DECOMPOSITION OF PURE SUBGROUPS OF TORSION FREE GROUPS

BY

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1. Introduction

Throughout this paper all groups are abelian. The notion of a cotorsion group, introduced by Harrison in [8], plays an important role. Some basic properties of cotorsion groups are listed in [4]. A torsion free group is called completely decomposable if it is isomorphic to a direct sum of torsion free groups of rank one. If G is a torsion free group and H is a subgroup of G, we use the symbol H_* to denote the minimal pure subgroup of G containing H. The symbols $\sum$ and $+$ will be used for direct sums; whereas the subgroup of a group G generated by subsets S and T will be denoted by $\{S, T\}$.

Recently, the author gave a negative answer [7] to a question posed by E. Weinberg [9] which asked: Does there exist a torsion free abelian group of cardinality greater than the continuum with the property that each pure subgroup is indecomposable? In this paper we use the techniques of [7] to generalize our result concerning Weinberg's question. In fact, if G is a torsion free group we show that there is a completely decomposable pure subgroup C of G such that $|G| \leq |C|^{\aleph_0}$. Our investigation of completely decomposable pure subgroups of torsion free groups requires the study of a distinguished class of independent subsets of a torsion free group. An independent subset S of a torsion free group G will be called quasi-pure independent if $\sum_{x \in S} \{x\}_*$ is a pure subgroup of G and $\{x\}_* = \{x\}$ whenever $\{x\}_*$ is cyclic and $x \in S$. Note that $\{S\}_* = \sum_{x \in S} \{x\}_*$ if S is a quasi-pure independent. We remark that quasi-pure independence is equivalent to pure independence if G is $\aleph_1$-free. In Section 2 we establish a number of remarkable properties of quasi-pure independent subsets.

2. Quasi-pure independence

We observe that, although nonvoid pure independent subsets may not exist, nonzero torsion free groups always have quasi-pure independent subsets. The proof of the following proposition can be accomplished by standard techniques.

Proposition 2.1. *Any quasi-pure independent subset S of a torsion free group G is contained in a maximal quasi-pure independent subset of G.*

One might hope that the cardinality of a maximal quasi-pure independent subset of a torsion free group is an invariant of the group. In [6] it was shown

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that a torsion free group may contain a finite maximal pure independent subset as well as a maximal pure independent subset of infinite cardinality. Unfortunately, the following example demonstrates that the same situation occurs for quasi-pure independence. Let $J = \prod_p I_p$ where p ranges over the primes and I_p denotes the p-adic group. We show that J contains maximal quasi-pure independent subsets S and T such that $|S| = 1$ and $|T| \geq \aleph_0$. For each prime p let x_p be an element of J whose p^{th} coordinate is a nonzero element of I_p and whose other coordinates are all zero. The set $[x_p \mid p \text{ is a prime}]$ is easily seen to be quasi-pure independent. Therefore, let T be a maximal quasi-pure independent subset of J containing $[x_p \mid p \text{ is a prime}]$. Since the additive group of integers Z can be embedded in J as a pure subgroup such that J/Z is divisible, it follows that J contains a maximal quasi-pure independent subset S of cardinality one. Although this example shows that the cardinality of a maximal quasi-pure independent subset is not an invariant of a torsion free group, we are able to establish a slightly weaker result. The proof of this next theorem is essentially the same as Chase's proof of Theorem 3.1 in [2]. For notational convenience we use the symbols $D(A)$ and tA to denote the minimal divisible group containing the group A and the torsion part of A, respectively.

THEOREM 2.2. *Let G be a torsion free group and suppose that S and T are infinite maximal quasi-pure independent subsets of G. Then $|S| = |T|$.*

Proof. It suffices to show that if X and Y are quasi-pure independent subsets of G where $|X| < |Y|$ and $\aleph_0 < |Y|$ then there is a quasi-pure independent subset X_1 containing X such that $|X_1| = |Y|$. Set $H = \sum_{x \in X} \{x\}_*$, $K = \sum_{y \in Y} \{y\}_*$, and $\beta = |Y|$. Then H and K are pure subgroups of G, $|H| < |K|$, and $|K| = \beta$. Let $\bar{G} = G/K$ and $\bar{H} = \{H, K\}/K$. Therefore $\bar{H} \subseteq \bar{G}$, $\bar{G}$ is torsion free, and $D(\bar{G}) = D(\bar{H}) + M$ where M is torsion free and divisible. $D(\bar{G})/H$ may be identified with $(D(\bar{H})/\bar{H}) + M$, in which case

$$t(\bar{G}/\bar{H}) \subseteq t(D(\bar{G})/\bar{H}) = t(D(\bar{H})/\bar{H}).$$

$t(D(\bar{H})/\bar{H})$ has cardinality less than β, since β is uncountable and since $|\bar{H}| \leq |H| < \beta$. Observing that $\bar{G}/\bar{H} \cong G/\{H, K\}$, we have shown that $t(G/\{H, K\})$ has cardinality less than β.

Since β is infinite, we may construct a free group F of rank less than β and an epimorphism

$$\psi : F \rightarrow t(G/\{H, K\}).$$

Then there is a homomorphism $\varphi : F \rightarrow G$ such that ψ is the composition of φ with the canonical map of G onto $G/\{H, K\}$. Since $|\{H, \varphi(F)\}| < \beta$, β is infinite and, K is completely decomposable, we may write $K = A + B$ where A and B are completely decomposable, $K \cap \{H, \varphi(F)\} \subseteq A$, and rank

$(B) = |B| = \beta$. Observing that

$$H \cap B \subseteq (H \cap K) \cap B \subseteq A \cap B = 0,$$

set $C = H + B$. Then clearly rank $(C) = \beta$ and C is completely decomposable. Since B is completely decomposable of cardinality β, B contains a quasi-pure independent subset V of cardinality β. Thus, $X \cup V$ will be a quasi-pure independent subset of G if $C = H + B$ is a pure subgroup of G. Suppose $nx \in C$ where $x \in G$ and n is a nonzero integer. Then $nx = h_1 + b_1$ where $h_1 \in H$ and $b_1 \in B$. Therefore, x maps onto an element of finite order in $G/\{H, K\}$. Hence, there is an element $y \in F$ such that $x - \varphi(y) \in \{H, K\}$. But then

$$x - \varphi(y) = h_2 + a + b_2$$

where $h_2 \in H$, $a \in A$, and $b_2 \in B$. We then have that

$$h_1 + b_1 = nx = n\varphi(y) + nh_2 + na + nb_2\,,$$

or that

$$h_1 - n\varphi(y) - nh_2 - na = nb_2 - b_1\,.$$

The left side of this equation is easily seen to be in A and the right side in $B\cdot$ Thus both sides are zero and we have that $b_1 = nb_2$. Therefore

$$h_1 = nx - nb_2 \in nG \cap H = nH.$$

It follows that $nx = h_1 + b_1 \in nC$, in which case $x \in C$. Hence, C is a pure subgroup of G and $X \cup V$ is a quasi-pure independent subset of G. Setting $X_1 = X \cup V$, we have that X_1 is a quasi-pure independent subset of G such that $|X_1| = \beta = |Y|$.

COROLLARY 2.3. (corollary to proof). *If a torsion free group G contains an uncountable quasi-pure independent subset, then any two maximal quasi-pure independent subsets of G have the same cardinality.*

If a torsion free group G contains maximal quasi-pure independent subsets S and T such that $|S| < |T|$, then Corollary 2.3 implies that any quasi-pure independent subset X of G is at most countable. In particular, $|T| \leq \aleph_0$.

Baer proved in [1] that a homogeneous torsion free group is separable if and only if every pure subgroup of finite rank is a direct summand. (For the definitions of a homogeneous group and a separable group, see [3].) Thus, for separable, homogeneous torsion free groups, we have the following corollary.

COROLLARY 2.4 *If G is a separable, homogeneous torsion free group, then the cardinality of a maximal quasi-pure independent subset of G is an invariant of G.*

Proof. Suppose S and T are maximal quasi-pure independent subsets of G. If $|S| = n < \aleph_0$, then, by Baer's theorem [1], $\{S\}_* = \sum_{x \in S} \{x\}_*$ is a direct summand of G, i.e., $G = \{S\}_* + M$. Clearly, if $M \neq 0$, we can choose $y \in M$ such that $S \cup [y]$ is quasi-pure independent. Therefore, $M = 0$ which implies that $| S | = \text{rank } (G) = n$. Since rank $(G) = n < \infty$, it follows that $| T | < \aleph_0$. By the same argument we have that $| T | = \text{rank } (G) = | S |$. If $\aleph_0 \leq | S |$, then $\aleph_0 \leq | T |$ since the rank of G cannot be finite. Hence, applying Theorem 2.2, we again have that $| S | = | T |$.

We now establish in the following theorem a remarkable relationship between the cardinality of a torsion free group and the cardinality of any maximal quasi-pure independent subset of the group.

THEOREM 2.5. *If G is a non-zero torsion free group and if S is a maximal quasi-pure independent subset of G, then $| G | \leq (| S | + 1)^{\aleph_0}$.*

Proof. Let $G = G_0 + D$ where G_0 is reduced and D is divisible. Since D is torsion free divisible, it is elementary to show that the cardinality of any maximal quasi-pure independent subset of D is rank (D). It is also clear that $S \cap D$ is a maximal quasi-pure independent subset of D whenever S is a maximal quasi-pure independent subset of G. Hence, it is enough to prove the theorem when $D = 0$, that is, when G is reduced.

Let E be the cotorsion completion of G and let H be the closure of $\{S\}_* = \sum_{x \in S} \{x\}_*$ in the n-adic topology on E. Since H must be pure, E/H is torsion free and reduced. It follows that H is a direct summand of E since $\text{Ext}\,(E/H, H) = 0$. Let $E = H + M$. Since E is torsion free, $E = H + M$, and G is pure in E, then $H \cap G + M \cap G$ is a pure subgroup of G. Therefore, if $M \cap G \neq 0$ we can choose $y \in M \cap G$ such that $S \cup [y]$ is a quasi-pure independent subset of G. But this contradicts the maximality of S. Therefore, $M \cap G = 0$ and the natural projection π of E onto H is a monomorphism when restricted to G. Hence, $| G | = | \pi(G) | \leq | H |$. Since $\{S\}_*$ is dense in H and since the n-adic topology on H is Hausdorff, we have that

$$| H | \leq | \{S\}_* |^{\aleph_0} = (| S | + 1)^{\aleph_0}.$$

Thus, $| G | \leq (| S | + 1)^{\aleph_0}$.

COROLLARY 2.6. (corollary of proof). *If S is a maximal quasi-pure independent subset of a torsion free group G, then G is isomorphic to a subgroup of the n-adic completion of $\sum_{x \in S} \{x\}_* = \{S\}_*$.*

With the aid of Theorem 2.4, we can establish a stronger version of Theorem 2.2 for torsion free groups of cardinality greater than the continuum.

THEOREM 2.7. *If G is a torsion free group of cardinality greater than the continuum, then any two maximal quasi-pure independent subsets of G have the same cardinality.*

Proof. Theorem 2.5 implies that any two maximal quasi-pure independent subsets of G are infinite. Hence, by Theorem 2.2, any two maximal quasi-pure independent subsets of G have the same cardinality.

The following theorem we shall need in Section 3.

THEOREM 2.8. *If E is a reduced torsion free cotorsion group and if S is a quasi-pure independent subset of E, then S is maximal (with respect to being quasi-pure independent) if and only if $E/(\sum_{x \in S} \{x\}_*)$ is divisible.*

Proof. If $E/(\sum_{x \in S} \{x\}_*)$ is divisible, then S is clearly a maximal quasi-pure independent subset of E. Hence, suppose S is a maximal quasi-pure independent subset of E. Let H be the closure of $\sum_{x \in S} \{x\}_*$ in the n-adic topology on E. Then $E = H + M$. If $M \neq 0$, we may choose $y \in M$ such that $S \cup [y]$ is quasi-pure independent. Therefore, $M = 0$ and $E = H$ which implies that $E/(\sum_{x \in S} \{x\}_*)$ is divisible.

3. Decomposition of pure subgroups of torsion free groups

Immediate consequences of Theorem 2.2 and Theorem 2.5 are the following theorems.

THEOREM 3.1. *If A and B are completely decomposable pure subgroups of infinite rank of a homogeneous torsion free group G, then there are isomorphic completely decomposable pure subgroups H and K of G such that A and B are direct summands of H and K, respectively.*

THEOREM 3.2. *If G is a torsion free group, then G contains a completely decomposable pure subgroup C such that $|G| \leq |C|^{\aleph_0}$.*

For a cardinal $\mu \geq 2$, we call a group G μ-indecomposable if in each direct decomposition of G the cardinal number of the set of non-trivial direct summands is less than μ. A group G will be called purely μ-indecomposable if every pure subgroup of G is μ-indecomposable. For $\mu = 2$, the above definitions correspond, respectively, to the definitions of indecomposability and pure indecomposability. L. Fuchs has established results concerning μ-indecomposable primary groups [3] and, as mentioned in the introduction, the author has given characterizations of purely indecomposable torsion free groups [7]. We conclude by generalizing a portion of the results in [7].

THEOREM 3.3. *If G is a torsion free purely μ-indecomposable group, then $|G| \leq \mu^{\aleph_0}$.*

Proof. By Theorem 3.2 there is a completely decomposable pure subgroup C of G such that $|G| \leq |C|^{\aleph_0}$. By hypothesis rank $(C) < \mu$. Therefore $|C|^{\aleph_0} \leq \mu^{\aleph_0}$.

THEOREM 3.4. *There is a purely μ-indecomposable torsion free group G of cardinality greater than or equal to μ if and only if there is a cardinal number α such that $\alpha < \mu \leq \alpha^{\aleph_0}$.*

Proof. The necessity follows from Theorem 3.3. Therefore, assume that μ and α are cardinals such that $\alpha < \mu \leq \alpha^{\aleph_0}$. If $\alpha < \mu \leq 2^{\aleph_0}$, set $G = \sum_\alpha I_p$ where I_p denotes the p-adic group. Hence $|G| = 2^{\aleph_0} \geq \mu$. If H is a pure subgroup of G such that $H = \sum_{i \in I} H_i$, then H contains a quasi-pure independent subset S of G such that $| S | = | I |$. We may also assume that each $x \in S$ has zero p-height in G, i.e., $\sum_{x \in S} \{x\}$ is p-pure in G. It follows that $\sum_{x \in S} \{x\}$ is a direct summand of a p-basic subgroup of G (For definition of a p-basic subgroup, see [5]). It is well known that any p-basic subgroup of G has rank α. Hence, $| I | = | S | \leq \alpha < \mu$. If $\mu > 2^{\aleph_0}$ then α must be infinite. Let G be the cotorsion completion of the free group $F = \sum_{\lambda \in \Lambda} \{x_\lambda\}$ where $| \Lambda | = \alpha$. Then $| G | = \alpha^{\aleph_0} \geq \mu$. Suppose that $H = \sum_{i \in I} H_i$ is a pure subgroup of G. Then H contains a quasi-pure independent subset S of G such that $| S | = | I |$. By Proposition 2.1 there is a maximal quasi-pure independent subset T which contains S. Theorem 2.8 implies that $X = [x_\lambda]_{\lambda \in \Lambda}$ is also a maximal quasi-pure independent subset of G. Applying Theorem 2.2, we have that $| I | = | S | \leq | T | \leq | X | = | \Lambda | = \alpha < \mu$.

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