SIMPLE TESTS FOR RECURRENCE OR TRANSIENCE OF INFINITE SETS IN RANDOM WALKS ON GROUPS

BY

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1. Introduction

It has been shown by R. A. Doney [3] that in the case of the simple three-dimensional random walk no condition of the type

$$\text{I} \qquad \sum_{a \in A} \phi(a) = \infty$$

with $\phi(a) \geq 0$ can be necessary and sufficient for a set A to be recurrent.

In this paper the analogous result is obtained for an arbitrary transient random walk on an Abelian group provided only that $G_{0i} \to 0$ as $i \to \infty$.

In what follows we will use the terminology and also some of the results of [6]. We assume that a countable group G is given with its elements numbered in some order $e = a_0, a_1, a_2, \cdots$. By a random walk on G we mean a Markov chain for which the probabilities

$$(1.1) \quad p_{ij}^{(n)} = \Pr(x_{m+n} = a_j : x_m = a_i) = \Pr(x_n = a_i^{-1} a_j : x_0 = e)$$

are functions of $a_i^{-1} a_j$, n, x_n denoting the element of G reached by the random walk at time n.

We also write

$$(1.2) \quad \begin{aligned} e(a, A) &= \Pr(x_n \notin A \quad \text{for } n > 0: x_0 = a) \\ f(a, A) &= \Pr(x_n \in A \quad \text{for some} \quad n \geq 0: x_0 = a) \\ f_{ij} &= f(a_i, \{a_j\}) && (i, j \geq 0) \\ G_{ij} &= \sum_{n=0}^{\infty} p_{ij}^{(n)} = f_{ij} G_{jj} = f_{ij} G_{00} && (i, j \geq 0) \end{aligned}$$

where G_{00}, and hence also each G_{ij}, is finite for a transient random walk. We say that a set A in G is recurrent if $f(a, A) = 1$ for all a in G, or equivalently,

$$(1.3) \quad \begin{aligned} 1 &= h(a, A) \\ &= \Pr(x_n \in A \quad \text{for infinitely many} \quad n \geq 0: x_0 = a) \quad (a \in G). \end{aligned}$$

A is said to be transient if $h(a, A) = 0$ for all a in G.

A set C in G is said to be almost closed if

$$(1.4) \qquad 0 \lneqq h(a, C) = 1 - h(a, G - C) \qquad (a \in G).$$

An almost closed set C is atomic if it does not contain two disjoint almost

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closed sets. If G is atomic the random walk is said to be simply atomic, if G contains no atomic almost closed set the random walk is simply non-atomic.

A random walk is aperiodic if G is the smallest group containing all a_i for which $G_{0i} > 0$.

2. Preliminary results

We say that there is a test for recurrence of type II if there is a finite-valued function $\phi(a)$ defined on G such that

$$\text{II} \qquad A \text{ is recurrent} \Leftrightarrow \textstyle\sum_{a \in A} \phi(a) \text{ is divergent.}$$

Similarly we say that there is a test for transience of type III if there is a finite-valued function $\phi(a)$ defined on G such that

$$\text{III} \qquad A \text{ is transient} \Leftrightarrow \textstyle\sum_{a \in A} \phi(a) \text{ is convergent.}$$

Let

$$A^+ = \{a : a \in A, \phi(a) \geq 0\}, \qquad A^- = A - A^+.$$

Then we have for tests of type II

$$\begin{aligned} \textstyle\sum_A |\phi(a)| = \infty &\Rightarrow \textstyle\sum_{A^+} \phi(a) = +\infty \quad \text{or} \quad \textstyle\sum_{A^-} \phi(a) = -\infty \\ &\Rightarrow A^+ \quad \text{or} \quad A^- \text{ is recurrent} \\ &\Rightarrow A \text{ is recurrent} \qquad (2.1) \\ &\Rightarrow \textstyle\sum_A \phi(a) \text{ is divergent} \\ &\Rightarrow \textstyle\sum_A |\phi(a)| = \infty, \end{aligned}$$

so that in this case $\phi(a)$ can be assumed without loss of generality to be non-negative. A similar argument can be used in the case of tests of type III. We shall therefore in what follows always assume that $\phi(a)$ is non-negative.

In [6] it is shown that an aperiodic random walk is either simply atomic or simply non-atomic. It follows from Theorems 8 and 10 of [6] that if tests of type III (II) exist for a random walk then G is the union of finitely many (one) atomic almost closed sets. Thus if such tests exist an aperiodic random walk is necessarily simply atomic and hence by [1] each set A is either transient or recurrent.

In this case therefore the existence of a test of type II or III implies the existence of both tests simultaneously.

We say that a random walk has property Δ if there is a positive η such that for any positive integer N we can find finite sets $A_1^N, \cdots, A_N^N$, with union B_N, and $\gamma_N \in G$ such that

$$f(X, A_t^N Y) > \eta f(\gamma_N X, B_N Y) \qquad (1 \leq t \leq N) \tag{2.2}$$

for all X, Y in G.

The basic lemma which we apply in the various types of Abelian groups to show that a test of type II does not exist is

LEMMA 1. *In a simply atomic random walk with property* Δ *for which* $G_{0i} \to 0$ as $i \to \infty$ *there is no valid test of type* II.

Before proving Lemma 1 it is necessary to obtain a preliminary

LEMMA 2. *If B is a finite set, C is a non-transient set and D is a transient set in a random walk on a countable group G then there is a non-transient set*

$$E = \bigcup_{r=1}^{\infty} B\alpha_r$$

for which $D \cap E = \phi$, $\alpha_1, \alpha_2, \cdots \in C$ *and*

$$C - \bigcup_{r=1}^{\infty} \{b^{-1}b'\alpha_r : b, b' \in B\}$$

is transient.

Proof. We first define the finite set

$$B^* = \{b^{-1}b' : b, b' \in B\}$$

and then choose integers $i_1 < i_2 < i_3 < \cdots$ by

$$i_r = \min \{i : a_i \in C, a_i \notin B^* a_{i_s} \ (1 \leq s < r)\}.$$

Denoting a_{i_r} by c_r and writing $C' = \{c_1, c_2, \cdots\}$ we see that the finite union

$$\bigcup_{b^* \in B^*} b^* C' = C$$

and hence one of the sets b^*C' is non-transient, which shows that C' is non-transient. From the definition of C' it follows that $BC' = \bigcup_1^{\infty} Bc_r$ is a disjoint union containing C' and so also non-transient. The set C'' consisting of those c_r for which $Bc_r \cap D \neq \phi$ is transient, since B is finite and D is transient, and so also is BC''. Denoting the elements of $\bar{C} = C' - C''$ by $\alpha_1, \alpha_2, \cdots$ we see that

$$E = \bigcup_1^{\infty} B\alpha_r$$

is also non-transient, $D \cap E = \phi$ and

$$C - \bigcup_{r=1}^{\infty}\{b^{-1}b'\alpha_r : b, b' \in B\} \subset \bigcup_{B^*} b^*C' - \bigcup_{B^*} b^*\bar{C} \subset \bigcup_{B^*} b^*C''.$$

Since the last set is transient we have proved the lemma as required.

Applying this lemma in the case of a simply atomic random walk we can proceed to the proof of the basic Lemma 1. We note that in this case the term "non-transient" can be replaced by "recurrent" in the statement of the lemma.

Proof of Lemma 1. We appeal to the construction of Lemma 7 in [6] in which disjoint finite sets $S_0 = \{e\}$, S_1, S_2, $\cdots$ are constructed such that $G - \bigcup_0^{\infty} S_\iota = V$ is transient and

$$\tfrac{2}{3} f_{ij} < f_{0j}, \qquad f_{ji} < \tfrac{2}{3} f_{0i} \tag{2.3}$$

for $i \in S_\iota, j \in S_m$, $m \geq \iota + 2$. (There is no loss of generality in putting $S_0 = \{e\}$ since in a group we can replace each S_ι by

$$S'_\iota = s^{-1} S_\iota = \{x : sx \in S_\iota\} \qquad (\iota \geq 0)$$

for which the inequalities (2.3) still hold.)

We note that at each stage of the construction, when $S_0, \cdots, S_{\iota-1}$ are known S_ι is taken to be an arbitrary finite set disjoint from $V, S_0, \cdots, S_{\iota-1}$ and containing at least a certain well-defined finite set of points

$$W_\iota = (U_\iota - T_\iota) \cup [\{\iota\} - V \cup T_\iota].$$

Assuming that $S_0, \cdots, S_{N-1}$ have been constructed we apply Lemma 2 with $B = B_N$, $C = G$ and

$$D = \bigcup \{B_N a : f(e, B_N a) \geq 1/N\} \cup \{a : f(\gamma_N, a) \leq \tfrac{1}{2} f(e, a)\} \tag{2.4}$$

to obtain $E = \bigcup_1^\infty B_N \alpha_r$ with $D \cap E = \phi$.

D is transient since, by assumption, $f(e, a_i) = G_{0i}/G_{00} \rightarrow 0$ as $i \rightarrow \infty$ and hence

$$f(e, B_N a_j) \leq \sum_{a_i \in B_N a_j} f(e, a_i) \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

and also by [6] the set

$$\{a : f(\gamma_N, a) \leq \tfrac{1}{2} f(e, a)\} \tag{2.5}$$

is transient.

From the recurrence of E, and the fact that $f(e, B_N \alpha_r) < 1/N$ it follows that an integer r_N can be chosen so that

$$1/N < f(e, E_N) < 2/N, \qquad E_N = \bigcup_{r=1}^{r_N} B_N \alpha_r. \tag{2.6}$$

We write

$$F_t^N = \bigcup_{r=1}^{r_N} A_t^N \alpha_r \qquad (1 \leq t \leq N),$$

and deduce from (2.2) that for any x in $\gamma_N^{-1} E_N$, and hence in some $\gamma_N^{-1} B_N \alpha_r$, we have

$$f(x, F_t^N) \geq f(x, A_t^N \alpha_r) > \eta f(\gamma_N x, B_N \alpha_r) = \eta.$$

From this it follows that

$$f(x, F_t^N) > \eta f(\gamma_N x, E_N)$$

for all x in $\gamma_N^{-1} E_N$ and thus also for all x in G, by the maximum principle [7]. Putting $x = e$ and noting that for a in E_N we have

$$f(\gamma_N, a) > \tfrac{1}{2} f(e, a)$$

we deduce from [6], (2.4) and (2.5) that for $1 \leq t \leq N$

$$\begin{aligned} f(e, F_t^N) &> \eta f(\gamma_N, E_N) = \eta \sum_{a \in E_N} f(\gamma_N, a) e(a, E_N) \\ &> \tfrac{1}{2} \eta \sum_{a \in E_N} f(e, a) e(a, E_N) = \tfrac{1}{2} \eta f(e, E_N). \end{aligned} \tag{2.7}$$

If we choose S_N to be any finite set in $G - V - \bigcup_{\iota=0}^{N-1} S_\iota$ containing E_N and W_N we can then state the Wiener's test [6] in the form:

$$A \text{ is recurrent} \Leftrightarrow \textstyle\sum_0^\infty f(e, A \cap S_\iota) = \infty. \tag{2.8}$$

In particular if A is taken to be any set of the form $A_L = \bigcup_{\iota \in L} E_\iota$ we see from (2.6) and (2.7) that

$$\textstyle\bigcup_L E_\iota \text{ is recurrent} \Leftrightarrow \sum_L \iota^{-1} = \infty \Leftrightarrow \bigcup_L F_{t_\iota}^\iota \text{ is recurrent} \tag{2.9}$$

for any choice of $\{t_\iota\}_{\iota=1}^\infty$ such that $1 \leq t_\iota \leq \iota$. Let us assume now that a test of type II is valid for the given random walk

$$\textstyle\sum_A \phi(a) = \infty \Leftrightarrow A \text{ is recurrent}. \tag{2.10}$$

Writing $\phi(S) = \sum_{a \in S} \phi(a)$ for each finite set S we deduce that

$$\textstyle\sum_L \phi(E_\iota) = \infty \Leftrightarrow \bigcup_L E_\iota \text{ is recurrent}. \tag{2.11}$$

$\phi(E_\iota)$ must necessarily approach 0 as $\iota \to \infty$ since otherwise we would have a positive δ and an infinite set Λ such that

$$\phi(E_\iota) > \delta > 0 \qquad (\iota \in \Lambda),$$

so that for a suitably chosen infinite subset Λ' of Λ $\bigcup_{\Lambda'} E_\iota$ would be recurrent by (2.11) and transient by (2.9).

We now choose $\{t_\iota\}_1^\infty$ so that

$$\phi(F_{t_\iota}^\iota) \leq \frac{1}{\iota} \phi(E_\iota) \qquad (\iota \geq 1). \tag{2.12}$$

From (2.9), (2.11) and (2.12) we see that

$$\textstyle\sum_L \phi(E_\iota) = \infty \Leftrightarrow \bigcup_L E_\iota \text{ is recurrent}$$
$$\textstyle\Leftrightarrow \bigcup_L F_{t_\iota}^\iota \text{ is recurrent} \Leftrightarrow \sum_L \phi(F_{t_\iota}^\iota) = \infty$$

and therefore

$$\sum_L \phi(E_\iota) = \infty \Leftrightarrow \sum_L \frac{1}{\iota} \phi(E_\iota) = \infty \tag{2.13}$$

for all subsequences of a sequence $\{\phi(E_\iota)\}_1^\infty$ of positive terms, with infinite sum, which converges to 0.

From (2.13) we immediately obtain a contradiction if we choose

$$1 < \iota_1 < \lambda_1 < \iota_2 < \lambda_2 < \cdots$$

such that

$$1/r < \textstyle\sum_{\iota_r}^{\lambda_r} \phi(E_\iota) < 2/r$$

and hence

$$\sum_{\iota_r}^{\lambda_r} \frac{1}{\iota} \phi(E_\iota) < \frac{2}{r \iota_r} \leq \frac{1}{r^2}$$

and put $L = \bigcup_{r=1}^\infty [\iota_r, \lambda_r]$. This completes the proof of Lemma 1.

3. Abelian random walk

In [5] it is shown that in a random walk on an Abelian group G we have $G_{0i} \to 0$ as $i \to \infty$ except possibly when G contains an infinite cyclic group $H = \{h^n\}_{-\infty}^{\infty}$ such that G/H is finite. If $\limsup_{i\to\infty} G_{0i} = \delta > 0$ then either

$$\text{(i)} \qquad G(e, h^n) \to \delta \quad \text{as } n \to \infty,$$

or

$$\text{(ii)} \qquad G(e, h^{-n}) \to \delta \quad \text{as } n \to \infty.$$

In case (i) any infinite subset of $H^+ = \{h^n\}_0^{\infty}$ is recurrent and $H^- = H - H^+$ is transient; in case (ii), H^+ and H^- have their roles interchanged.

Denoting the elements of G/H by $g_1 H, \cdots, g_J H$ we deduce in case (i) that any infinite subset of $g_j H^+$ is recurrent and $g_j H^-$ is transient. Thus an infinite subset of $G^+ = \bigcup_1^J g_j H^+$ is recurrent and $G^- = G - G^+$ is transient. Writing

$$(3.1) \qquad \begin{aligned} \phi_1(a) &= 1 \qquad a \in G^+ \\ &= 0 \qquad a \in G^- \end{aligned}$$

we see that the corresponding test of type II is valid in case (i), in case (ii) we only interchange the roles of G^+ and G^- and put $\phi_2(a) = 1 - \phi_1(a)$ to obtain a valid test of type II.

Before proceeding to consider the possible existence of a type II test for a random walk on a countable Abelian group we will first need to prove

LEMMA 3. *If H is a subgroup of G then the existence of a type II test in G implies the existence of a type II test in H if H is a recurrent set or in G/H if H is a transient normal divisor of G.*

Proof. Consider first the case when H is recurrent. We can then define transition probabilities

$$(3.2) \quad \bar{p}_{ij} = \Pr(x_1, \cdots, x_{r-1} \notin H, x_r = a_j, \text{ for some } r > 0 : x_0 = a_i)$$

for any pair a_i, a_j in H. These are obviously the transition probabilities of a random walk in the group H, termed the imbedded random walk.

A subset A of H is recurrent in the imbedded random walk if and only if it is recurrent in the original random walk and hence the same type II test is valid, restricted to sets in H.

In the case of a transient normal divisor H we define

$$(3.3) \qquad \bar{p}_{ij} = \textstyle\sum_{a_\alpha \in a_j H} p_{i\alpha}$$

for any pair of cosets $a_i H, a_j H$, which is independent of the particular element of $a_i H$ which is chosen. These are easily seen to be the transition probabilities of a random walk in G/H, termed the image random walk.

Suppose that a type II test is valid in G. Since each $a_i H$ is transient then

$$\phi(a_i H) = \sum_{a \epsilon a_i H} \phi(a) \tag{3.4}$$

is finite. A set $B = \bigcup_{a \epsilon A}\{aH\}$ is recurrent in G/H if an only if the corresponding set $C = \bigcup_{a \epsilon A} aH$ is recurrent in G and hence if and only if

$$\infty = \sum_C \phi(c) = \sum_A \phi(aH) = \sum_B \phi(b)$$

Thus we have also a type II test in the image random walk if H is a transient normal divisor.

COROLLARY. *If G is the direct product of two groups*

$$G = G_1 \otimes G_2$$

then there is a type II *test in G_1 or in G_2 if there is such a test in G.*

Proof. If G_1 is recurrent then there is a type II test for the imbedded random walk in G_1. If G_1 is transient then there is a type II test for the image random walk in G/G_1. However $G/G_1 \cong G_2$ so that this implies that there is a type II test in G_2.

We will now proceed to show that random walks on certain special types of Abelian groups have property Δ. We adopt the terminology of [4] and write our groups additively in the remainder of this paper.

LEMMA 4. *A simply atomic random walk on a subgroup G of the rationals Q, containing the integers Z, has property Δ.*

Proof. Either the positive rationals G^+ in G or the negative rationals $G^- = G - G^+$ form a recurrent set. We may suppose without loss of generality that the former is true. We then write

$$L_i = \bigcup_{j=0}^{\infty}[i - 1 + 2jN, i + 2jN) \qquad (1 \le i \le 2N) \tag{3.5}$$

and note that each $L_i \cap G$ is recurrent. Let

$$G_{i,k} = [0, 2kN) \cap L_i \cap \{x : x \in G, xk\,! \in Z\}. \tag{3.6}$$

Since $G_{i,k}$ increases monotonically to $L_i \cap G$ as $k \to \infty$ we can find an integer k such that

$$f(0, G_{i,k}) > \tfrac{1}{2} \qquad (1 \le i \le 2N). \tag{3.7}$$

Similarly by the recurrence of $L_i \cap G - G_{i,k}$ we can find an integer K such that

$$f(0, G_{i,K} - G_{i,k}) > \tfrac{1}{2} \qquad (1 \le i \le 2N). \tag{3.8}$$

Finally we write

$$\begin{aligned} A_i^N &= G_{2i,K} \cup G_{2i-1,K} \qquad (1 \le i \le N), \\ B_N &= \textstyle\bigcup_1^N A_i^N, \end{aligned} \tag{3.9}$$

and check immediately from (3.5) — (3.9) that

$$f(x - 2kN, A_i^N) > \tfrac{1}{2} f(x, B_N) \qquad (1 \le i \le N) \tag{3.10}$$

for all x in B_N. From the maximum principle [7] it then follows that (3.10) is true for all x in G and so the random walk on G has property Δ, as required.

LEMMA 5. *A simply atomic random walk in the plane lattice $Z \oplus Z$ has property Δ if*

$$\begin{aligned} 1 &> \lim_{|\xi|\to\infty} \sup f((0, 0), \{(x, y) : x = \xi\}) > 0, \\ 1 &> \lim_{|\eta|\to\infty} \sup f((0, 0), \{(x, y) : y = \eta\}) > 0. \end{aligned} \tag{3.12}$$

Proof. Writing $X = Z \oplus \{0\}$, $Y = \{0\} \oplus Z$, $G = Z \oplus Z$ we see from [7] that the image random walks in G/X and in G/Y must both be transient with finite non-zero means μ_1, μ_2, respectively, and hence that the given random walk has finite mean

$$(\mu_1, \mu_2) = \sum\nolimits_G (x, y) p_{(0,0),(x,y)} \tag{3.13}$$

The projection of this random walk on the orthogonal unit vector (α, β) with transition probabilities

$$\pi_{0,t} = \sum\nolimits_{\alpha x + \beta y = t} p_{(0,0),(x,y)} \tag{3.14}$$

is a random walk in a subset S of a real line, with mean zero. Therefore by [2] any interval on the line $\mu_1 x + \mu_2 y = 0$ is recurrent, in particular the two sets

$$S \cap \{(\alpha t, \beta t) : 0 \le t < |\beta|\}, \qquad S \cap \{(\alpha t, \beta t) : 0 \ge t > -|\beta|\}$$

are recurrent. Interpreting this first in terms of the given random walk we see that the sets

$$\begin{aligned} U_1 &= \{(x, y) : \mu_2 x \ge \mu_1 y > -\mu_1 + \mu_2 x\} \cap G \\ U_2 &= \{(x, y) : \mu_2 x \le \mu_1 y < \mu_1 + \mu_2 x\} \cap G \end{aligned} \tag{3.15}$$

are recurrent sets in $Z \oplus Z$.

From this point we proceed in a manner similar to that of Lemma 4. We first write

$$\begin{aligned} L_{2i-1} &= U_1 \cap \textstyle\bigcup_{j=0}^{\infty} \{(x, y) : x = i + jN\} \qquad (1 \le i \le N) \\ L_{2i} &= U_2 \cap \textstyle\bigcup_{j=0}^{\infty} \{(x, y) : x = i + jN\} \qquad (1 \le i \le N) \end{aligned} \tag{3.16}$$

and note that each L_i is also recurrent, since a finite number of translates of each L_i suffice to cover the recurrent U_1 or U_2.

We then let

$$G_{i,k} = \{(x, y) : 0 \le x < kN\} \cap L_i \qquad (1 \le i \le 2N) \tag{3.17}$$

and proceed exactly as in Lemma 4 to obtain k, K, satisfying

$$f((0,0), G_{i,k}) > \tfrac{1}{2} < f((0,1), G_{i,k}) \qquad (1 \leq i \leq 2N) \tag{3.18}$$

and also

$$f((0,0), G_{i,K} - G_{i,k}) > \tfrac{1}{2} < f((0,1), G_{i,K} - G_{i,k}) \qquad (1 \leq i \leq 2N). \tag{3.19}$$

We then define

$$A_i^N = G_{2i-1,K} \cup G_{2i,K}, \qquad B_N = \bigcup_1^N A_i^N \qquad (1 \leq i \leq N) \tag{3.20}$$

and choose (kN, y_N) in U_1. From (3.15)–(3.20) it then follows that

$$f((x - kN, y - y_N), A_i^N) > \tfrac{1}{2} f((x, y), B_N) \quad (1 \leq i \leq N) \tag{3.21}$$

for all (x, y) in B_N and therefore for all (x, y) in $Z \oplus Z$ as well. This completes the proof of the lemma.

LEMMA 6. *A simply atomic random walk on $Z(p^\infty) \oplus H = G$ has property Δ if H is a torsion group.*

Proof. If H is finite we write $H = H_n$ for all positive integers n. If H is infinite, with elements $h_1, h_2, \cdots$, we denote by H_n the finite group generated by $h_1, \cdots, h_n$. For given N we choose m so that $p^{m-1} \geq N$ and define the finite set

$$G_{i,k} = \{[(i - 1)/p^m, i/p^m) \cap Z(p^k)\} \oplus H_k. \tag{3.22}$$

As $k \to \infty$ the set $G_{i,k}$ increases monotonically to a recurrent set and therefore K can be chosen so that

$$f((0, 0), G_{i,K}) > \tfrac{1}{2} \qquad (1 \leq i \leq p^m). \tag{3.23}$$

We then define

$$A_t^N = \bigcup_{i=(t-1)p+1}^{tp} G_{i,K} \qquad (1 \leq t < N) \tag{3.24}$$

$$A_N^N = \bigcup_{i=(N-1)p+1}^{p^m} G_{i,K}, \qquad B_N = \bigcup_1^N A_t^N, \tag{3.25}$$

and deduce immediately from (3.22)–(3.25) that

$$f(x, A_t^N) > \tfrac{1}{2} f(x, B_N) \qquad (1 \leq t \leq N) \tag{3.26}$$

for all x in B_N and hence also for all x in G. This completes the proof of the lemma.

LEMMA 7. *If G is a torsion group which contains finite direct summands of arbitrarily large order then G has property Δ.*

Proof. For given N we write $G = D \oplus H$ with D finite of order at least N and denote by H_k the group generated by the first k elements of H. If

$d_1, \cdots, d_M$ are the elements of D we write

$$G_{i,k} = \{d_i\} \oplus H_k \qquad (1 \le i \le M). \tag{3.27}$$

Since $\{d_i\} \oplus H$ is recurrent for each i we can choose K so that (3.23) is satisfied for each i in $[1, M]$. In a manner analogous to the previous proof we define $B_N = \bigcup_1^N A_t^N$, where

$$\begin{aligned} A_t^N &= G_{t,K} \qquad (1 \le t < N) \\ &= \bigcup_N^M G_{i,K} \qquad (t = N). \end{aligned} \tag{3.28}$$

It then follows immediately that (3.26) is satisfied for all x in B_N and hence also for all x in G, so that Lemma 7 is proved.

From the previously proved lemmas and also from the results of [4] it is a fairly easy matter to deduce our principal

THEOREM 1. *There is a simple test for recurrence or transience (type II or type III) of sets A in an aperiodic random walk on an Abelian group if and only if*

$$\lim_{i\to\infty} \sup G_{0i} > 0. \tag{3.29}$$

Proof. In the case when (3.29) holds we have already shown that there are simple tests (type II or III) for recurrence or transience.

We will now assume that $G_{0i} \to 0$ as $i \to \infty$ and quote certain results of [4] which together with Lemmas 1, 3, 4, 5, 6, 7 will prove the theorem. We also assume that there is a simple test of either type so that the random walk is simply atomic and tests of both types are valid.

Let T be the torsion subgroup of G then by Lemma 3 there is either

(i) a type II test in the torsion group T if T is recurrent, or

(ii) a type II test in the torsion-free group G/T if T is transient.

In the latter case it is known that a torsion-free group is isomorphic to a subgroup $\bar{G}$ of a countable direct sum of copies of the rationals $\oplus_{i=1}^{\alpha} Q_i$. If the group $\bar{G}$ (or G/T) contains at least three independent elements, or equivalently if the direct sum necessarily contains at least three terms, we define $\bar{H}$ as the subgroup of elements in $\{0\} \oplus \{0\} \oplus \{0\} \oplus \oplus_{i=4}^{\alpha} Q_i$. From the type II test in $\bar{G}$ (or G/T) it then follows by Lemma 3 that there is a type II test in $\bar{G}/\bar{H}$. (Since the random walk on $\bar{G}/\bar{H}$ is a genuinely three-dimensional one it is transient and so $\bar{H}$ is transient in the random walk on $\bar{G}$.) Thus in case (ii) we are reduced to three possible sub-cases

(a) a type II test in a subgroup G_1 of $Q \oplus Q \oplus Q$ containing $Z \oplus Z \oplus Z$,

(b) a type II test in a subgroup G_2 of $Q \oplus Q$ containing $Z \oplus Z$, or

(c) a type II test in a subgroup G_3 of Q containing Z for which $G_{0i} \to 0$ as $i \to \infty$.

In case (ii) (a) either the subgroup $H_1 = G_1 \cap [Q \oplus \{0\} \oplus \{0\}]$ is recurrent, in which case we are reduced to case (ii) (c) for the imbedded random walk in H_1, or H_1 is transient, in which case we are reduced to case (ii) (b) for the image random walk in G_1/H_1.

In case (ii) (b) if either of the one-dimensional subgroups

$$H_2^x = G_2 \cap [Q \oplus \{0\}] \quad \text{or} \quad H_2^y = G_2 \cap [\{0\} \oplus Q]$$

is recurrent then we are reduced to case (ii) (c) for the corresponding imbedded random walk. If both H_2^x and H_2^y are transient and

$$(3.30) \quad 0 < \lim_{|\xi|,|\eta| \to \infty} \sup f((0, 0), G_2 \cap [\{\xi\} \oplus Q]) f((0, 0), G_2 \cap [Q \oplus \{\eta\}])$$

then the image random walks must both have non-zero means, and $G_2 \cong Z \oplus Z$. Therefore, by Lemma 5, no type II test can exist for this random walk. On the other hand if the inequality (3.30) does not hold then for one of the image random walks we are reduced to case (ii) (c) again. Since by Lemma 4, case (ii) (c) can never occur we see that no type II test is valid in any torsion-free Abelian group.

Turning now to the torsion case (i) we know from [4] that any torsion group is the direct sum of a divisible torsion group D and a reduced torsion group R. Thus in this case, by the corollary to Lemma 3, there is either (i) (a) a type II test in a divisible torsion group, or (i) (b) a type II test in a reduced torsion group.

In case (i) (a) we can write D as a direct sum of groups of the form $Z(p^\infty)$ for various primes p, and thus this case is immediately excluded by Lemma 6.

In case (i) (b) the group R can be written as a direct sum of reduced primary groups. If all these latter groups are of bounded order then each of them, and hence also R is a direct sum of finite cyclic groups, if any one is of unbounded order then it contains cyclic direct summands of arbitrarily large order. Whichever of these is true we can always say in case (i) (b) that R contains direct summands of arbitrarily large order and so by Lemma 7 no type II test is valid.

Since we have now disposed of all possible cases which can arise for countable Abelian groups we have completed the proof of the theorem.

Note. In the case of a random walk which is not aperiodic the existence of a type II (or III) test in G implies the existence of such a test in the subgroup H generated by those elements a_i for which $G_{0i} > 0$. The theorem then shows that this can only happen if $\lim_{i \to \infty} \sup G_{0i} > 0$ so that the requirement of aperiodicity can be dropped in the statement of the theorem. As remarked at an earlier stage we can say in addition that in the case of a type III test the group G is simply atomic and hence is identical with H and in the case of a type II test the group G is a union of finitely many atomic almost closed sets aH. In either case we immediately deduce that G contains an infinite cyclic group of finite index in G.

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