LINEAR GROUPS OF DEGREE EIGHT WITH NO ELEMENTS OF ORDER SEVEN

BY

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1. Finite linear groups of degree eight

The finite quasiprimitive linear groups of degree less than eight have been determined in [3], [6], [16], [21]. Feit has recently determined the quasiprimitive linear groups of degree eight which contain a noncentral element of order 7. In this paper we apply the results of [12], [13], [14] to determine the remaining quasiprimitive linear groups of degree eight. Specifically, we prove the following theorem.

THEOREM. *Suppose G is a finite quasiprimitive unimodular linear group of degree 8 for which $7 \nmid |G|$. Then $G/Z(G)$ is one of the following groups where $|Z(G)| \mid 8$.*

I. *A_6, $Z(G)$ splits, or an extension of degree 2 in which $G/Z(G)$ is an extension of A_6 induced by an automorphism from $GL_2(9)$.*

II. *Subgroups of $A \times B$ where A, B have projective quasiprimitive representations of degrees 2 and 4.*

III. *$O_2(G/Z(G)) \neq 1$, $G/O_2(G) \cong$ subgroup of $Sp_6(2)$, $Z(G)$ has a nonsplitting center.*

IV. *$G/Z(G)$ is an extension of $A_5 \times A_5 \times A_5$ by a group isomorphic to Z_3 or S_3, $2 \mid |Z(G)|$.*

V. *$G \cong SL_2(17)\, Z(G)$.*

Notation is standard as in [13]. We let $\omega = e^{2\pi i/3}$. We assume G is a unimodular quasiprimitive linear group of degree eight with natural representation X. By definition, X quasiprimitive means that if $H \lhd G$, $X|H$ has similar constituents.

If G has a normal subgroup H for which $X|H$ is reducible we may apply [15] to see II holds. Each of the tensor product components must be quasiprimitive or X would not be quasiprimitive. Suppose G has a minimal noncentral solvable normal subgroup H. By quasiprimitivity and unimodularity, $HZ(G)/Z(G)$ is a 2-group. Then $K = O_2(G)$ has no rank 2 characteristic abelian subgroups and by [10 Th 5.4.9] is $Q \circ Z$ where Q is extraspecial and Z has order 1, 2, 4, 8. As $X|Q$ is irreducible, $|Q| = 2^7$. Now $C(Q) = Z(G)$ as $X|Q$ is irreducible and so $G/O_2(G) \cong$ subgroup of $Sp_6(2)$ by [11]. This is Case III. Now let $E = E(G)$ be the product of all quasisimple subnormal subgroups. Suppose E has one

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component. Note that by [6, 2D], no primes larger than 7 divide $|G|$ except for V. We are assuming $7 \nmid |G|$. If $5^2 \nmid |G|$, $E/Z(E)$ is $\cong A_5$, A_6, $Sp_4(3)$ by [6, 9A]. The only possible group of degree 8 is A_6 giving Case I. If there are several components let E_1 be a minimal normal nonsolvable subgroup. Clearly $X|E_1$ is a direct sum of three 2-dimensional linear groups each nonsolvable and so $SL_2(5)$ by [3]. Now there must be an element of order 3 permuting each giving Case IV. We are reduced to finding a quasisimple linear group with $|G/Z(G)| = 2^a \cdot 3^b \cdot 5^c$ where $c \geq 2$. By [6, 3E], $a \leq 14$, $b \leq 9$, $c \leq 8$. By [17], a Sylow 5-group is abelian. We assume from now on that G is a minimal counterexample to the theorem. We have seen $E(G)$ is a quasisimple group and in particular can assume $G = E(G)$. Note $Z(G)$ is cyclic of order 1, 2, 4 or 8 and $F(G) = Z(G)$ where $F(G)$ is the Fitting subgroup.

2. Special eigenvalue arguments

In this section we collect information about the possible eigenvalue structure of elements $X(g)$, $g \in G$. Note first that no element $X(g)$ can have an eigenspace of codimension 2 by [14]. (Such elements will be called special elements.) In particular any noncentral involution in $X(G)$ has trace 0. As $F(G) = Z(G)$, the 3- or 5-modular core is trivial. For definitions and properties of these cores see [6, 3A]. The 2-modular core is $Z(G)$. Occasionally we force some noncentral element into some modular core to provide a contradiction. We also note that by the quadratic pairs paper [20] no element of order 5 can have a quadratic minimal polynomial mod 5. By Lindsey [17] no element of order 5 can have an eigenspace of codimension 3.

We examine elements $X(g)$ with two equal eigenvalues. This is done in two lemmas. The first is general.

LEMMA 2.1. *Suppose X is a quasiprimitive representation of a group G in which for some g in G, $X(g)$ has only eigenvalues ε and $\bar{\varepsilon}$ where $\varepsilon = e^{2\pi i/5}$. Then any two conjugates of g either commute or generate $SL_2(5)$ with the center of $SL_2(5)$ in the center of G. Such groups are described in* [1] *and none are the alleged quasisimple group of degree* 8.

Proof. If h is a conjugate of g, $X|\langle g, h\rangle$ has at most 2-dimensional constituents by the argument of Blichfeldt in [3, page 143] for higher dimensions. If g and h do not commute there must be some irreducible 2-dimensional constituent. Let $X|\langle g, h\rangle = \sum_{i=1}^{r} X_i + \sum_{i=1}^{s} \lambda_i$ where X_i are irreducible of degree 2 and λ_i are linear. By [3], $X_i|\langle g, h\rangle$ must be isomorphic to $SL_2(5)$.

Let H be a minimal nonsolvable subgroup of $\langle g, h\rangle$. As $X|H$ has at most 2-dimensional constitutents $H \cong SL_2(5)$. If there are any linear constituents, there is an element of order 6 in $H = H'$ with at least one eigenvalue 1, the remaining eigenvalues 1, $-\omega$, or $-\bar{\omega}$. This contradicts Blichfeldt [3 p. 96]. It follows that $X|H$ and $X|\langle g, h\rangle$ have no linear constituents.

Let $X|\langle g, h\rangle = X_1 \oplus \cdots \oplus X_t$ where $t = n/2$. We have shown $X_i(\langle g, h\rangle) \cong$

$SL_2(5)$. Suppose $\langle g, h\rangle \not\cong SL_2(5)$ and so $1 \neq H = \ker X_1 \lhd \langle g, h\rangle$. As $X|H$ has linear constituents H is solvable and so as $X_i(H) \lhd X_i(\langle g, h\rangle)$, $X_i(H) = \pm I$. In particular H is an abelian 2-group in the center of $\langle g, h\rangle$. As $\langle g, h\rangle/H \cong SL_2(5)$ and the multiplier of A_5 has order 2, $H \cap \langle g, h\rangle' = e$ and this contradicts the fact that $\langle g, h\rangle$ is generated by elements of order 5. The central element of $SL_2(5)$ is scalar $-I$ in the center of G. This proves the lemma.

An alternate proof of this result uses the fact that over **C**, $X(g)$ has a quadratic minimal polynomial. The same is true mod 5 and so the group is known by [20]. However, we need the methods of this proof for our next lemma.

We return to our linear group G of degree 8.

LEMMA 2.2. *There is no element g in G for which $X(g)$ has eigenvalues*

$$\{\omega, \omega, \omega, \omega, \bar{\omega}, \bar{\omega}, \bar{\omega}, \bar{\omega}\}$$

where $\omega = e^{2\pi i/3}$.

Proof. We show that any two conjugates of g either commute or generate $SL_2(3)$ or $SL_2(5)$. If they generate $SL_2(5)$ the central element is in the center of G. This contradicts [19] or [2]. If g and h are noncommuting conjugates of g, as in Lemma 2.1, $X|\langle g, h\rangle$ has at most 2-dimensional constituents. If $X|\langle g, h\rangle$ has a 2-dimensional constituent representing $SL_2(5)$ we argue as in Lemma 2.1 to conclude $\langle g, h\rangle \cong SL_2(5)$ and the center of $\langle g, h\rangle$ is in $Z(G)$. Suppose then all nonlinear constituents X_i of $X|\langle g, h\rangle$ represent $SL_2(3)$. We want to show $\langle g, h\rangle \cong SL_2(3)$. Let $X_i|\langle g, h\rangle$ be represented on the 2-dimensional space V_i. As $X_i(\langle g, h\rangle) \cong SL_2(3)$, either $X_i(g)X_i(h)$ has order 6 or 4 by inspection in $SL_2(3)$.

If $X_i(g)X_i(h) = X_i(gh)$ has order 6, $X_i(gh^2)$ has order 4. Suppose there is an i and a j such that $X_i(gh)$ has order 6 and $X_j(gh)$ has order 4. If there is only one such j, $X((gh)^6)$ is a special 2-element. Consequently there are at least two such j's. Using gh^2 there are two such i's as well. Now $(gh)^6(gh^2)^4$ has eigenvalues $\{-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1, 1, 1, 1\}$ contradicting Blichfeldt. Suppose then that for all i, $X_i(gh)$ has order 6 and $\langle g, h\rangle$ is larger than$SL_2(3)$. As above there must be two or four such i's or $\pm X((gh^2)^2)$ would be a special 2-element. Note as in [2, Section 3] that as $X_i(\langle g, h\rangle)$ is $SL_2(3)$ and $X_i(gh)$ has order 3 for each i, $X_i(w(g, h)) = I$ iff $X_j(w(g, h)) = I$ for any word $w(g, h)$. Consequently if $\langle g, h\rangle \not\cong SL_2(3)$ there is an element k in $\langle g, h\rangle$ such that $X_i(k) = I$ and $k \neq i$. This means there is a linear constituent ξ on which $\xi(k) \neq 1$. It follows that there are two nonlinear constituents say X_1 and X_2 and four linear constituents of $X|\langle g, h\rangle$. Now

$$X((gh)^3) = \operatorname{diag}(-1, -1, -1, -1, 1, 1, 1, 1)$$

and

$$X(k) = \operatorname{diag}(1, 1, 1, 1, \omega_1, \omega_2, \omega_3, \omega_4)$$

where the ω_j are cube roots of 1. As $X(k)$ is not a special 3-element all ω_i are nontrivial and now $X(k(gh)^3)$ contradicts Blichfeldt. Note by the form of $X(g)$ and $X(h)$ if ω_i is 1, ω_j is 1 also for some $i \neq j$ as k is a word in g and h.

3. Sylow 5-subgroup intersections

In this section we show there must be some nontrivial Sylow 5-subgroup intersections. In particular we show there is some 5-subgroup A for which $C(A)$ contains more than one Sylow 5-subgroup. The two statements are equivalent as the Sylow 5-subgroup is abelian by [17].

Note first that if P is a Sylow 5-subgroup $|G: N(P)| = 2^\alpha 3^\beta$ where $\alpha \leq 14$, $\beta \leq 9$ is the number of Sylow 5-subgroups. As X is quasiprimitive, $2^\alpha 3^\beta \neq 1$. If $|P| \geq 5^3$, no number $2^\alpha 3^\beta$ with $\alpha \leq 14$, $\beta \leq 9$ is congruent to 1 mod 125. Consequently there must be some nontrivial Sylow 5-subgroup intersections. The only integers $2^\alpha \cdot 3^\beta$, $\alpha \leq 14$, $\beta \leq 9$, congruent to 6 mod 125 are 6 and 2^8. Certainly $|G: N(P)| \neq 6$ here as G is quasisimple. If $|G: N(P)| = 2^8$ and $P \cong Z_{25} \times Z_5$ an element π in $\mho^1(P)$ would have 2^β conjugates contradicting [10, Th 4.3.3]. This proves the following lemma.

LEMMA 3.1. *If P is a Sylow 5-subgroup and $|P| \geq 5^3$ there is an A in P such that $C(A)$ has more than one Sylow 5-subgroup. Suppose $P \cong Z_{25} \times Z_5$ and $\mho^1(P) = \langle \pi \rangle$. If $C(\pi)$ has 6 Sylow 5-subgroups, there must be some Sylow 5-subgroup Q such that $P \cap Q$ does not contain $\langle \pi \rangle$ and $P \cap Q \neq e$.*

We turn to the case in which a Sylow 5-subgroup of G has order 5^2. If P is strongly selfcentralizing a result of Sibley [18] shows G does not exist. This can also be handled by the results of [8] in this special case. This means there is an element $\pi \neq e$ in P such that $C(\pi)$ is not $PZ(G)$. An argument of J. Leon appearing below and in Leon's Caltech thesis shows $\langle \pi \rangle$ is a defect group and so is a Sylow intersection.

LEMMA 3.2. *Suppose G is a simple group of order $2^a \cdot 3^b \cdot 5^c$. Suppose σ is an element of order $r = 2$ or 3 in $C(\pi)$, π an element of order 5 in P. Then $B_0(r)$ contains a character χ of degree $5 \cdot r^\alpha$, $\chi(\pi\sigma) \neq 0$, $\langle \pi \rangle$ is the 5-defect group for the 5-block containing χ. In particular $C(\pi)$ has a 5-block with defect group $\langle \pi \rangle$ and $\langle \pi \rangle$ is a Sylow 5-subgroup intersection.*

Proof. As $\pi\sigma$ is an r-singular element the orthogonality relations for modular characters give $0 = \sum \chi(1)\chi(\pi\sigma) = 1 + \sum' \chi(1)\chi(\pi\sigma)$ where the first sum is over all characters in $B_0(r)$, the second is over the characters with the exception of the trivial character. There must be some nontrivial character χ in $B_0(r)$ with $\chi(\pi\sigma) \neq 0$, $\chi(1) \not\equiv 0 \pmod{r'}$ where if $r = 2$, $r' = 3$ and if $r = 3$, $r' = 2$. This means $\chi(1) = r^\alpha$, $r^\alpha \cdot 5$, or $r^\alpha \cdot 5^2$. Now $\chi(1) \neq r^\alpha$ by [7] as G is simple and there is only one character of degree 1. If $\chi(1) = r^\alpha \cdot 5^2$, $\chi(\pi\sigma) = 0$ as χ has 5-defect 0 [5]. Therefore $\chi(1) = 5 \cdot r^\alpha$. By [4], χ belongs to a 5-block of defect 1. The defect group must be $\langle\pi\rangle$ by [4] as $\chi(\pi\sigma) \neq 0$. The remaining statements follow by [5].

We apply this in the following special way.

LEMMA 3.3. *Under the conditions of Lemma* 3.2 *if* $C(\pi)$ *has elements of order* 3 *and* 2, *either* $C(\pi)$ *has more than one* 5-*block with defect group* $\langle\pi\rangle$ *or* π *is conjugate to all of its nontrivial powers.*

Proof. By Lemma 3.2, $C(\pi)$ has at least one 5-block with defect group $\langle\pi\rangle$. If there is only one, G has characters of degree $5 \cdot 2^\alpha$ and $5 \cdot 3^\beta$ in the 5-block of G corresponding to the 5-block of $C(\pi)$ by Lemma 3.2. Let $s = |N\langle\pi\rangle/C\langle\pi\rangle|$. Clearly $s = 1, 2$, or 4. If s is 1, all characters have the same degree impossible in this situation as Lemma 3.2 ensures faithful characters of degrees $5 \cdot 2^\alpha$ and $5 \cdot 3^\beta$. Neither α nor β can be 0 by [6]. If s is 2, the degree equation is

$$\chi_1(1) \pm \chi_2(1) \pm \chi_3(1) = 0.$$

Here we have $5 \cdot 2^\alpha \pm 5 \cdot 3^\beta \pm 5 \cdot 2^\gamma 3^\delta = 0$. This implies a character of G of degree 5 contradicting [6]. This shows $s = 4$ and π is conjugate to all its powers.

We conclude this section with a Lemma examining how an element of order 5 normalizes a nontrivial 2-group or 3-group Q. Except in special cases, the element must centralize Q.

LEMMA 3.4. *Suppose* π *is a* 5-*element of* G *which normalizes but does not centralize a group* Q *which is either a* 2-*group or a* 3-*group. Then* Q *is a nonabelian* 2-*group and* $X|Q$ *is either irreducible or has two constituents of degree* 4.

Proof. Assume Q is a minimal group on which π acts nontrivially. By [10, Theorem 5.3.6], $[Q, \pi] = Q$. If Q is abelian, $X|\langle Q, \pi\rangle$ has an irreducible constituent of degree 5 and 3 linear characters trivial when restricted to Q. This means Q is elementary of rank 4 and contains a special 2 or 3-element depending on whether Q is a 2-group or a 3-group.

This means Q is nonabelian. By [10, Theorem 5.3.7], Q is special. If Q is a 3-group, $X|Q$ has at most 3-dimensional constituents. An irreducible 3-dimensional 3-group is induced and the Frattini factor has rank 2. This means an element of order 5 centralizes Q contradicting our situation as $[Q, \pi] = Q$. If Q is a 2-group, an irreducible constituent in which $X(\pi)$ acts nontrivially must have degree 4 or 8. If there is only one constituent of degree 4, $X|Q$ has an irreducible 4-dimensional constituent plus four trivial constituents. There must be a special 2-element. This proves the lemma.

4. The structure of $C(A)$

Let H be $O^{5'}(C(A))$ where $A \neq 1$ is a 5-group. Assume H has more than one Sylow 5-group. Such an H exists by Lemma 3.1, Lemma 3.2 and the remarks preceding Lemma 3.2. Note $H = O^{5'}(H)$. Let $F^*(H) = F^* = E(H)F(H)$

be the generalized Fitting subgroup of H. As H has abelian Sylow 5-groups, H centralizes $O_5(F(H)) \supseteq A$. As $H = O^{5'}(H)$ and 5-elements of H centralize $O_3(H)$ by Lemma 3.4, H centralizes $O_3(H)$. As $A \neq 1$, A centralizes $O_2(H)$, and elements $X(a)$, $a \in A^{\#}$, have at least three distinct eigenvalues, $X|O_2(H)$ has at least three irreducible constituents. This means 5-elements in H must centralize $O_2(H)$ by Lemma 3.4. As $H = O^{5'}(H)$, H centralizes $O_2(H)$. Now $F(H) \subseteq Z(H)$ and so $F(H) = Z(H)$. Now H is not abelian as H has more than one Sylow 5-group and so $E(H) \neq 1$. We now let $K = AE(H)$ and consider the possibilities of $X|K$. Denote $E = E(H)$.

Note first that $A \subseteq Z(K)$ and as $X(a)$, $a \in A^{\#}$, has at least three distinct eigenvalues $X|K$ must have at least three distinct constituents. As no element $X(a)$, $a \in A^{\#}$ can have 5 identical eigenvalues, $X|K$ must have constituents of degree at most four. Assume first $X|K$ has a 4-dimensional constituent and $X|K = Y \oplus W$ where Y is irreducible of degree 4. Here W must be reducible. As $E = E^{\infty}$, all constituents represent faithfully some homomorphic image of E which is some central product of quasisimple groups. All constituents are of course unimodular. In particular $Y(E)$ is listed in [3]. Note E is generated by 5-elements and so $Y(E)$ could not be imprimitive.

We see then $Y(E)$ is isomorphic to A_5, $SL_2(5)$, a central extension over a center of order 2 of A_6, a central extension of $Sp_4(3)$ or Y is a tensor product of two 2-dimensional groups isomorphic to $SL_2(5)$. Let L_1 be the elements of E for which $W(L_1) = I$ and L_2 those for which $Y(L_2) = I$. If $W(L_2)$ is nonabelian W must represent A_5 or a sum of 1 or 2 representations of $SL_2(5)$. Either there is a special 2-element or a Blichfeldt element with eigenvalues 1, 1, 1, 1, $-\omega$, $-\bar{\omega}$, $-\omega$, $-\bar{\omega}$ [2, p. 96]. Now Y represents E faithfully except possibly for central elements.

However, if Y represents $Sp_4(3)$, W cannot represent it except trivially as W is reducible and no central extension of $Sp_4(3)$ can be represented faithfully in 3 dimensions. Consequently W is trivial and there are blatant special elements. If Y represents a cover of A_6, W is either trivial in which case there are special elements or $W(E)$ is a central extension of A_6 over a center of order 3. Now the central 3-element in $W(E)$ is represented by eigenvalues 1, 1, 1, 1, ω, ω, ω, 1. A Sylow 5-group has distinct linear characters and so this element is in the 3-modular core a contradiction as in Section 2. If Y is a tensor product of two 2-dimensional representations of $SL_2(5)$, W is a direct sum of either one or two subgroups representing $SL_2(5)$ or W contains a 3-dimensional constituent representing A_5. In any case, a subgroup U of E with $E/U \cong SL_2(5)$ or A_5 has at least a 2-dimensional trivial constituent when represented by W. Now $X|U$ is a sum of two or three 2-dimensional representations of $SL_2(5)$. There is an element with eigenvalues $-\omega$, $-\bar{\omega}$, $-\omega$, $-\bar{\omega}$, 1, 1, 1, 1 or $-\omega$, $-\bar{\omega}$, $-\omega$, $-\bar{\omega}$, $-\omega$, $-\bar{\omega}$, 1, 1 contradicting Blichfeldt [3, p. 96].

We are left with $Y(E) \cong SL_2(5)$ or A_5. Suppose first it is $SL_2(5)$. As A_5 has multiplier of order 2, $W(E) \cong A_5$ or $SL_2(5)$ as if W is trivial there is a special 3-element. If it is A_5, $W|E$ has a 3-dimensional constituent and a trivial constituent. Now X restricted to a Sylow 5-group has distinct linear characters. This means the involution in E must be in the 2-modular core. This is a contradiction as described in Section 2.

This means $W(E) \cong SL_2(5)$ and to avoid special elements, W has two 2-dimensional constituents. Let $W|K = W_1 \oplus W_2$ where W_i are irreducible of degree 2. As elements $X(a)$, $a \in A^{\#}$, have at least three distinct eigenvalues $W_1|A$ is not similar to $W_2|A$. This means X restricted to a Sylow 5-group P has distinct linear constituents. Again only elements of the 2 or 3-modular core can centralize P and so only central elements in G can centralize P.

We now show, for suitable A, $C(A) = AEZ(G)$. Recall $H = O^{5'}(C(A))$. If $\tau \in C(A)$, $X(\tau)$ acts on the spaces that W_1, W_2, and Y act on as these are distinct eigenspaces for the action of $X(A)$. As $SL_2(5)$ is maximal finite unimodular in two dimensional groups, $X(\tau)$ must act as an inner automorphism times a scalar on the spaces W_1 and W_2 act on. By multiplying by an element of E it can be assumed $X(\tau)$ is a scalar when restricted to the space W_1 acts on. If $X(\tau)$ is nonscalar on either of the other invariant spaces, conjugates of τ by elements of E together with A generate a group containing special elements. This shows we may choose any elements in $C(A)$ not in E to centralize E and so centralize a Sylow 5-group P of H. Now $C(A) = AEZ(G)S$ where S is a 5-group centralizing E and A. This applies to elements in a Sylow 5-group P of G containing A. We may therefore replace A by $O_5(C(A))$ and obtain $C(A) = AEZ(G)$.

We note that if A has order 5, $C(\pi) \cong \langle \pi \rangle \times SL_2(5)Z(G)$. Here π is not conjugate to all its powers as it has exactly three distinct eigenvalues. This contradicts Lemma 3.3. Suppose $|A| > 5$. If A is noncyclic let $B \cong \langle b \rangle \times \langle a \rangle \subseteq A$ where $Y(b) = I$. Now some element ab^i has five equal eigenvalues. Consequently A is cyclic of order 5^2 by [6, 3B]. If $A = \langle a \rangle$, $\langle a^5 \rangle = \mho^1(P)$ where again P is a Sylow 5-group. We can replace $\langle a \rangle$ by $\langle a^5 \rangle$ to obtain

$$C(a^5) = C(\mho^1(P)) = AEZ(G).$$

To apply Lemma 3.1 we look below for a Sylow 5-group Q such that $1 \neq P \cap Q$ and $\langle a^5 \rangle = \mho^1(P) \nsubseteq Q$. None of the cases allow this and Lemma 3.1 is contradicted.

Assume $V(E) \cong A_5$. Now either $W(E) \cong A_5$ or $W(E) \cong SL_2(5)$. In the latter case to avoid special elements, W has two irreducible constituents. Again the central element in E centralizes a Sylow 5-group with distinct linear characters putting it in the 2-modular core, a contradiction. In the first case $W|E$ has a 3-dimensional constituent. Again an element τ in $C(A)$ must normalize E and as the 3-dimensional representation of A_5 does not extend to S_5, $\tau\varepsilon$ must centralize E for some ε in E. It must then centralize a Sylow 5-group which again has distinct linear characters and so it is in some modular core. Again replace A by $O_5(C(A))$. As before, if $|A| = 5$, Lemma 3.3 is contradicted. If $|A| > 5$, A is cyclic of order 5^2 and if $A = \langle a \rangle$, $C(a^5)$ has 6 Sylow 5-groups, $\langle a^5 \rangle = \mho^1(P)$, P a Sylow 5-group. By Lemma 3.1 there is a Sylow 5-group Q such that $P \cap Q \neq 1$ and $a^5 \notin Q$. We continue to look for it.

We suppose now $X|E$ has at most 2-dimensional constituents. Each must represent $SL_2(5)$. By taking a minimal normal subgroup E_1 of E we get $X|E_1 \cong SL_2(5)$. Some constituent represents it faithfully. If there are fewer than four constituents there is a Blichfeldt element of order 6. If there are four irreducible constituents there is a 3-element with eigenvalues $\omega, \omega, \omega, \omega, \bar{\omega}, \bar{\omega}, \bar{\omega}, \bar{\omega}$. This contradicts Lemma 2.2.

The remaining case is $X|E$ has a constituent of degree 3 and none of degree 4. Let $X|E = Y \oplus W$ where Y is irreducible of degree 3. We see immediately that $Y(E)$ is A_5 or an extension $\tilde{A}_6$ of A_6 by a center of order 3 by [3].

Suppose first $Y(E)$ represents $\tilde{A}_6$. To avoid special elements, W must have a 3-dimensional constituent also representing $\tilde{A}_6$ and $E \cong \tilde{A}_6$. Now $X|E$ has two 3-dimensional constituents and two trivial constituents. There can be no elements with eigenvalues $\omega, \omega, \omega, \omega, \omega, \omega, 1, 1$ and if Q is a Sylow 3-group of E, $X|Q$ has two distinct 3-dimensional constituents Y_1 and Y_2 and two trivial constituents. Now A centralizes Q and so in a suitable base mod 5, $X(a)$ has at most a quadratic minimal polynomial, $a \in A^{\#}$. This can be seen by reducing mod 5 and choosing a base in which $X|Q = Y_1 \oplus Y_2 \oplus 2 \cdot 1_Q$. Then $X(a) = I_3 \oplus I_3 \oplus B$ where B is a 2×2 matrix. Now G is known by [20].

We now assume $Y(E) \cong A_5$. Suppose first W has a 3-dimensional constituent which can be taken isomorphic to A_5 by the above. Now by the subdirect product theorem $E \cong A_5$ or $SL_2(5)$. If it is $SL_2(5)$, $W|E$ is faithful and, $W|E$ has a 2-dimensional faithful constituent. The center is a special 2-element. Consequently $E \cong A_5$ and $X|K = Y_1 \oplus Y_2 \oplus Y_3$ where Y_1 and Y_2 are irreducible of degree 3 and where Y_3 is a direct sum of two linear characters, $Y_3|E = 2 \cdot 1_E$. If τ is an element of $C(A)$, $X(\tau)$ acts on each of the 3-dimensional spaces and the 2-dimensional space as $Y_1(a) \neq Y_2(a)$ for $a \in A^{\#}$. As no outer automorphism in 3 dimensions lifts to a 3-dimensional group, $X(\tau\varepsilon)$ with ε in E is scalar on one and hence both 3-dimensional subspaces. Assuming $X(\tau\varepsilon)$ is not scalar or an element with eigenvalues not allowed,

$$X(\tau\varepsilon) = \operatorname{diag}(\lambda, \lambda, \lambda, \mu, \mu, \mu, \alpha, \beta).$$

Note $\mu \neq \lambda$ unless $X(\tau\varepsilon)$ is scalar. If $\tau\varepsilon$ is a 2-element let S be a Sylow 2-group of $E \cdot (\tau\varepsilon)$. Now $X|S = \oplus \sum_{i=1}^{8} \lambda_i$; all λ_i are distinct except possibly λ_7 and λ_8. As A centralizes S, elements in A mod 5 have a quadratic minimal polynomial counter to [20]. A similar argument applies if $\tau\varepsilon$ is a 3-element. If $\tau\varepsilon$ is a 5-element we may augment A with $\tau\varepsilon$. Take A maximal centralizing E. If $|A| = 5$, $C(A) = A \times E \times Z(G)$ and Lemma 3.3 applies. Here $a \in A^{\#}$ cannot be conjugate to all its powers as the eigenvalues are inconsistent. If $|A| > 5$ again A is cyclic and we get if $A = \langle a \rangle$, $C(a^5) = A \times E \times Z(G)$. As $\langle a^5 \rangle = \mho^1(P)$, P a Sylow 5-group, Lemma 2.1 gives a Sylow 5-group Q with $P \cap Q \neq 1$ and $a^5 \notin Q$ still to be found.

We are left with $Y(E) \cong A_5$ and W has no 3-dimensional constituents. This means W has a 2-dimensional constituent representing $SL_2(5)$ or there are special elements. It follows as above $E \cong SL_2(5)$ and $W = W_1 \oplus W_2 \oplus \lambda$ where W_i are irreducible of degree 2 and λ is linear.

There is now an element γ of E for which $X(\gamma)$ has eigenvalues

$$\{1, \omega, \bar{\omega}, -\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1\}.$$

Each eigenvalue has multiplicity 1 or 2 and in a suitable basis over a field of characteristic 5, an element $a \in A^{\#}$ has a quadratic minimal polynomial. This contradicts [20]. This completes the proof by showing G does not exist. Consequently the theorem is proved.

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