, \dots, i_{n-k-1}}^{1, 4, \dots, n-k+1} \varphi_1 + A_{i_1, \dots, i_{n-k-1}}^{2, 4, \dots, n-k+1} \varphi_2 + A_{i_1, \dots, i_{n-k-1}}^{3, 4, \dots, n-k+1} \varphi_3 = 0, \\ 0 \leq i_1 < i_2 < \dots < i_{n-k-1} \leq n. \end{cases}$$

Denote by  $F$  a common factor of degree  $k$  of  $\varphi_1, \varphi_2$ . If  $F \nmid \varphi_3$ , then there is a factor  $G$  of  $F$  such that  $G$  divides  $A_{i_1, \dots, i_{n-k-2}}^{3, 4, \dots, n-k}$  for any choice of  $i_1, \dots, i_{n-k-2}$ . This means that  $G$  divides both  $A_{i_1, \dots, i_{n-k-1}}^{1, 3, 4, \dots, n-k}$  and  $A_{i_1, \dots, i_{n-k-1}}^{2, 3, \dots, n-k}$ . Then  $\varphi_1$  and  $\varphi_2$  have a common factor of degree  $\geq k + \deg G$ , and we can check whether this new polynomial of higher degree and  $\varphi_3$  have a common factor of degree  $k$  or not. If the answer is negative, we can repeat the construction until we find such a factor, after deleting all common factors of  $A_{i_1, \dots, i_{n-k-1}}^{1, 3, 4, \dots, n-k}$  and  $A_{i_1, \dots, i_{n-k-1}}^{2, 3, \dots, n-k}$ .  $\square$

REMARK 4. If there are more than  $n$  focal points on a line  $r \in B$ , then  $r$  is a focal line.

Theorem 0.1 allows us to give a rough description of the focal locus of a variety  $X$  ruled by an  $n$ -dimensional family of lines, once we know the dimension of the constant tangent space along a general line.

DEFINITION 3.1. Let  $B \subset \mathbb{G}(1, N)$  be a subvariety of the Grassmannian such that its general element is not focal. Let  $k$  be the degree of the focal locus on a general element  $r \in B$ . Let us denote by  $U \subset B$  the open set consisting

of the lines on which the focal locus is a proper subscheme of length  $k$ . Then the closure in  $\mathbb{P}^N$  of the union of the focal loci on the lines of  $U$  is called the *strict focal locus* of  $B$ .

REMARK 5. If  $B \subset \mathbb{G}(1, N)$ ,  $\dim B = n$ , and  $n = k$ , then  $U$  is the open set of non-focal lines, and the strict focal locus is simply the closure in  $\mathbb{P}^N$  of the union of focal points on the non-focal lines of  $B$ .

The study of the strict focal locus enables us to formulate a pattern of classification of varieties of dimension  $n + 1$  ruled by an  $n$ -dimensional family of lines. First, any such variety is characterized by the number and the multiplicity of the distinct focal points on a general line. Then we can study the strict focal locus of  $X$  and, in particular, the number of components and their dimensions. The maximal possible dimension for a component of the strict focal locus is  $n$ . If there is a component of dimension  $< n$ , then for every point of it there are infinitely many lines of  $B$  passing through this point. Hence this component must be contained in the fundamental locus of the family  $B$ . If there are components of dimension  $n$  of the focal locus, then every line of  $B$  is tangent to them. This is a particular case of a property of the focal locus that also holds for varieties ruled by linear subspaces of dimension  $\geq 1$ . We will establish this property in the general case.

THEOREM 3.2. *Let  $B \subset \mathbb{G}(k, N)$  be a family of  $k$ -spaces in  $\mathbb{P}^N$  of dimension  $n \leq N - k$ . Suppose that the union of the  $k$ -planes belonging to  $B$  is a variety  $X$  of dimension  $k + n$ , and that the focal locus has codimension 1 in  $X$ . Then every general subspace  $\Lambda$  belonging to  $B$  is tangent to  $F$  at all the focal points on  $\Lambda$  that are not fundamental points.*

*Proof.* Let us consider  $\mathcal{I}$ , the desingularization of the incidence correspondence of  $B$ , and the natural projections  $f, g$  defined by the following diagram:

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{f} & \mathbb{P}^N \\ g \downarrow & & \\ & & B \end{array}$$

Let  $p$  be a general point of  $g^{-1}(\Lambda)$ , belonging to the focal subvariety  $V(\chi) \subset \mathcal{I}$ . By definition, the differential of  $f$  in  $p$ ,

$$d_p f : T_p \mathcal{I} \longrightarrow T_{f(p)} \mathbb{P}^N,$$

has a non-trivial kernel. Let us consider the restriction of  $f$  to the focal subvariety,

$$f : \begin{array}{ccc} \mathcal{I} & \longrightarrow & \mathbb{P}^N \\ \cup & & \cup \\ V(\chi) & \longrightarrow & F. \end{array}$$

As we assumed  $\text{char } \mathbb{K} = 0$ , the algebraic geometry analogue of Sard's Theorem holds ([H, p. 176]), so for a general  $p \in V(\chi)$  the homomorphism  $d_p f|_{V(\chi)} : T_p V(\chi) \rightarrow T_{f(p)} F$  is surjective. Since not all focal points are fundamental points and  $f$  has finite fibres,  $\dim V(\chi) = \dim F$ . Thus  $d_p f|_{V(\chi)}$  is an isomorphism. Now consider again the differential of  $f$  in a general point  $p$  of  $V(\chi) \cap g^{-1}(\Lambda)$ ,

$$d_p f : T_p \mathcal{I} \rightarrow T_{f(p)} \mathbb{P}^N.$$

We know that  $d_p f$  has a non-trivial kernel and that its image contains  $T_{f(p)} F$ . Since  $V(\chi)$  is a codimension 1 subvariety of  $\mathcal{I}$ ,  $T_p V(\chi)$  is a linear subspace of codimension 1 in  $T_p \mathcal{I}$ . Hence  $d_p f(T_p \mathcal{I}) = T_p F$ . As  $g^{-1}(\Lambda) \subset T_p \mathcal{I}$ , we have  $\Lambda = d_p f(g^{-1}(\Lambda)) \subset T_{f(p)} F$ .  $\square$

Returning to varieties with constant tangent space along lines, assume that on a general line  $r \in B$  there are focal points which are not fundamental points. Then the strict focal locus has a component  $Y$  of pure codimension 1 in  $X$ , and every line in  $U$  is tangent to  $Y$  at its focal, non-fundamental points. In Theorem 0.1 we can consider the two extremal cases, namely,  $k = 0$  and  $k = n$ . In the first case, the theorem implies that a variety ruled by lines has no focal point on a general line if and only if the only fixed tangent space along a general line is the line itself. In the second case, we obtain a characterization of the varieties whose degenerate Gauss map has 1-dimensional fibres, i.e., varieties of dimension  $n + 1$  with tangent space constant along lines.

**COROLLARY 3.3.** *Let  $B \subset \mathbb{G}(1, N)$  be a family of lines in  $\mathbb{P}^N$  of dimension  $n$ ,  $n \leq N - 2$ . Suppose that the union of the lines belonging to  $B$  is an algebraic variety  $X$  of dimension  $n + 1$ . Then the following conditions are equivalent:*

- (i) *The focal locus on the general element  $r \in B$  consists of  $n$  points (counting multiplicity).*
- (ii) *The tangent space to  $X$  is constant along the lines of  $B$ .*

**REMARK 6.** If  $X$  is not ruled by linear subspaces of dimension  $\geq 2$ , then condition (i) implies that  $B$  is the family of the fibres of the Gauss map. If  $X$  possesses a higher dimensional ruling, then the fibres of the Gauss map can have dimension greater than 1.

#### 4. Varieties ruled by linear subspaces

In this section we investigate whether the results established in the previous section can be extended to varieties ruled by linear subspaces of dimension  $> 1$ . In particular, we expect that in the case of a family of linear subspaces of dimension  $k$  the existence of constant tangent spaces gives a focal hypersurface on the general  $k$ -space. This is true for varieties with degenerate Gauss mapping, for which Theorem 0.2 yields a straightforward generalization of Corollary 3.3.

*Proof of Theorem 0.2.* Let  $X \subset \mathbb{P}^N$  be a projective variety of dimension  $n + k$ , with Gauss map whose fibres have dimension  $k$ . We want to prove that condition (ii) holds for the family  $B \subset \mathbb{G}(k, N)$  of fibres of the Gauss map of  $X$ . Let  $\Lambda$  be a general element of  $B$ . Let us consider the characteristic map of  $B$  relative to  $\Lambda$ ,

$$\begin{array}{ccc} \chi(\Lambda) : T_\Lambda B \otimes \mathcal{O}_\Lambda & \longrightarrow & \mathcal{N}_{\Lambda|\mathbb{P}^N} \\ \wr \Big\| & & \wr \Big\| \\ \mathcal{O}_\Lambda^n & \longrightarrow & \mathcal{O}_\Lambda^{N-k}(1) \end{array}$$

The map  $\chi(\Lambda)$  is represented by an  $n \times (N - k)$  matrix, whose entries are linear forms on  $\Lambda$ . The columns of this matrix evaluated in a point  $p \in \Lambda$  can be regarded as vectors  $L_1(p), \dots, L_n(p)$  in  $V/\hat{\Lambda}$ . Let us denote by  $\Pi$  the fixed tangent space to  $X$  along  $\Lambda$ . Then the image of  $\chi(\Lambda)$  in any point  $p \in \Lambda$  is contained in  $\hat{\Pi}/\hat{\Lambda}$ , which is a fixed subspace of  $V/\hat{\Lambda}$  of dimension  $n$ . If we consider the coordinates of the normal vectors  $L_1(p), \dots, L_n(p)$  in  $\hat{\Pi}/\hat{\Lambda}$ , we get a matrix  $(m_{ij})_{i,j=1,\dots,n}$ . Then the condition defining the focal locus on  $\Lambda$  is

$$\det(m_{ij}) = 0,$$

which in general gives a hypersurface on  $\Lambda$  of degree  $n$ , even though in special cases it is possible that all of  $\Lambda$  is focal. Suppose now that  $B \subset \mathbb{G}(n, k)$  satisfies condition (ii). If we fix a general  $\Lambda \in B$ , the focal variety on  $\Lambda$  is a hypersurface of degree  $n$ . Then on the general line  $r \subset \Lambda$  there are  $n$  (not necessarily distinct) focal points, which are the points where the morphism

$$\lambda : T_\Lambda B \otimes \mathcal{O}_r \longrightarrow (\mathcal{N}_{\Lambda|\mathbb{P}^N})|_r,$$

given by the restriction of the characteristic map, has not maximal rank. We can adapt to  $\lambda$  the procedure applied in the proof of the implication (ii)  $\implies$  (i) of Theorem 0.1. In this way, we see that there is a fixed subspace  $W(r)$  of dimension  $n$  contained in the image of the characteristic map in any point of  $r$ . Now choose a general point  $p$  in  $\Lambda$ , so that, in particular,  $p$  is non-focal and smooth. We restrict to an affine open set  $U_0 \subset \mathbb{P}^N$  and consider a system of affine coordinates on  $U_0$  such that  $p$  is the point  $(0, \dots, 0)$ . Then  $\Lambda_0 = \Lambda \cap U_0$  is a linear space of dimension  $k$ . We can fix  $k$  lines  $r_1, \dots, r_k$  through  $p$  spanning  $\Lambda_0$ , such that for all  $j$  on  $r_j$  there are  $n$  focal points (considered with multiplicity). On any  $r_j$  there is a fixed tangent subspace, spanned by  $r_j$  and  $W(r_j)$ . Thus the tangent space to  $X$  in  $p$  contains all lines  $r_1, \dots, r_k$  (spanning  $\Lambda_0$ ) and all linear subspaces  $W(r_1), \dots, W(r_k)$ . For dimensional reasons, this implies that all the spaces  $W(r_j)$  must coincide. We thus have found a fixed linear space  $W$  of dimension  $n$ , such that in any smooth point of  $\Lambda_0$  the tangent space to  $X$  is spanned by  $\Lambda_0$  and  $W$ .  $\square$

In the general case of varieties ruled by lines it was possible to find non-focal lines on which there were more than the general number of focal points. Under the hypotheses of Theorem 0.2, the open set of subspaces on which the focal locus has degree  $k$  coincides with the set of non-focal subspaces. Thus Theorem 0.2 allows us to describe the possible characterizations of the strict focal locus for varieties with degenerate Gauss mapping. In this case, the strict focal locus is defined as the closure in  $\mathbb{P}^N$  of the union of the focal points on non-focal subspaces. For varieties with degenerate Gauss mapping, the focal locus is contained in the singular locus of the variety. The converse is not true in general.

**THEOREM 4.1.** *Let  $X$  be a variety with degenerate Gauss mapping, and denote by  $B$  the family of fibres of the Gauss map of  $X$ . Then the focal points of  $B$  are singular points of  $X$ .*

*Proof.* Recall that the focal points are the ramification points of the projection  $f : \mathcal{I} \rightarrow X$  from the desingularization of the incidence correspondence of  $B$  to  $X$ . As the degree of  $f$  is 1, either the focal points are points where  $f$  is not finite, or they are necessarily non-normal points of  $X$ . In the former case, they are fundamental points of  $B$ ; in the latter case, they are a fortiori singular points of  $X$ . Since through a fundamental point there pass at least two different fibres of the Gauss map, the fundamental points are also always singular.  $\square$

**REMARK 7.** This theorem can be extended to every variety  $X$  ruled by a family  $B$ , such that the projection from the desingularized incidence correspondence to  $X$  has degree 1. In this case all focal points not belonging to the fundamental locus are non-normal points, but nothing can be said about fundamental points.

Theorem 0.2 suggests that the following more general equivalence holds: Given a variety  $X$  of dimension  $n + k$  covered by a family  $B$  of  $k$ -spaces with  $\dim B = n$ ,  $X$  possesses a constant tangent space of dimension  $k + h$  along a general  $\Lambda \in B$  if and only if the focal locus of  $B$  on a general  $\Lambda \in B$  is a hypersurface of degree  $h$ . Unfortunately, there are counterexamples to this equivalence even for the first possible non-trivial case, namely for varieties ruled by a family of planes with focal lines. Observe that this case is the simplest possible case that is not covered by Theorem 0.1 or by Theorem 0.2.

**EXAMPLE 4.2.** We give two examples of varieties of dimension 4 ruled by a 2-dimensional family  $B$  of planes, with a focal line on the general  $\Lambda \in B$ . We show that the tangential properties along the planes of the ruling are not the same in the two cases.

Let us consider a variety  $Y$  of dimension 3 ruled by lines, with a fixed tangent plane along the general line of the ruling, but no higher dimensional

constant tangent space. Then the family of tangent planes has a focal line on the general element, and this line is precisely the line of the ruling of  $Y$ . In this case it is possible to prove that the union of the family of tangent planes is a variety  $X$  of dimension 4 with a fixed tangent  $\mathbb{P}^3$  along every plane. Hence, for the variety  $X$  the relationship between the dimension of the fixed tangent space along the planes of the ruling and the degree of the focal locus holds.

Now, let  $Z$  be a variety of dimension 3 ruled by lines, with constant tangent space along the lines of the ruling. Denote by  $B$  the 2-dimensional family of such lines, i.e., (in general) the family of the fibres of the Gauss map. Then we can choose a family  $C \subset \mathbb{G}(2, N)$  of planes such that, for every line  $r$  in  $B$ , there is a plane in  $C$  containing  $r$  and lying in the constant tangent space to  $Z$  along  $r$ . On a general plane  $\Pi$  in  $C$  the line  $r$  of  $B$  such that  $r \subset \Pi$  is a focal line. Assume that the union of the planes of  $C$  is a variety  $X$  of dimension 4. It is possible to prove that along a general line in  $\Pi$  there is a constant  $\mathbb{P}^3$  tangent to  $X$ , but that this  $\mathbb{P}^3$  depends on the chosen line, so that there is no constant  $\mathbb{P}^3$  tangent to  $X$  along  $\Pi$ . This example shows therefore that the equivalence mentioned above is not always valid.

In conclusion, all we know in general cases is that if a variety  $X$  of dimension  $n+k$ , ruled by an  $n$ -dimensional family of  $k$ -spaces, possesses a fixed space of dimension  $k+h$  tangent along a general  $k$ -space, then the focal locus on the general  $k$ -space of the ruling must contain a hypersurface of degree  $\geq h$ . If we know the degree of the focal locus, we only know the maximal dimension of a space tangent to  $X$  along the general line lying in a space of the ruling, which can vary with the choice of the line.

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