WIENER'S TESTS FOR ATOMIC MARKOV CHAINS[1]

BY
B. H. MURDOCH

1. Introduction

In a paper of Itô and McKean [1] a criterion, known as Wiener's test, is obtained for deciding whether a set of states in the simple d-dimensional random walk will be visited finitely often with probability 0 or 1. Lamperti has shown [2] that a similar test is valid for a class of Markov chains which includes all d-dimensional random walks with zero means and finite second moments.

In this paper we obtain necessary and sufficient conditions for the existence of Wiener-type tests in arbitrary discrete parameter Markov chains with stationary transition probabilities.

We follow closely the terminology of Chung [3]. The state space I is taken as the set of positive integers. $P = (p_{ij})$ $(i, j \in I)$ is the matrix of (one-step) transition probabilities, its n^{th} power $P^n = (p_{ij}^{(n)})$ is then the matrix of n-step transition probabilities. Ω is the set of infinite sequences $\omega = (i_0, i_1, \cdots)$ with $i_t \in I$ $(t \geq 0)$. The probability measure $\Pr(\cdot)$ on Ω is fully determined once the initial probability distribution of states i_0 is known, $p_j = \Pr(i_0 = j)$. Usually we will only be interested in conditional probabilities $\Pr(\cdot \mid i_0 = j)$ where the initial state is fixed. The successive states of a sample path are labelled $x_0, x_1, \cdots$ and we say that the Markov chain is in state i at time n if $x_n = i$. For any element $\omega = (i_0, i_1, \cdots)$ of Ω we write $x_n(\omega) = i_n$.

We define the Green's function of the chain

$$G_{ij} = \textstyle\sum_{n=0}^{\infty} p_{ij}^{(n)} \tag{1.1}$$

where for convenience we put $p_{ij}^{(0)} = \delta_{ij}$. If G_{ii} is finite we say that the state i is transient, otherwise it is recurrent. The set R of recurrent states of I can be divided into disjoint recurrent classes $R_u = \{i : G_{ui} > 0\}$ corresponding to certain recurrent states u.

For any set of states A we define the functions

$$f_m(i, A) = \Pr(x_n(\omega) \in A \quad \text{for at least} \quad m \quad \text{values of} \quad n \mid x_0(\omega) = i) \tag{1.2}$$

$$\begin{aligned} h(i, A) &= f_\infty(i, A) \\ &= \lim_{m\to\infty} f_m(i, A) = \Pr(x_n(\omega) \in A \quad \text{infinitely often} \quad \mid x_0(\omega) = i) \end{aligned} \tag{1.3}$$

$$\begin{aligned} e_m(i, A) &= f_m(i, A) - f_{m+1}(i, A) \\ &= \Pr(x_n(\omega) \in A \quad \text{exactly} \quad m \quad \text{times} \quad \mid x_0(\omega) = i). \end{aligned} \tag{1.4}$$

Received November 8, 1966.

[1] This research was supported in part by a U. S. Army contract.

We put $e(i, A) = e_0(i, A)$, $e_{ij} = e(i, \{j\})$, $f(i, A) = f_1(i, A)$, $f_{ij} = f(i, \{j\})$ and note that $G_{ij} = f_{ij} G_{jj}$. By expanding $f_m(i, A)$ according to the position j and time n of the m^{th} last entry into A we obtain

$$f_m(i, A) = h(i, A) + \textstyle\sum_{n=0}^{\infty} \sum_{j \in A-R} p_{ij}^{(n)} e_m(j, A) \tag{1.5}$$
$$= h(i, A) + \textstyle\sum_{j \in A-R} G_{ij}\, e_m(j, A).$$

We say that a set A is transient (recurrent) from state i if $h(i, A) = 0\ (1)$. If A is transient (recurrent) from each state i then A is transient (recurrent), without qualification. If $f_{ij} > 0$ we say that state j is accessible from state i.

For any set A we denote by $L(A)$ and $\Gamma(A)$ the events

$$\lim_{n\to\infty} \inf \{x_n(\omega) \in A\} \quad \text{and} \quad \lim_{n\to\infty} \sup \{x_n(\omega) \in A\},$$

respectively, so that $h(i, A)$ is just $\Pr\{\Gamma(A) \mid x_0(A) = i\}$. The set A is said to be almost closed [3], [4] if

$$\Pr\{\Gamma(A)\} = \Pr\{L(A)\} > 0$$

for some initial distribution $\{p_j\}_1^{\infty}$ in which all $p_j > 0$. This is equivalent to saying that

$$\textstyle\sum_{j=1}^{\infty} p_j\, h(j, A) = \sum_{j=1}^{\infty} p_j \Pr\{L(A) \mid x_0(\omega) = j\} > 0$$

and hence to the requirement that

$$h(j, A) = \Pr\{L(A) \mid x_0(\omega) = j\} \not\equiv 0 \qquad (j \geq 1). \tag{1.6}$$

An almost closed set is said to be atomic if it does not contain two disjoint almost closed sets. It is non-atomic if it contains no atomic almost closed set. If I is atomic (non-atomic) the Markov chain is said to be simply atomic (simply non-atomic). If I contains no non-atomic almost closed set the Markov chain is said to be atomic, countably or multiply, according to whether I contains infinitely many disjoint atomic almost closed sets or not.

By considering the subdivision of Ω into its atoms and its non-atomic part Blackwell [4] has shown that I can be decomposed into a finite or countable number of disjoint almost closed sets $C_1, C_2, \cdots$ of which at most one is non-atomic and the rest are atomic and also $\sum_k \Pr\{L(C_k)\} = 1$, or equivalently

$$\textstyle\sum_k h(j, C_k) = 1 \qquad (j \geq 1). \tag{1.7}$$

2. Preliminary results

Before proving our main theorem for transient atomic chains we will first need several lemmas.

LEMMA 1. *If A is a subset of*

$$E = \{j : f(s, j) \leq \gamma f(r, j)\} \tag{2.1}$$

then

$$f_m(s, A) \leq \gamma f_m(r, A) \qquad \textit{for} \quad m \geq 1$$

and also

$$h(s, A) \leq \gamma h(r, A).$$

Proof. We first put $I_\iota = \{1, 2, \cdots, \iota\}$ and define

$$A_\iota = A \cap (R \cup I_\iota) \qquad (\iota \geq 1). \tag{2.2}$$

Let $R_1, R_2, \cdots, R_\alpha$ be the recurrent classes which have elements, $u_1, u_2, \cdots, u_\alpha$ (say), in common with A. Since states in a recurrent class R_u are not accessible from states in any other recurrent class we see that $\Gamma(A \cap R)$ is the disjoint union of $\Gamma(R_1), \cdots, \Gamma(R_\alpha)$. Also $(A_\iota - R)$ consists of finitely many transient states and so is transient; therefore

$$h(i, A_\iota) = h(i, A \cap R) = \sum_{k=1}^{\alpha} h(i, R_k) = \sum_{k=1}^{\alpha} f(i, u_k). \tag{2.3}$$

The latter equality follows from the fact that a sample path which passes through u_k must be in R_k infinitely often, and conversely, with probability 1.

Using the expansion (1.5) and (2.3) we now obtain

$$f_m(i, A_\iota) = \sum_{k=1}^{\alpha} f(i, u_k) + \sum_{j \in A_\iota - R} f(i, j) G_{jj} e_m(j, A_\iota). \tag{2.4}$$

From the definition of E it then follows that

$$f_m(s, A_\iota) \leq \gamma f_m(r, A_\iota) \tag{2.5}$$

for all positive finite ι, m. Letting $\iota \to \infty$ we derive the desired result for finite m. Finally letting $m \to \infty$ we obtain the corresponding result for $h(\cdot, A)$.

In the case of a simply atomic chain we deduce the

COROLLARY. *In a simply atomic Markov chain the set E is transient (recurrent) if $\gamma < 1$ $(\gamma > 1)$.*

Proof. In such a chain every set is either transient or recurrent [4]. If $\gamma < 1$ then

$$h(s, E) \leq \gamma h(r, E) < 1$$

and the latter possibility is excluded, so that E is transient. If $\gamma > 1$ then $I - E$ is contained in the transient set

$$\{j : f(r, j) \leq \gamma^{-1} f(s, j)\}.$$

Thus $I - E$ is transient and so E is recurrent.

Remark. Analogous results to those in the lemma and its corollary can be obtained in a similar manner with E replaced by

$$E_1 = \{j : f(s, j) < \gamma f(r, j)\}.$$

If in addition all states of the chain are transient then E, E_1, are the same sets as

$$E_2 = \{j : G_{sj} \leq \gamma G_{rj}\}, \qquad E_3 = \{j : G_{sj} < \gamma G_{rj}\},$$

and the lemma and its corollary could be equally well stated in terms of them. In the general case, when some states may be recurrent, both the lemma and its corollary would be false if stated in terms of E_2, the corollary also being false for E_3.

In the case $\gamma = 1$ it is a simple matter to construct examples of E_1, E_3, which are transient and others which are recurrent. It is undecided whether the same is true for E or E_2.

In general the union of a finite number of transient sets is transient but the union of a countable number of such sets may not be. In the latter case we can prove the weaker

LEMMA 2. *If $T_1, T_2, \cdots$ are transient sets in a Markov chain then we can construct a transient set T such that $T_k - T$ is a finite set for each $k \geq 1$.*

Proof. For each positive i, k, we have

$$\lim_{\iota\to\infty} f(i, T_k - I_\iota) = h(i, T_k) = 0 \tag{2.6}$$

and therefore there exist positive integers ι_{ik} such that

$$f(i, T_{ik}) < 2^{-k} \qquad (i, k \geq 1) \tag{2.7}$$

where

$$T_{ik} = T_k - I_{\iota_{ik}} \qquad (i, k \geq 1).$$

If we now put

$$U_k = \bigcap_{i=1}^k T_{ik}, \qquad T = \bigcup_{k=1}^\infty U_k,$$

we see firstly that $f(i, U_k) < 2^{-k}$ for $1 \leq i \leq k$, and hence that

$$h(i, T) = h(i, \textstyle\bigcup_{k=\iota}^\infty U_k) \leq f(i, \bigcup_{k=\iota}^\infty U_k) \leq \sum_{k=\iota}^\infty f(i, U_k) < 2^{1-\iota} \tag{2.8}$$

for $i \leq \iota$, since each U_k is transient, and so $h(i, T) = 0$ for all positive i. The proof is completed by noting that $T_k - T$ contains at most the states $1 \leq j \leq \sup_{1\leq i\leq k} \iota_{ik}$.

Remark. In a similar, but simpler, manner one can show that if $T_1, T_2, \cdots$ are transient from a fixed state i then a set T can be constructed which is also transient from i and such that $T_k - T$ is finite for each positive k.

Before proceeding further we need to state some of the properties of the almost closed sets which we shall need in later proofs. These will constitute

LEMMA 3. *Any Markov chain can be decomposed into disjoint almost closed sets $C_1, C_2, \cdots$ of which at most one C_k is non-atomic, the rest being atomic, such that*

(i) *for any set A we have*

$$h(i, A) = \sum_k h(i, A \cap C_k) \qquad (i \geq 1) \tag{2.9}$$

and if C_k is atomic then $h(i, A \cap C_k)$ is either $h(i, C_k)$ or 0 identically;

(ii) *for any $\gamma < 1$ the sets*

$$D_k = \{j : h(j, C_k) \leq \gamma, j \in C_k\} \tag{2.10}$$

are transient and hence also is $D = \bigcup D_k$;

(iii) *if C_k contains recurrent states then*

$$f(j, C_k) = h(j, C_k) = h(j, \iota) \tag{2.11}$$

for any recurrent state ι in C_k;

(iv) $f(j, C_k) > 0$ *if and only if* $h(j, C_k) > 0$.

Proof. (i) Since the sets C_k are almost closed the intersection of any pair of events $\Gamma(A \cap C_k)$, $\Gamma(A \cap C_\iota)$, has probability 0. Therefore

$$\begin{aligned} h(i, A) &= \Pr(\Gamma\{\bigcup (A \cap C_k)\} \mid x_0(\omega) = i) \\ &= \sum \Pr(\Gamma(A \cap C_k) \mid x_0(\omega) = i) = \sum h(i, A \cap C_k). \end{aligned} \tag{2.12}$$

By the definition of atomicity the probability

$$\Pr(\Gamma(A \cap C_k) \mid x_0(\omega) = i)$$

must be either $\Pr(\Gamma(C_k) \mid x_0(\omega) = i)$, or 0, identically.

(ii) This is just a restatement of a corollary on page 109 of [3] with the invariant set $\Lambda = \Gamma(C_k)$. The transience of D follows from part (i).

(iii) As pointed out by Blackwell [4] the sets of the decomposition are only unique modulo transient sets, also the recurrent classes R_u of the Markov chain can be chosen as some of the sets of the decomposition. We assume then that we have a particular decomposition $\tilde{C}_1, \tilde{C}_2, \cdots$ containing all the R_u among its members. We then remove from each $\tilde{C}_k$ the transient set D_k defined in (2.10) with $\gamma = \frac{1}{2}$. The states of D we then add to the various $C_k' = \tilde{C}_k - D_k$ as follows.

Firstly, to any recurrent class C_k' we add those states j for which $h(j, C_k') = 1$, to obtain the corresponding C_k. We then divide the remaining states j of D among the other sets C_k' in any manner such that j is added to C_k' if $h(j, C_k') > \frac{1}{2}$ and otherwise j is added to any C_k' for which $h(j, C_k') > 0$. The resulting sets are then the required C_k. Since we have only added or subtracted transient sets all the functions $h(j, C_k)$, $h(i, A \cap C_k)$, are unaltered.

It then follows that (2.11) holds for any recurrent ι in C_k since a sample path which enters C_k enters every ι in $\tilde{C}_k$ infinitely often with probability one.

(iv) For any j, k, for which $f(j, C_k) > 0$ it follows that there is a state ι in C_k such that $f(j, \iota) > 0$. Combining this with the fact that by our construction $h(\iota, C_k) > 0$ we immediately deduce that $h(j, C_k) > 0$. Since we always have $f(j, C_k) \geq h(j, C_k)$ we see that the proof of (iv) is complete.

We will now proceed to prove a group of lemmas concerning the behaviour of the functions $f(r, j)$ for large j in the various types of Markov chains. The basic result is

LEMMA 4. *Let $0 < \gamma < 1$; then a Markov chain is atomic if and only if the state space I can be divided into a transient set T and almost closed sets $A_1, A_2, \cdots$ such that, for each positive k, r, s, for which $h(r, A_k)h(s, A_k) > 0$ the inequality*

$$f(s, j)h(r, A_k) > \gamma f(r, j)h(s, A_k) \tag{2.13}$$

holds for all recurrent j in A_k and all sufficiently large j in A_k.

Proof. We suppose first that the chain is atomic and take $C_1, C_2, \cdots$ as the sets of the decomposition of Lemma 3. For each k, r, s for which

$$h(r, C_k)h(s, C_k) > 0 \tag{2.4}$$

we define

$$T_{rs}^k = \{j : f(s, j)h(r, C_k) \leq \gamma f(r, j)h(s, C_k), j \in C_k\}$$

From Lemma 1 we deduce that

$$h(s, T_{rs}^k)h(r, C_k) \leq \gamma h(r, T_{rs}^k)h(s, C_k). \tag{2.15}$$

From Lemma 3 and the atomicity of C_k it follows that either $h(i, T_{rs}^k) = h(i, C_k)$, or 0, identically. Since the first possibility contradicts (2.15) we deduce that T_{rs}^k is transient when (2.14) holds.

By Lemma 2 we can construct a transient set T_k in C_k such that $(T_{rs}^k - T_k)$ is finite for each positive triple k, r, s, satisfying (2.14). We then put $A_k = C_k - T_k$ and $T = \bigcup T_k$. From Lemma 3 we see that T is transient and each A_k is an atomic almost closed set. From the definitions of T, T_{rs}^k, and the fact that $h(i, A_k) = h(i, C_k)$ for all positive i, k, we can deduce (2.13) for all sufficiently large j in A_k when $h(r, A_k)h(s, A_k) > 0$. If j is a recurrent state in A_k then $f(i, j) = h(i, A_k)$ for all positive i and so also in this case (2.13) holds when $h(r, A_k)h(s, A_k) > 0$.

Conversely if (2.13) holds as required we suppose that some set A_k is not atomic, that is to say that A_k contains two disjoint almost closed sets B, C. If $h(r, A_k)h(s, A_k) > 0$ we see from Lemma 1 that

$$h(s, B')h(r, A_k) \geq \gamma h(r, B')h(s, A_k)$$

where B' is the subset of B where (2.13) holds. Since in this case $(B - B')$ is a finite set of transient states we can immediately deduce that

$$h(s, B)h(r, A_k) \geq \gamma h(r, B)h(s, A_k) \tag{2.16}$$

This is also trivially true when $h(r, A_k)h(s, A_k) = 0$.

By the corollary on page 109 of [3] which we used to prove Lemma 3(ii) we can find states r, s, such that $h(r, B)$, $h(s, C)$ both exceed $(1 - \gamma/4)$ and hence $h(s, B) < \gamma/4$. Since these contradict (2.16) we see that A_k does not contain disjoint almost closed sets and so A_k is atomic. This completes the proof of the lemma.

Remarks. From Lemma 3(iv) it follows that if $h(r, A_k) = 0$ then $f(r, A_k) = 0$ and hence also $f(r, j) = 0$ for all j in A_k, and similarly with s instead of r. This means that, in general, either (2.13) holds under the required conditions or both sides of (2.13) are 0.

By suitably modifying the proof of the lemma in the case of a multiply atomic chain we can show that the lemma remains true for such chains with "almost closed sets $A_1, A_2, \cdots$" replaced by "a finite sequence of sets

$A_1, A_2, \cdots, A_n$". The A_k's in this case can only be shown to be subsets of almost closed sets.

From Lemma 4 we can obtain a sequence of transient sets T_ι corresponding to $\gamma_\iota = 1 - 2^{-\iota}$. By the method of Lemma 2 we can construct a set $\tilde{T}$ such that $(T_\iota - \tilde{T})$ is finite for positive ι. From (2.13) we then deduce the following corollary which could also be obtained from Martin boundary theory.

COROLLARY. *A Markov chain is atomic if and only if the state space I can be divided into a transient set $\tilde{T}$ and almost closed sets $\tilde{A}_1, \tilde{A}_2, \cdots$ such that for each positive k, r, s, we have*

$$f(s, j)/f(r, j) \to h(s, \tilde{A}_k)/h(r, \tilde{A}_k) \tag{2.17}$$

as $j \to \infty$ in $\tilde{A}_k$ (with the understanding that for recurrent states j in $\tilde{A}_k$ there is equality in (2.17) and also that we have a 0 in numerator and/or denominator on the right of (2.17) if and only if it occurs in the same position on the left of (2.17) for all j).

Proof. The sufficiency of (2.17) follows trivially from the lemma. To prove the necessity we put $\tilde{A}_k = C_k - \tilde{T}$ for each k. For any positive k, ι, r, s for which $h(r, \tilde{A}_k)h(s, \tilde{A}_k) > 0$ we have

$$\frac{f(s, j)}{f(r, j)} > (1 - 2^{-\iota}) \frac{h(s, \tilde{A}_k)}{h(r, \tilde{A}_k)}$$

for all recurrent j and all sufficiently large j in $\tilde{A}_k$, and similarly with r and s interchanged. Therefore (2.17) holds as required when $h(r, \tilde{A}_k)h(s, \tilde{A}_k) > 0$.

As in the above remarks in the case when $h(s, A_k)h(r, A_k) = 0$ it follows from Lemma 3(iv) that any zero on the right of (2.17) is matched by one in the same position on the left for all j.

We can also prove results similar to Lemma 4 and its corollary giving criteria for individual sets C_k to be atomic. In (2.13) and (2.17), k is then fixed and r, s, are in C_k, the condition (2.14) now being superfluous. We can also prove a somewhat simpler

LEMMA 5. *Let $0 < \gamma < 1$; then the almost closed set C_k is atomic if and only if it is the union of a transient set T_k and an almost closed set A_k such that for each r, s, in A_k the inequality*

$$f(s, j) > \gamma f(r, j) \tag{2.18}$$

holds for all recurrent j in A_k and for all sufficiently large j in A_k.

Proof. By Lemma 3 the set

$$D_k = \{j : h(j, C_k) \leq \gamma, j \in C_k\}$$

is transient. For each r, s, in $(C_k - D_k)$ we define

$$T_{rs} = \{j : f(s, j) \leq \gamma f(r, j), j \in C_k\}.$$

From Lemma 1 we then deduce that

$$h(s, T_{rs}) \leq \gamma h(r, T_{rs}). \tag{2.19}$$

From Lemma 3 and the atomicity of C_k it follows that either $h(i, T_{rs}) = h(i, C_k)$, or 0, identically. Since the first possibility contradicts (2.19) with s in $(C_k - D_k)$ we see that T_{rs} is transient.

We then proceed exactly as in Lemma 4 to construct a transient set T, such that $(T_{rs} - T)$ is finite for each r, s, in $(C_k - D_k)$, and put $T_k = T \cup D_k$, $A_k = C_k - T_k$. We can easily check that (2.18) is satisfied as required.

The proof of the sufficiency of (2.18) follows similar lines to those of Lemma 4. Instead of (2.16) we get the inequality

$$h(s, B) \geq \gamma h(r, B)$$

for subsets B of A_k, r, s in A_k, and a contradiction follows, as before, if A_k is not atomic.

Remark. We can similarly prove a criterion for a Markov chain to be atomic with (2.18) holding in each of a sequence of almost closed sets A_k and the state space being the union of a transient set T and the sets $A_1, A_2, \cdots$.

We now prove a lemma concerning the behaviour of $f(i, j)$ as $i \to \infty$, rather than j, which is really a restatement of a result of Doob [5].

LEMMA 6. *If $0 < \delta < 1$ then the state space I in a Markov chain can be divided into a transient set V and a recurrent set W such that for each transient state j the inequality*

$$f(i, j) < \delta \tag{2.20}$$

holds for all sufficiently large i in W.

Proof. From the corollary to Theorem (2.1) in [5] it follows that for any transient state j we have $f(x_n, j) \to 0$ as $n \to \infty$ for almost all sample paths in Ω, which is equivalent to saying that

$$V_j = \{i : f(i, j) \geq \delta\}$$

is transient for each transient j. We then use Lemma 2 to construct a transient set V such that $(V_j - V)$ is finite for each transient j. Putting $W = I - V$ we see immediately that (2.20) holds as required.

Combining Lemmas 5 and 6 we can obtain a division of each atomic almost closed set C_k into "shells" analogous to those used by Lamperti in [2].

LEMMA 7. *Let $0 < \delta < 1$, $\frac{1}{2} < \gamma < 1$, then the almost closed set C is atomic, containing no recurrent states, if and only if C is the union of a transient set B and an infinite sequence of disjoint finite non-empty sets ("shells") $S_0 = \{s\}, S_1, S_2, \cdots$* such that

$$(C - S_0 - S_1 - \cdots - S_\iota)$$

is not transient and

$$(2.21) \qquad \gamma f(i, j) < f(s, j), \qquad f(j, i) < \delta f(s, i)$$

for $i \in S_\iota$, $j \in S_m$, $m \geq \iota + 2$.

Proof. We will first prove the necessity in the case when C is one of the atomic closed sets C_k of the decomposition of Lemma 3. We then choose A and T as the corresponding sets defined in Lemma 5 and take V, W as the sets defined in Lemma 6. Let s be an element of A so that, by Lemma 5, we have $f(s, j) > 0$ for all sufficiently large j in A. Therefore, putting

$$(2.22) \qquad E = \{i : i \in A \cap W, f(s, i) > 0\}, \qquad B = C - E,$$

we see immediately that B is transient.

Assuming that S_0, S_1, $\cdots$, $S_{\iota-1}$ have been defined as sets in E satisfying the required conditions we then put

$$(2.23) \qquad \begin{aligned} T_\iota &= \textstyle\bigcup_{n=0}^{\iota-1} S_n, \\ U_\iota &= \textstyle\bigcup_{i \in T_\iota} \{j : \gamma f(i, j) \geq f(s, j) \text{ and/or } f(j, i) \geq \delta f(s, i); j \in E\}. \end{aligned}$$

This latter set U_ι is finite since T_ι is finite and Lemmas 5, 6, imply that for i in E each of the inequalities

$$\gamma f(i, j) \geq f(s, j), \qquad f(j, i) \geq \delta f(s, i)$$

is satisfied for only finitely many j in E. We can then take S_ι as any finite non-empty set such that

$$U_\iota - T_\iota \subset S_\iota \subset I - T_\iota, \qquad \iota \in T_\iota \cup S_\iota \cup (I - E).$$

From the method of construction of the sets S_ι we see immediately that $E = \bigcup_0^\infty S_\iota$, each S_ι is transient and so $(C - S_0 - \cdots - S_\iota)$ is not transient, and the inequalities (2.21) are satisfied as required.

In the general case of an atomic almost closed set C' not among the C_k of Lemma 3 we remark that by [4] the decomposition is unique modulo transient sets and so $C' \triangle C_k$ is transient for some k. To obtain sets satisfying the conditions of the lemma we need only choose s in $A \cap C'$ in the above argument, then put

$$B' = (C' \cap B) \cup (C' - C)$$

and choose as our shells the sequence of non-empty sets of the form $S_\iota \cap C'$, in the natural order. This will still be an infinite sequence since each set $(S_\iota \cap C')$ is transient and their union

$$\textstyle\bigcup_{\iota=0}^\infty (S_\iota \cap C') = E \cap C'$$

is not transient.

The proof of the sufficiency of (2.21) follows similar lines to those of Lemma 4. We suppose that C is not atomic and so contains two almost closed sets

F, G. Since

$$h(s, F) + h(s, G) \leq h(s, C)$$

we must have one of the terms, $h(s, F)$ say, $\leq \frac{1}{2}$.
As in Lemma 4 we deduce from (2.21) that

$$\gamma\, h\,(i, F) \leq h\,(s, F) \leq \tfrac{1}{2}$$

for all i in $C - B$. As in Lemma 4 we need only choose i in $(F - B)$ such that $h(i, F) > (2\gamma)^{-1}$ to get a contradiction. Therefore C is atomic.

If C contains a recurrent state, $i \in S_\iota$ say, then it also contains the recurrent class R_i. If R_i is finite then $R_i \subset T_k$ for some k and so by atomicity $(C - T_k)$ is transient, contradicting one of the assumptions of the Lemma. On the other hand, if R_i is infinite then $R_i \cap S_m$ is non-empty for some $m \geq \iota + 2$ and hence there is a state j in S_m such that $f(j, i) = 1$ contradicting the second part of (2.21). This completes the proof of the lemma.

Remarks. In the simple case of a d-dimensional random walk with zero mean and finite second moments, as treated by Spitzer in [6], the construction of the shells given above simplifies for suitable δ, γ, to give us I divided up into concentric spherical shells with radii increasing geometrically, as in [6].

3. Wiener's test for atomic chains

At this point we can proceed to establish our form of Wiener's test either by using an analytical or a probabilistic argument as in [2] or [6]. We choose the analytical approach and note that the same idea could also be used in proving Lamperti's form of Wiener's test. Our results cannot in general be stated in terms of capacities as are those in [1], [2] and [6].

We will first obtain a criterion for transience or recurrence for simply atomic chains and then show how similar results can be obtained for more general atomic chains. In the following section of the paper we will consider arbitrary Markov chains and show that the tests we have obtained are not valid for any more general types of chains, so that our tests are, in a sense, best possible.

THEOREM 1. *In a Markov chain an atomic almost closed set C which does not contain recurrent states can be divided into shells* $\Sigma_0 = \{s\}, \Sigma_1, \Sigma_2, \cdots$ *such that an arbitrary set A in C is transient if and only if the series*

$$\sum_{\iota=0}^{\infty} f(s, A \cap \Sigma_\iota) \tag{3.1}$$

is convergent.

Proof. Let us assume that the set A is transient and take $B, S_0 = \{s\}, S_1, \cdots$ as the sets defined in Lemma 7. The sets

$$A_m = A \cap \textstyle\bigcup_{k=0}^{\infty} S_{m+2k}$$

are also transient for all $m \geq 0$ and we can find an integer p such that $f(s, A_m)$

$< \gamma/2$ for $m \geq p$. For such m we expand

$$f(i, A_m) = \sum_{j \in A_m} e(j, A_m) f(i, j) G_{jj} \tag{3.2}$$

as in (1.5). From (3.2) and (2.21) it then follows that

$$\sum_{j \in A_m - S_{m+2k}} e(j, A_m) f(i, j) G_{jj} < \gamma^{-1} f(s, A_m) < \tfrac{1}{2} \tag{3.3}$$

for $i \in S_{m+2k}$, $k \geq 0$, $m \geq p$, and so also

$$2 \sum_{j \in A \cap S_{m+2k}} e(j, A_m) f(i, j) G_{jj} \geq f(i, A \cap S_{m+2k}) \tag{3.4}$$

for $i \in A \cap S_{m+2k}$. From the minimal property of the function $f(i, A \cap S_{m+2k})$ among those functions which are regular and non-negative outside $A \cap S_{m+2k}$ and ≥ 1 in $A \cap S_{m+2k}$ we deduce that (3.4) holds for all i in I. Putting $i = s$ in (3.4) and summing from $k = 0$ to ∞ we obtain

$$\begin{aligned} 2f(s, A_m) &= 2 \sum_{j \in A_m} e(j, A_m) f(s, j) G_{jj} \\ &\geq \sum_{k=0}^{\infty} f(i, A \cap S_{m+2k}) \end{aligned} \tag{3.5}$$

for all $m \geq p$ which immediately shows that the series

$$\sum_{\iota=0}^{\infty} f(s, A \cap S_\iota) \tag{3.6}$$

is convergent for any transient set A.

Since B is transient we can choose integers $n_0 = 1 < n_1 < n_2 < \cdots$ such that

$$f(i, B \cap \{n_\iota, n_\iota + 1, \cdots\}) < 2^{-\iota} \tag{3.7}$$

for $1 \leq i \leq \iota$. If we now put

$$\Sigma_0 = S_0, \qquad \Sigma_\iota = S_\iota \cup (B \cap \{n_{\iota-1}, \cdots n_\iota - 1\}) \tag{3.8}$$

we can deduce that

$$f(s, A \cap \Sigma_\iota) \leq f(s, A \cap S_\iota) + 2^{1-\iota} \tag{3.9}$$

for $s \leq \iota$ and therefore from the convergence of (3.6) it follows that the series (3.1) is also convergent.

Conversely, if we are given a sequence of sets for which the series (3.1) converges the Borel-Cantelli Lemma immediately implies that A is transient from s and hence is transient, by Lemma 3, since s is chosen in Lemma 7 to satisfy $h(s, C) > 0$.

Remark. In the course of the proof of Theorem 1 we have really proved more than stated. From (3.4) we can also deduce that

$$\sum_{\iota=0}^{\infty} f(i, A \cap S_\iota)$$

is convergent for all i and from (3.7) and (3.8) it follows that

$$f(i, A \cap \Sigma_\iota) \leq f(i, A \cap S_\iota) + 2^{-\iota}$$

for $\iota \geq i$. Therefore the series

$$\sum_{\iota=0}^{\infty} f(i, A \cap \Sigma_\iota) \tag{3.10}$$

is convergent for all i if and only if A is transient, the converse trivially following from the theorem.

The function $h(i, A)$ can be expressed by Lemma 3 in the form

$$h(i, A) = \sum h(i, A \cap C_k) \tag{3.11}$$

where $C_1, C_2, \cdots$ is some decomposition of the state space I and for each atomic C_k either $h(i, A \cap C_k)$ is 0, if $A \cap C_k$ is transient, or $h(i, C_k)$ otherwise.

For an arbitrary set A and any atomic almost closed set C we define the function

$$\begin{aligned} \chi(A, C) &= 0 \quad \text{if the corresponding series (3.1) converges,} \\ &= 0 \quad \text{if } C \cap R \neq \emptyset, \quad A \cap C \cap R = \emptyset, \\ &= 1 \quad \text{otherwise,} \end{aligned} \tag{3.12}$$

where R is the set of recurrent states in I. We can then state for any decomposition $I = C_1 \cup C_2 \cup \cdots$ the

THEOREM 2. *In an atomic Markov chain the function $h(i, A)$ can be expressed in the form*

$$h(i, A) = \sum \chi(A, C_k) h(i, C_k) \tag{3.13}$$

or all sets A and all states i in I.

Proof. From (3.11) we see that it is only necessary to show that

$$h(i, A \cap C_k) = \chi(A, C_k) h(i, C_k) \tag{3.14}$$

for all $A \subset I$, $i \in I$, and all C_k. This follows from the above remarks and the definition (3.12) since $\chi(A, C_k) = 0$ if and only if $A \cap C_k$ is transient.

From Theorems 1 and 2 we can now deduce our general Wiener's tests for transience or recurrence of arbitrary sets in atomic Markov chains. Firstly the test for a simply atomic chain is

THEOREM 3. *In a transient simply atomic Markov chain there is a subdivision of the state space I into shells $\Sigma_0 = \{s\}, \Sigma_1, \Sigma_2, \cdots$ such that an arbitrary set A is transient or recurrent according to whether the series (3.1) is convergent or divergent.*

Proof. This is obtained directly from Theorem 1 by putting $C = I$ and noting that in a simply atomic chain a set A is either transient or recurrent.

In the case of a multiply or countably atomic Markov chain we take a decomposition of the state space into the recurrent classes $R_1, R_2, \cdots$ (with $\bigcup R_k = R$) and atomic almost closed sets $Q_1, Q_2, \cdots$ which do not contain recurrent states. We can then prove

THEOREM 4. *In an atomic Markov chain there is a subdivision of the state space I into the set R of recurrent states and the shells $\{\Sigma_\iota^1\}_{\iota=0}^\infty$, $\{\Sigma_\iota^2\}_{\iota=0}^\infty$, $\cdots$ such that an arbitrary set A is transient if and only if $A \cap R$ is empty and each series*

$$\sum_{\iota=0}^\infty f(s_k, A \cap \Sigma_\iota^k) \tag{3.15}$$

is convergent.

Proof. From Theorem 2 it follows that A is transient if and only if all the functions $\chi(A, Q_k) = 0$ and $\chi(A, R_k) = 0$. From Theorem 1 it follows that $\chi(A, Q_k) = 0$ if and only if the corresponding series (3.15) converges for a suitable s_k. From (3.12) it follows that each $\chi(A, R_k) = 0$ if and only if each $A \cap R_k$ is empty, which is equivalent to saying that $A \cap R$ is empty. This completes the proof of the theorem.

The corresponding test for recurrence is

THEOREM 5. *In an atomic Markov chain there is a subdivision of the state space I into sets R_1, R_2, $\cdots$ and shells $\{\Sigma_\iota^1\}_{\iota=0}^\infty$, $\{\Sigma_\iota^2\}_{\iota=0}^\infty$, $\cdots$ such that an arbitrary set A is recurrent if and only if each series (3.15) is divergent and each set $A \cap R_k$ is non-empty.*

Proof. From Theorem 2 it follows that A is recurrent if and only if all the functions $\chi(A, Q_k) = 1$ and $\chi(A, R_k) = 1$. From Theorem 1 it follows that $\chi(A, Q_k) = 1$ if and only if the corresponding series (3.15) diverges for a suitable s_k. From (3.12) it follows that $\chi(A, R_k) = 1$ if and only if each $A \cap R_k$ is non-empty. This completes the proof of the theorem.

Remarks. As in the remark after Theorem 1 we can show in Theorem 3 that A is transient if and only if the series (3.11) is convergent for all $i \geq 1$. In Theorem 4 we can similarly show that A is transient if and only if $A \cap R$ is empty and the series

$$\sum_{\iota=0}^\infty f(i, A \cap \Sigma_\iota^k) \tag{3.16}$$

is convergent for all k and all $i \geq 1$.

On the other hand we can also show that A is recurrent in Theorem 3 if and only if the series (3.11) is divergent for some positive i. Similarly in Theorem 5 the set A is recurrent if and only if each set $A \cap R_k$ is non-empty and each series (3.16) is divergent for some positive i.

In Theorem 3 we can still show that a set A is transient if and only if the series (3.11) is convergent for some positive i. This follows from the fact that the convergence of the series implies that A is transient from i, by the Borel-Cantelli Lemma, and by atomicity we see then that A is transient.

Theorems 1, 3, 4, 5 could also be stated and proved as tests for transience or recurrence from a fixed state s_0, by replacing s, s_1, s_2, $\cdots$ in series (3.1) and (3.15) by s_0. A set which is transient (recurrent) from s_0 is also transient (recurrent) from each state accessible from s_0. Thus, if all states in I, except possibly in a transient set, are accessible from s_0 then Theorems 4 and 5 can be stated with all the s_k in the series (3.15) replaced by the single state s_0.

4. Tests for general Markov chains

In a general Markov chain, which is not atomic, we will show that there are not tests for transience or recurrence of the type obtained in Theorems 3, 4, 5. Theorem 1 is of course true for any atomic almost closed set C in such a chain and Theorem 2 can be modified to remain true in the general case, by the addition of a single term $h(i, C_0)$ on the right of (3.13) where C_0 is the non-atomic part of I.

One can obtain similar tests to Theorems 1, 3, 4, 5 but with shells $\Sigma_0, \Sigma_1, \cdots$ which are not fixed in advance. The test for transience, with $R_1, R_2, \cdots, R$ defined as before, is

THEOREM 6. *A set A is transient if and only if $A \cap R$ is empty and $(I - R)$ can be divided into finite sets $\Sigma_0, \Sigma_1, \cdots$ such that*

$$\sum_{\iota=0}^{\infty} f(i, A \cap \Sigma_\iota) \tag{4.1}$$

is convergent for all positive i.

Proof. From the convergence of (4.1) we see immediately by the Borel-Cantelli Lemma that $(A - R)$ is transient from each state i and hence is transient. Since $A \cap R$ is empty then A is also transient.

Conversely, if A is transient, then $A \cap R$ is empty and we can choose integers $1 = n_0 < n_1 < n_2 < \cdots$ such that

$$f(i, A \cap \{n_\iota, n_\iota + 1, \cdots\}) < 2^{-\iota} \tag{4.2}$$

for $1 \leq i \leq \iota$, for each $\iota \geq 1$, as in the proof of Theorem 1. If we now define

$$\sum\nolimits_{\iota-1} = \{n_{\iota-1}, n_{\iota-1} + 1, \cdots, n_\iota - 1\} - R$$

for $\iota \geq 1$ we see that

$$f(i, \Sigma_\iota \cap A) < 2^{-\iota}$$

for $\iota \geq i$ and therefore the series (4.1) is convergent for all positive i. Obviously $I - R = \bigcup_{\iota=0}^{\infty} \Sigma_\iota$ and so the proof of the theorem is complete.

The corresponding test for recurrence is

THEOREM 7. *A set A is recurrent if and only if each set $A \cap R_k$ is non-empty and also the series (4.1) is divergent for all i in I and all sequences of finite sets $\Sigma_0, \Sigma_1, \cdots$ for which*

$$\Pr(i, L\{\textstyle\bigcup_{\iota=0}^{\infty} \Sigma_\iota\}) > 0. \tag{4.2}$$

Proof. Let A be a recurrent set, then it follows, as in the proof of Theorem

5, that each set $A \cap R_k$ is non-empty. Also we can deduce from the recurrence that

$$h(i, A \cap \{\bigcup_{\iota=0}^{\infty} \Sigma_\iota\}) \geq \Pr(i, L\{\bigcup_{\iota=0}^{\infty} \Sigma_\iota\}) > 0$$

and so the series (4.1) must diverge by the Borel-Cantelli Lemma, for all $i, \Sigma_0, \Sigma_1, \cdots$ for which (4.2) holds, as required.

If A is not recurrent then either some set $A \cap R_k$ is empty or, failing that, $\Pr(i, L\{I - A - R\}) > 0$.

In the latter case we choose $\Sigma_0, \Sigma_1, \cdots$ as any sequence of finite sets whose union is $(I - A - R)$ and let i be any state for which $h(i, I - A - R) > 0$. The series (4.1) is then trivially convergent and (4.2) is satisfied. The proof of the theorem is now complete.

We will say that there is a simple (multiple, countable) Wiener test for transience of a set if there is a sequence of disjoint finite sets S_ι^k, and real-valued functions $\phi_k(\cdot)$ defined for subsets of all the S_ι^k, such that an arbitrary set A is transient if and only if the series

$$\sum_{\iota=0}^{\infty} \phi_k(A \cap S_\iota^k) \tag{4.3}$$

is [are] convergent for $k = 1$ [$1 \leq k \leq N, 1 \leq k < \infty$]. We similarly say that there is a simple [multiple, countable] Wiener test for recurrence with the series (4.3) required to be divergent. By convention the series (4.3) is said to be divergent if any one term is infinite.

In the case of transience tests there is no loss of generality in assuming that each $\phi_k(\cdot)$ is non-negative. If A is non-transient then the divergence of (4.3) for some k implies the divergence of

$$\sum_{\iota=0}^{\infty} | \phi_k(A \cap S_\iota^k) |. \tag{4.4}$$

Applying (4.3) to the empty set $\emptyset$ we see that $\phi_k(\emptyset) = 0$ for all k. For an arbitrary transient set A we put

$$A^+ = \bigcup_{\phi_k(A \cap S_\iota^k) \geq 0} (A \cap S_\iota^k), \qquad A^- = A - A^+$$

from the transience of A^+ and A^- it follows that the series of positive, and of negative, terms of (4.3) are each convergent and hence (4.4) is convergent. Thus the tests for transience are still valid with $\phi_k(\cdot)$ replaced by $| \phi_k(\cdot) |$. We may also drop the subscript k on $\phi_k(\cdot)$ since the only subset common to any S_ι^k and $S_{\iota'}^{k'}$, with $k \neq k'$, is the empty set $\emptyset$ for which $\phi_k(\emptyset) \equiv 0$.

THEOREM 8. *There is a simple Wiener test for transience if and only if $(I - R)$ is the union of finitely many atomic almost closed sets.*

Proof. If $(I - R)$ satisfies the given condition we can choose sets Σ_ι^k by Theorem 4 with $1 \leq k \leq N$. We choose an infinite sequence of disjoint finite sets $\{S_m^1\}_{m=0}^{\infty}$ containing each set which is either of the form

$$\bigcup_{k=1}^{N} \Sigma_\iota^k$$

or consists of a single recurrent state $\{r\}$. We then put

$$\phi(\{r\}) = \infty, \qquad \phi(\emptyset) = 0$$

and

$$\phi(A \cap \{\textstyle\bigcup_{k=1}^{N} \Sigma_\iota^k\}) = \textstyle\sum_{k=1}^{N} f(s_k, A \cap S_\iota^k).$$

Since there are only finitely many terms in the latter sum, the series $\sum_{\iota=0}^{\infty} \phi(A \cap S_\iota^1)$ converges if and only if each series (3.15) is convergent and also A contains no recurrent states, that is to say, if and only if A is transient.

Conversely let us assume that $(I - R)$ contains infinitely many disjoint almost closed sets $B_1, B_2, \cdots$ and that there is a simple Wiener test for transience, with non-negative $\phi(A \cap S_\iota)$, of the form

$$\text{“}A \text{ is transient} \Leftrightarrow \textstyle\sum_{\iota=0}^{\infty} \phi(A \cap S_\iota) < \infty\text{”}.$$

Since none of the sets $B_1, B_2, \cdots$ is transient and each set $B_k \cap S_\iota$ is transient the series

$$\textstyle\sum_{\iota=0}^{\infty} \phi(B_k \cap S_\iota)$$

is a divergent series of finite terms for each $k \geq 1$. Thus we can choose integers $1 = \iota_0 < \iota_1 < \iota_2 < \cdots$ such that

$$\textstyle\sum_{\iota=\iota_{k-1}}^{\iota_k - 1} \phi(B_k \cap S_\iota) > 1 \qquad (k \geq 1) \tag{4.5}$$

If we now define

$$A = \textstyle\bigcup_{k=1}^{\infty} \bigcup_{\iota=\iota_{k-1}}^{\iota_k - 1} (B_k \cap S_\iota)$$

we see immediately from (4.5) that the series $\sum_{\iota=0}^{\infty} \phi(A \cap S_\iota)$ is divergent and hence A is not transient.

However from the transience of each $B_k \cap S_\iota$ we deduce that

$$\begin{aligned} \Pr(\Gamma(A)) &= \Pr(\Gamma\{A - \textstyle\bigcup_0^{\iota_M} S_\iota\}) \\ &\leq \textstyle\sum_{k=M}^{\infty} \Pr(\Gamma(B_k)) \end{aligned}$$

for each $M \geq 1$. From Lemma 3 it follows that the latter series converges and so $\Pr(\Gamma(A)) = 0$ and A is transient. We thus have a contradiction and it follows that $(I - R)$ is the union of finitely many atomic almost closed sets.

COROLLARY 1. *In a transient Markov chain there is a simple Wiener test for transience if and only if the chain is simply or multiply atomic.*

Proof. We need only note in this case that R is empty and hence I decomposes into finitely many atoms.

Remarks. In the proof of the theorem and its corollary we have tacitly assumed an initial probability distribution on I with $\Pr(x_0(\omega) = i) > 0$ for all positive i. If instead we assume $\Pr(x_0(\omega) = s) = 1$ for a particular s we can obtain similar results for transience from s. In particular there is a simple Wiener test for transience from s in a transient Markov chain if and only if only a finite number of disjoint atomic almost closed sets are accessible from

s. We can also prove results similar to those of the theorem giving necessary and sufficient conditions for the existence of Wiener tests for transience of subsets of a fixed set S.

We now prove the corresponding theorem on the existence of a countable Wiener test for transience. We need not concern ourselves with multiple Wiener tests for transience since these are seen to be equivalent to simple transience tests by replacing several series (4.3) by their sum.

THEOREM 9. *There is a countable Wiener test for transience if and only if I is atomic.*

Proof. If I is atomic we can choose sets Σ_ι^k by Theorem 4 whose union is $I - R$. For the case already covered by Theorem 8 we can then define S_ι^1 as before and put $S_\iota^k = \emptyset$ for all $\iota \geq 0$, $k \geq 2$, and obtain trivially a countable Wiener test for transience. In the other case when $(I - R)$ is the union of an infinite sequence $\{Q_k\}_{k=1}^{\infty}$ of atomic almost closed sets we then put $S_\iota^k = \Sigma_\iota^k$ for $\iota \geq 0$ when k is not a recurrent state and $S_0^k = \{k\}$, $S_{\iota+1}^k = \Sigma_\iota^k$ for $\iota \geq 0$ when k is recurrent. We then put $\phi(\{k\}) = \infty$ if k is recurrent, $\phi(\emptyset) = 0$ and otherwise

$$\phi(A \cap \Sigma_\iota^k) = f(s_k, A \cap \Sigma_\iota^k).$$

All the series (4.3) are then convergent if and only if A contains no recurrent states and also each series (3.15) is convergent, that is to say, if and only if A is transient.

Conversely let us assume that I contains a non-atomic almost closed set C and that there is a countable Wiener test for transience with non-negative $\phi_k(A \cap S_\iota^k)$. The series

$$\sum_{\iota=0}^{\infty} \phi_k(C \cap S_\iota^k) \tag{4.6}$$

must then diverge for some $k = \kappa$, say. For subsets A of

$$C' = C \cap (\textstyle\bigcup_{\iota=0}^{\infty} S_\iota^\kappa)$$

the convergence of (4.3), with $k = \kappa$, is then a necessary and sufficient condition for transience of A since the series (4.3) is trivially convergent to 0 for other values of k. C' is not transient, by the divergence of (4.6) with $k = \kappa$. Since C is non-atomic we can write it as the disjoint union of non-atomic almost closed sets $C_1, \cdots, C_m$ such that

$$\Pr(\Gamma(C_i)) < \Pr(\Gamma(C')) \qquad (1 \leq i \leq m).$$

Writing $C_i' = C_i \cap C'$ for $1 \leq i \leq m$ we deduce that at least two sets C_i', C_j', are non-transient. Repeating the same argument we can show that there is an infinite sequence of disjoint non-atomic almost closed sets $A_1, A_2, \cdots$ for which each $B_k = A_k \cap C'$ is non-transient. Applying the same argument as used in Theorem 8 we can construct a transient subset A of C' for which the series

$$\sum_{\iota=0}^{\infty} \phi(A \cap S_\iota^\kappa)$$

diverges, which gives us a contradiction. Therefore I contains no non-atomic almost closed set C, that is to say, I is atomic. This completes the proof of the Theorem.

We now turn to the question of the existence of Wiener tests for recurrence. In this case we need to assume that our set functions $\phi_k(\cdot)$ are subadditive, that is to say, that

$$\phi_k(A \cup B) \leq \phi_k(A) + \phi_k(B) \tag{4.7}$$

for any subsets A, B, of the same S_ι^k. Then trivially $\phi_k(\cdot)$ is non-negative for each k.

In the case of recurrence tests we get a more precise correspondence between the type of test and the type of atomic chain.

THEOREM 10. *There is a multiple* (N*-ple*) *Wiener test for recurrence if and only if* I *is multiply* (N*-ply*) *atomic.*

Proof. If I is N-ply atomic we can divide I into N atomic almost closed sets $C_1, C_2, \cdots, C_N$ by Lemma 3. Each C_k is then either a recurrent class or can be written, by Theorem 1, as a disjoint union of sets $\{\Sigma_\iota^k\}_{\iota=0}^\infty$. In the former case we put $S_0^k = C_k$, $S_\iota^k = \emptyset$ ($\iota \geq 1$), and $\phi_k(A \cap S_\iota^k) = 0$ or ∞ depending on whether $A \cap S_\iota^k$ is empty or not. In the latter case we put $S_\iota^k = \Sigma_\iota^k$ and

$$\phi_k(A \cap S_\iota^k) = f(s_k, A \cap S_\iota^k).$$

The divergence of (4.3) for each k in $[1, N]$ is then, by Theorem 5, a necessary and sufficient condition for the recurrence of A.

Conversely if there is a N-ple Wiener test for recurrence we suppose first that I contains $(N + 1)$ disjoint almost closed sets $B_1, \cdots, B_{N+1}$. Since the complementary sets $B_1^c, B_2^c, \cdots, B_{N+1}^c$ are not recurrent there is for each i in $[1, N + 1]$ an integer k_i in $[1, N]$ such that

$$\textstyle\sum_{\iota=0}^\infty \phi_{k_i}(B_i^c \cap S_\iota^{k_i}) < \infty.$$

Therefore $k_i = k_j = \kappa$, say, for some pair of distinct integers i, j. From the subadditivity of $\phi_\kappa(\cdot)$ we deduce that

$$\textstyle\sum_{\iota=0}^\infty \phi_\kappa((B_i^c \cup B_j^c) \cap S_\iota^\kappa) < \infty$$

and hence $I = B_i^c \cap B_j^c$ is not recurrent. This contradiction implies that I is M-ply atomic, with $M \leq N$.

Suppose now that $M < N$ and that I is the disjoint union of the atomic almost closed sets $D_1, \cdots, D_M$. By the recurrence test the sets $T_i = I - \bigcup_{\iota=0}^\infty S_\iota^i$, with $1 \leq i \leq N$, are not recurrent and hence $T_i \cap D_{h_i}$ is transient for some h_i in $[1, M]$. Therefore $h_i = h_j = \gamma$, say, for some pair of distinct integers i, j. We deduce that

$$(T_i \cap D_\gamma) \cup (T_j \cap D_\gamma) = D_\gamma$$

is transient, which is impossible. Hence $M = N$ and the proof of the theorem is complete.

The corresponding result for countably atomic chains is contained in

THEOREM 11. *There is a countable Wiener test for recurrence if and only if I is countably atomic.*

Proof. If I is countably atomic we can show as in Theorem 10 that there is a countable Wiener test for recurrence.

Conversely we can show that if there is a countable Wiener test then, by the method of Theorem 10, I cannot be expressed as the union of M atomic almost closed sets for any finite M. Suppose that I contains a non-atomic almost closed set C, so that C^c is not recurrent and hence the series (4.6) must converge for some $k = \kappa$, say. Therefore by (4.3) the set

$$D = C^c \cup (I - \bigcup_{\iota=0}^{\infty} S_\iota^\kappa)$$

is also not recurrent, so that

$$\Pr(\Gamma(D)) < 1.$$

Since C is non-atomic it can be written as a finite union $\bigcup_{i=1}^n C_i$ of non-atomic almost closed sets C_i for which

$$\Pr(\Gamma(C_i)) < 1 - \Pr(\Gamma(D)) \qquad (1 \leq i \leq n).$$

Hence we have also

$$\Pr(\Gamma(D \cup C_i)) < 1 \qquad (1 \leq i \leq n)$$

which means that each set $D \cup C_i$ is non-recurrent. By subadditivity

$$\sum_{\iota=0}^{\infty} \phi_k((D \cup C_i) \cap S_\iota^k) \geq \sum_{\iota=0}^{\infty} \phi_k(I \cap S_\iota^k) = \infty \qquad (k \neq \kappa)$$

since I is recurrent, so that the convergent series (4.3) must be

$$\sum_{\iota=0}^{\infty} \phi_\kappa((D \cup C_i) \cap S_\iota^\kappa) < \infty \qquad (1 \leq i \leq n).$$

From the subadditivity of $\phi_\kappa(\cdot)$ we then derive

$$\sum_{\iota=0}^{\infty} \phi((D \cup C) \cap S_\iota^\kappa) = \sum_{\iota=0}^{\infty} \phi_\kappa(I \cap S_\iota^\kappa) < \infty,$$

which is impossible, by the recurrence of I. Therefore I contains no non-atomic almost closed set and, together with the fact that I is not multiply atomic, this shows that I is countably atomic and the proof of the Theorem is complete.

We turn finally to the question which initiated this research, the existence of a Wiener test for a Random Walk on a group [7]. By such a random walk we mean a Markov chain in which the state space I forms a group under an operation $\circ$, with identity 1 and inverse $\cdot^{-1}$ such that $1 \circ i = i \circ 1 = i$, $i \circ i^{-1} = i^{-1} \circ i = 1$ for all i in I, and for which the transition probabilities p_{ij}, and hence also $p_{ij}^{(n)}$ and G_{ij}, are functions of $i^{-1} \circ j$.

We say that a Markov chain is indecomposable [3] if it is not possible to find two non-empty sets J, K in I such that $G_{jk} = 0 = G_{kj}$ for any pair $j \in J$, $k \in K$. In the case of a random walk on a group [7] this is equivalent to the definition of aperiodicity in [6], that I is the smallest group containing all states i for which $p_{1i} > 0$. This implies that for each i in I there are integers $n, i_1, i_2, \cdots, i_n, 1 = j_1, j_2, \cdots, j_n$ such that

$$p_{j_1 i_1} p_{j_2 i_1} p_{j_2 i_2} p_{j_3 i_2} \cdots p_{j_n i_n} p_{i i_n} > 0, \tag{4.8}$$

since the set of i for which this is true form a group which is contained in any group containing all i for which $p_{1i} > 0$.

We can now prove our

THEOREM 12. *An aperiodic random walk on a group, considered as a Markov chain, is either simply atomic or simply non-atomic.*

Proof. Suppose that I contains an atomic almost closed set C. We then choose a state ι for which

$$\delta = \Pr\{\Gamma(C) \mid x_0(\omega) = \iota\} > 0.$$

From the group property it then follows that

$$D = \iota^{-1} \circ C = \{\iota^{-1} \circ i : i \in C\}$$

is an atomic almost closed set for which

$$\Pr\{\Gamma(D) \mid x_0(\omega) = 1\} = \delta > 0.$$

Let $0 < \varepsilon < \frac{1}{2}\delta$ and choose by Lemma 3 a state m for which

$$\Pr\{\Gamma(m^{-1} \circ D) \mid x_0(\omega) = 1\} = \Pr\{\Gamma(D) \mid x_0(\omega) = m\} > 1 - \varepsilon > \tfrac{1}{2}.$$

Since D and $m^{-1} \circ D$ are both atomic almost closed sets it follows from Lemma 3 that $(m^{-1} \circ D)$ and D differ only by a transient set and so

$$\delta = \Pr\{\Gamma(D) \mid x_0(\omega) = 1\} = \Pr\{\Gamma(m^{-1} \circ D) \mid x_0(\omega) = 1\} > 1 - \varepsilon$$

for each positive $\varepsilon < \frac{1}{2}\delta$. Therefore

$$\Pr\{\Gamma(D) \mid x_0(\omega) = 1\} = 1. \tag{4.9}$$

If either $p_{jk} > 0$ or $p_{kj} > 0$ then

$$\Pr\{\Gamma(D) \mid x_0(\omega) = j\} = 1 \Rightarrow \Pr\{\Gamma(D) \mid x_0(\omega) = k\} > 0$$

and a similar argument shows immediately that

$$\Pr\{\Gamma(D) \mid x_0(\omega) = j\} = 1 \Rightarrow \Pr\{\Gamma(D) \mid x_0(\omega) = k\} = 1. \tag{4.10}$$

Due to the aperiodicity of the random walk we can find, for each i in I, integers $n, i_1, \cdots, i_n, 1 = j_1, \cdots, j_n$ satisfying (4.8). Therefore by a repeated application of (4.10) we deduce that

$$\Pr\{\Gamma(D) \mid x_0(\omega) = 1\} = 1 \Rightarrow \Pr\{\Gamma(D) \mid x_0(\omega) = i_1\}$$

$$(4.11) \qquad = 1 \Rightarrow \cdots \Rightarrow \Pr\{\Gamma(D) \mid x_0(\omega) = i_n\} = 1 \Rightarrow \Pr\{\Gamma(D) \mid x_0(\omega) = i\} = 1$$

for each i in I, which implies that I is simply atomic. Therefore either (a) I contains an atomic almost closed set and is itself simply atomic or (b) I contains no atomic almost closed set and so is simply non-atomic. This completes the proof of the theorem.

We can immediately deduce the

COROLLARY. *An aperiodic random walk on a group has either no Wiener test of any type or else has simple Wiener tests for transience and for recurrence.*

Proof. By Theorem 12 the random walk is either simply atomic and so by Theorems 8, 10 has simple Wiener tests for transience and recurrence or is simply non-atomic and so by Theorems 8–11 has no Wiener tests of any type.

Remark. We can also prove more generally that a general random walk on a group is either simply non-atomic or else it is atomic with one atomic almost closed set consisting of the smallest group G containing all states i for which $p_{1i} > 0$ and the other atomic almost closed sets being merely the cosets of G in I. We can then deduce that there is either no Wiener test for transience or recurrence from a state i or else there are simple Wiener tests for recurrence and for transience from each state i, since from i only states of the atomic almost closed set $i \circ G$ are accessible.

Note. By an application of the zero-or-one law for symmetric events [8] one can show that an aperiodic random walk on an Abelian group is necessarily simply atomic. On the other hand an aperiodic random walk on a non-Abelian group G may be either simply non-atomic, as for instance if G is the free group on $m \geq 2$ generators [9], or simply atomic, as in the following example.

Let G be the group generated by three elements a, b, c of infinite order for which

$$(4.12) \qquad ab = ba^{-1}, \qquad ac = ca, \qquad bc = cb.$$

We define a random walk on this group by putting

$$(4.13) \qquad \begin{aligned} p_{gh} &= 1/6 \quad \text{if} \quad g^{-1}h = a, a^{-1}, b, b^{-1}, c \quad \text{or} \quad c^{-1}, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Let A be an arbitrary subset of G and put

$$(4.14) \qquad \phi(x, y, z) = h(a^x b^y c^z, A).$$

From (4.12), (4.13) and (4.14) we can then deduce that

$$(4.15) \qquad \begin{aligned} 6\phi(x, y, z) = {}& \phi(x+1, y, z) + \phi(x-1, y, z) + \phi(x, y+1, z) \\ &+ \phi(x, y-1, z) + \phi(x, y, z+1) + \phi(x, y, z-1), \end{aligned}$$

since $a^x b^y c^z a^{\pm 1} = a^{x \pm (-1)^y} b^y c^z$. From the simple atomicity of the simple 3-dimensional random walk it follows that any bounded solutions of (4.15) are necessarily constant and hence by [4] the random walk on G is simply atomic.

REFERENCES

1. K. ITÔ AND H. J. MCKEAN JR., *Potentials and the random walk*, Illinois J. Math., vol. 4 (1960), pp. 119–132.
2. J. LAMPERTI, *Wiener's test and Markov chains*, J. Math. Anal. Appl., vol. 6 (1963), pp. 58–66.
3. K. L. CHUNG, *Markov chains with stationary transition probabilities*, Springer, Berlin, 1960.
4. D. BLACKWELL, *On transient Markov processes with a countable number of states and stationary transition probabilities*, Ann. Math. Statist., vol. 26 (1955), pp. 654–658.
5. J. L. DOOB, *Discrete potential theory and boundaries*, J. Math. Mech., vol. 8 (1959), pp. 433–458.
6. F. SPITZER, *Principles of random walk*, D. Van Nostrand, Princeton, 1964.
7. H. KESTEN, *Symmetric random walks on groups*, Trans. Amer. Math. Soc., vol. 92 (1959), pp. 336–354.
8. E. HEWITT AND L. J. SAVAGE, *Symmetric measures on Cartesian products*, Trans. Amer. Math. Soc., vol. 80 (1955), pp. 470–501.
9. E. B. DYNKIN AND M. B. MALJUTOV, *Random walk on groups with a finite number of generators*, Soviet Mathematics, vol. 2 (1961), pp. 399–402.

IMPERIAL COLLEGE
LONDON, ENGLAND
TRINITY COLLEGE
DUBLIN, IRELAND