COMPLETELY REDUCIBLE ACTIONS OF CONNECTED ALGEBRAIC GROUPS ON FINITE-DIMENSIONAL ASSOCIATIVE ALGEBRAS

BY

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Introduction

Let G be a group which acts completely reducibly by algebra automorphisms on a finite-dimensional associative K-algebra A, which is separable modulo its radical.

When the characteristic of K is zero, Mostow showed in [4], using the representation theory of reductive algebraic groups, that there is a G-invariant separable subalgebra of A complementary to the radical (a G-invariant Wedderburn factor).

Taft in [5] conjectured that there is a G-invariant Wedderburn factor when characteristic K is $p \neq 0$.

In this paper, we verify the conjecture when K is perfect and the image of G in the algebraic group of algebra automorphisms of $A \otimes_K \bar{K}$ has connected closure [see Theorem 1].

Relevant facts about separable algebras may be found in [1, Section 72], and about algebraic groups in [3].

Let A be a finite-dimensional associative algebra over a field K, with radical R. Suppose that A/R is a separable algebra and that S is a separable subalgebra of A complementary to R. S will be called a Wedderburn factor of A. Let $p\colon A \to R$ be the projection of the sum $A = S \oplus R$ onto the factor R; let $\pi\colon A \to A/R$ be the quotient map.

Let G be a group which acts completely reducibly on A by algebra automorphisms. Write gb for the image of $b \in A$ under $g \in G$.

All mappings are K-linear.

Section 1

Throughout this section, $R^2 = (0)$.

Let V be a G-invariant subspace of A complementary to R, and let $h\colon A \to R$ be the projection of the sum $A = V \oplus R$ onto the factor R. Let $f = h \mid S\colon S \to R$.

We introduce a second action $(*)$ of G on A which stabilizes S. The two actions coincide if and only if S is G-invariant under the original action.

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DEFINITION 1. For $g \in G$, let

$$g * b = \begin{cases} gb & \text{if } b \in R \\ gb - p(gb) & \text{if } b \in S. \end{cases}$$

When a subspace W of A is invariant under the $(*)$-action of G, we will say that W is G_*-invariant.

LEMMA 1. *Under $(*)$, G acts completely reducibly on A by algebra automorphisms.*

Proof. Section 4.1.

Lemmas 2–4 and Proposition 1 below describe some properties of f relative to the two actions of G on A. Let $\mathrm{Hom}_{G_*}(S, R)$ be the G-module homomorphisms relative to the $(*)$-action.

LEMMA 2. *For $a \in S$, $ga - g * a = g * f(a) - f(g * a)$.*

Proof. Section 4.2.

Thus, S is G-invariant if and only if $f \in \mathrm{Hom}_{G_*}(S, R)$.

The following Hochschild cohomology sequence will be convenient for our purposes.

DEFINITION 2.

$$R \xrightarrow{\delta_1} \mathrm{Hom}\,(S, R) \xrightarrow{\delta_2} \mathrm{Hom}\,(S \otimes S, R)$$

is the exact [2, Theorem 4.1] sequence such that:

(a) For $r \in R$, $s \in S$, $\delta_1 r(s) = sr - rs$.

(b) For $f \in \mathrm{Hom}\,(S, R)$ and $s, s' \in S$,

$$\delta_2 f(s \otimes s') = sf(s') + f(s)s' - f(ss').$$

The kernel of δ_2 is the space $\mathrm{Der}\,(S, R)$ of derivations in $\mathrm{Hom}\,(S, R)$; since the sequence is exact, $\delta_1 R = \mathrm{Der}\,(S, R)$, i.e., every derivation is inner.

Let G act on $S \otimes S$ by the diagonal $*$-action:

$$g * (s \otimes s') = g * s \otimes g * s'.$$

For $N = S$ or $S \otimes S$, let G act on $\mathrm{Hom}\,(N, R)$ by

$$(gf)(n) = g * (f(g^{-1} * n)) \quad \text{for } f \in \mathrm{Hom}\,(N, R) \text{ and } n \in N.$$

$\mathrm{Hom}_{G_*}(N, R)$ is then the space of G-fixed elements in $\mathrm{Hom}\,(N, R)$; furthermore, a straightforward verification shows that δ_1 and δ_2 are G-module morphisms.

Since $R^2 = (0)$, $\{(1 + r)S(1 - r) \mid r \in R\}$ is the set of Wedderburn factors in A. In what follows, $f = h \mid S$.

LEMMA 3. *$(1 + r)S(1 - r)$ is G-invariant if and only if $f - \delta_1 r$ is in $\mathrm{Hom}_{G_*}(S, R)$.*

Proof. Section 4.3.

As a consequence of Lemma 3, we have the following proposition:

PROPOSITION 1. *There is a G-invariant Wedderburn factor in A if and only if f is in* $\mathrm{Hom}_{G_*}(S, R) + \delta_1 R$.

LEMMA 4. $\delta_2 f \in \mathrm{Hom}_{G_*}(S \otimes S, R)$.

Proof. Section 4.4.

PROPOSITION 2. (a) *The condition*

$$\delta_2(\mathrm{Hom}_{G_*}(S, R)) = \mathrm{Hom}_{G_*}(S \otimes S, R) \cap \delta_2(\mathrm{Hom}(S, R))$$

is sufficient for the existence of a G-invariant Wedderburn factor.

(b) *Let F be a field extension of K.*

$$\delta_2(\mathrm{Hom}_{G_*}(S, R)) = \mathrm{Hom}_{G_*}(S \otimes S, R) \cap \delta_2(\mathrm{Hom}(S, R))$$

if and only if

$$\delta_2(\mathrm{Hom}_{G_*}(S \otimes F, R \otimes F)) = \mathrm{Hom}_{G_*}(S \otimes F \otimes_F S \otimes F, R \otimes F) \cap \delta_2(\mathrm{Hom}(S \otimes F, R \otimes F)).$$

Proof. (a) From the condition and Lemma 4, we have

$$\delta_2 f \in \delta_2(\mathrm{Hom}_{G_*}(S, R)),$$

i.e., there exists $f_1 \in \mathrm{Hom}_{G_*}(S, R)$ such that $f - f_1 \in \mathrm{Ker}\,\delta_2 = \delta_1 R$.

(b) This is a straightforward verification which we omit.

PROPOSITION 3. *If there is a G-invariant complement M to* $\mathrm{Der}(S, R)$ *in* $\mathrm{Hom}(S, R)$, *then*

$$\delta_2(\mathrm{Hom}_{G_*}(S, R)) = \mathrm{Hom}_{G_*}(S \otimes S, R) \cap \delta_2(\mathrm{Hom}(S, R)).$$

Proof. The proof is group-theoretical. We have

$$\begin{aligned}
&\delta_2(\mathrm{Hom}(S, R)) \cap \mathrm{Hom}_{G_*}(S \otimes S, R) \\
&= \delta_2(M) \cap \mathrm{Hom}_{G_*}(S \otimes S, R) \\
&= \delta_2(M \cap \mathrm{Hom}_{G_*}(S, R)) \quad \text{since } \delta_2 \mid M \text{ is an injective } G\text{-module morphism,} \\
&= \delta_2(\mathrm{Hom}_{G_*}(S, R)).
\end{aligned}$$

Section 2

We give circumstances under which the hypothesis of Proposition 3 holds.

2.1. Let $R^2 = (0)$. Let K be a perfect field, $\bar{K}$ the algebraic closure of K, and $\bar{A}$ the $\bar{K}$-algebra $A \otimes \bar{K}$. Let $\mathrm{Aut}(\bar{A})$ be the algebraic group of $\bar{K}$-algebra automorphisms of $\bar{A}$.

Let $t_*\colon G \to \operatorname{Aut}(\bar{A})$ be the group homomorphism determined by the $(*)$-action of G on $\bar{A}$, and $\overline{t_*(G)}$ the closure of $t_*(G)$ in $\operatorname{Aut}(\bar{A})$.

PROPOSITION 4. *If $\overline{t_*(G)}$ is connected, then there is a G-invariant complement to* Der (S, R) *in* Hom (S, R).

Proof. Section 4.5.

2.2. Let n be the index of nilpotency of R. Let $t\colon G \to \operatorname{Aut}(\bar{A})$ be the group homomorphism determined by the original action of G on A.

THEOREM 1. *K a perfect field.*

If $\overline{t(G)}$ is a connected subgroup of Aut $(\bar{A})$, *then there is a G-invariant Wedderburn factor in A.*

In particular, if G is a connected algebraic group which acts rationally on A, then there is a G-invariant Wedderburn factor in A.

Proof. By induction on n. Denote $\overline{t(G)}$ by H.

By [3, Proposition 1.4], since K is perfect and G acts completely reducibly on A, G acts completely reducibly on $\bar{A}$. Since G and H stabilize the same subspaces of $\bar{A}$, H acts completely reducibly on $\bar{A}$.

Let $j\colon \operatorname{Aut}(\bar{A}) \to \operatorname{Aut}(\bar{A}/\bar{R}^2)$ be the natural morphism of algebraic groups. j induces a completely reducible rational action of H on $\bar{A}/\bar{R}^2$. The $(*)$-action of H on $\bar{A}/\bar{R}^2$ is also rational, since $\bar{S}/\bar{R}^2$, as an H_*-module, is canonically isomorphic to the rational H-module $\bar{A}/\bar{R}$. Thus the natural map $t_*\colon H \to \operatorname{Aut}(\bar{A}/\bar{R}^2)$ is a morphism of algebraic groups, and hence $t_*(H)$ is a connected algebraic subgroup of $\operatorname{Aut}(\bar{A}/\bar{R}^2)$. By Proposition 4, there is an H-invariant (hence G-invariant) complement to $\operatorname{Der}(\bar{S}/\bar{R}^2, \bar{R}/\bar{R}^2)$ in $\operatorname{Hom}(\bar{S}/\bar{R}^2, \bar{R}/\bar{R}^2)$; therefore by Proposition 3,

$$\delta_2(\operatorname{Hom}_{G_*}(\bar{S}/\bar{R}^2, \bar{R}/\bar{R}^2)) = \operatorname{Hom}_{G_*}(\bar{S}/\bar{R}^2 \otimes \bar{S}/\bar{R}^2, \bar{R}/\bar{R}^2) \cap \delta_2(\operatorname{Hom}(\bar{S}/\bar{R}^2, \bar{R}/\bar{R}^2)).$$

Hence by Proposition 2(b), (a), there is a G-invariant Wedderburn factor T in A/R^2.

Let $p\colon A \to A/R^2$ be the quotient G-module morphism. $p^{-1}(T)$ is a G-invariant subalgebra of A with radical R^2, which has index of nilpotency less than n. The action of H on $\overline{p^{-1}(T)}$ is completely reducible and the image of H in $\operatorname{Aut}(\overline{p^{-1}(T)})$ is connected since H is connected. Therefore, by induction, there is an H-invariant (hence G-invariant) Wedderburn factor S in $p^{-1}(T)$. S is also a Wedderburn factor in A.

Section 3

Here more information is given on the significance of the condition of Proposition 2(a) with regard to the existence of G-invariant Wedderburn factors.

Let $R^2 = (0)$. Let $(*)$ be any completely reducible action of G on A which stabilizes S. A completely reducible action of G on A is called a twisting of $(*)$ if the action induces $(*)$ according to Definition 1.

PROPOSITION 5. *There is a G-invariant Wedderburn factor for each twisting of* $(*)$ *if and only if*

$$\delta_2\,(\mathrm{Hom}_{G*}\,(S, R)) = \mathrm{Hom}_{G*}\,(S \otimes S, R) \cap \delta_2\,(\mathrm{Hom}\,(S, R)).$$

Proof. $\Leftarrow$ Proposition 2(a).

$\Rightarrow$ Let $f \in \mathrm{Hom}\,(S, R)$ have the property $\delta_2 f \in \mathrm{Hom}_{G*}\,(S \otimes S, R)$. The following action is a twisting of $(*)$:

$$gb = g * b \quad \text{if } b \in R,$$

$$gb = g * b + g * f(b) - f(g * b) \quad \text{if } b \in S.$$

By the hypothesis, there is a G-invariant (relative to the twisted action) Wedderburn factor $(1 + r)S(1 - r)$. As in the proof (Section 4.3) of Lemma 3, one can compute that $f - \delta_1 r \in \mathrm{Hom}_{G*}\,(S, R)$. Hence,

$$\mathrm{Hom}_{G*}\,(S \otimes S, R) \cap \delta_2\,(\mathrm{Hom}\,(S, R)) \subset \delta_2\,(\mathrm{Hom}_{G*}\,(S, R)).$$

The other inclusion holds since δ_2 is a G-module morphism.

Using an induction on the index of nilpotency of R and Proposition 5, we have:

COROLLARY. *Let G be a group. There are G-invariant Wedderburn factors for all algebras and all completely reducible actions of G if and only if*

$$\delta_2\,(\mathrm{Hom}_G\,(S, R)) = \mathrm{Hom}_G\,(S \otimes S, R) \cap \delta_2\,(\mathrm{Hom}\,(S, R))$$

holds for all algebras with radical of square zero and completely reducible actions of G which stabilize a Wedderburn factor S.

4.1. *Proof of Lemma* 1. We have:

(a) Via $\pi \mid S$, S under $(*)$ is G-(and algebra) isomorphic to A/R under the original action.

(b) The two actions agree on R.

Since G acts completely reducibly on A, G acts completely reducibly on A/R. Therefore, by (a) and (b) G acts (via $(*)$) completely reducibly on S and R, and so on A.

The $(*)$-action is by algebra automorphisms: let $a, a' \in S$; $b, b' \in R$; and $g \in G$.

$$g * ((a + b)(a' + b'))$$
$$= g * (aa' + ab' + ba') \quad \text{since } R^2 = (0),$$
$$= g * (aa') + g(ab') + g(ba')$$
$$= (g * a)(g * a') + (ga)(gb') + (gb)(ga') \quad \text{by (a) above,}$$
$$= (g * a)(g * a') + (g * a)(g * b') + (g * b)(g * a')$$
$$\text{from the definition of } * \text{ and the fact that } R^2 = (0),$$
$$= (g * (a + b))(g * (a' + b')).$$

4.2. *Proof of Lemma* 2.

$$g * f(a) - f(g * a)$$
$$= g * f(a) - h(g * a - ga + ga)$$
$$= g * f(a) - h(-p(ga) + ga)$$
$$= g * f(a) + p(ga) - g(h(a))$$
$$\text{since } h \mid R = \text{id and } h \text{ is a } G\text{-module morphism,}$$
$$= p(ga) \quad \text{since } f(a) = h(a) \in R$$
$$= ga - g * a.$$

4.3. *Proof of Lemma* 3. Let $s \in S$.

$$g((1 + r)s(1 - r)) = gs + (gr)(gs) - (gs)(gr)$$
$$= g * s + (gs - g * s) + (gr)(gs) - (gs)(gr).$$

Comparing the S and R components, we have that

$$g((1 + r)s(1 - r) \in (1 + r)S(1 - r)$$

if and only if

$$(gs - g * s) + (gr)(gs) - (gs)(gr) = r(g * s) - (g * s)r$$

if and only if

$$g * f(s) - f(g * s) + (gr)(gs) - (gs)(gr) = \delta_1 r(g * s) \quad \text{by Lemma 2}$$

if and only if

$$(f - \delta_1 r)(g * s) = g * ((f - \delta_1 r)(s)).$$

4.4. *Proof of Lemma* 4.

$$
\begin{aligned}
&g * f(ss') - f(g * (ss')) \\
&\quad = (gs)(gs') - (g * s)(g * s') \quad \text{by Lemmas 2 and 1,} \\
&\quad = (g * s + g * f(s) - f(g * s))(g * s' + g * f(s') - f(g * s')) - (g * s)(g * s') \\
&\qquad\qquad \text{by Lemma 2,} \\
&\quad = g * (f(s)s') - (g * s)f(g * s') + g * (sf(s')) - f(g * s)(g * s').
\end{aligned}
$$

Therefore, $g * (\delta_2 f(s \otimes s')) = \delta_2 f(g * (s \otimes s'))$.

4.5. *Proof of Proposition* 4. Since S is G_*-invariant, S is $\overline{t_*(G)}$-invariant. $\overline{t_*(G)}$ permutes the simple components of S, and the isotropy subgroup of a component has finite index in $\overline{t_*(G)}$. Since $\overline{t_*(G)}$ is connected, each isotropy subgroup is $\overline{t_*(G)}$, i.e., each simple component of S is $\overline{t_*(G)}$ (hence G_*)-invariant.

S is the direct sum of full matrix algebras $\{M_i\}_{i=1}^n$, since K is algebraically closed. Let e_i be the identity element of M_i; $1 \in S$ is the orthogonal direct sum $\sum_{i=1}^n e_i$ and each e_i is G_*-fixed.

R has the G_*-invariant decomposition $\sum e_i R e_j$, where each $e_i R e_j$ is an S-bimodule. Therefore, Hom (S, R) has the two G-invariant decompositions $\sum$ Hom $(S, e_i R e_j)$ and

$$\sum_i \operatorname{Hom}(M_i, e_iRe_i) \oplus \sum_{i \neq j} \operatorname{Hom}(M_i \oplus M_j, e_iRe_j) \oplus \sum_{k \neq i, j} \operatorname{Hom}(M_k, e_iRe_j).$$

The derivations, which are all inner, have the G-invariant decomposition

$$\operatorname{Der}(S, R) = \sum_i \operatorname{Der}(M_i, e_iRe_i) \oplus \sum_{i \neq j} \operatorname{Der}(M_i \oplus M_j, e_iRe_j),$$

since Der $(M_k, e_iRe_j) = 0$ for $k \neq i, j$ by the orthogonality of $\{e_k\}$.

To prove the proposition we show that there are G-invariant complements to

(1) Der $(M_i \oplus M_j, e_iRe_j)$ in Hom $(M_i \oplus M_j, e_iRe_j)$ for $i \neq j$,

(2) Der (M_i, e_iRe_i) in Hom (M_i, e_iRe_i).

Let $M_i = M$ and $e_iRe_j = T$.

Lemma A. *$f_1: M \otimes T \to$ Hom (M, T), defined by $f_1(m \otimes t)(n) = mnt$ for $m, n \in M$, $t \in T$, is an isomorphism of G-modules.*

Here $M \otimes T$ has the diagonal G_*-module structure.

Proof. Let $m \times m$ be the size of M. Let $T = \sum_k V_k$ be a decomposition of T into simple left M-modules. Since V_k is isomorphic to K^m as M-modules, it will suffice to show

$$M \otimes K^n \xrightarrow{\cong} \operatorname{Hom}(M, K^n).$$

This is readily checked by linear algebra.

Similarly, f_2: $T \otimes M_j \to \operatorname{Hom}(M_j, T)$, defined by $f_2(t \otimes m)(n) = tnm$, is a G-module isomorphism.

For (1) above, it follows from the lemma that

$$M_i \otimes T \oplus T \otimes M_j \xrightarrow{\cong} \operatorname{Hom}(M_i \oplus M_j, T).$$

Under this isomorphism, Der $(M_i \oplus M_j, T)$ and $\{(e_i \otimes r, -r \otimes e_j) \mid r \in T\}$ correspond.

Let W be a G_*-invariant complement to $K \cdot e_j$ in M_j. Then, $M_i \otimes T \oplus T \otimes W$ is a G_*-invariant complement to

$$\{(e_i \otimes r, -r \otimes e_j) \mid r \in R\}$$

in $M_i \otimes T \oplus T \otimes M_j$. This completes the proof of (1).

(2) Let $M = M_i$ and $T = e_i R e_i$. Let M° be the algebra opposite to M. $M \otimes M^\circ$ is a full matrix algebra and T is a left-$M \otimes M^\circ$-module, where $(N \otimes N')r = NrN'$ for $N \in M$, $N' \in M^\circ$ and $r \in R$. Let $T = \sum_k V_k$ be the decomposition of T into simple $M \otimes M^\circ$-modules. Each V_k is isomorphic to M with the natural $M \otimes M^\circ$-module structure. Therefore, we may identify each V_k with a copy $M^{(k)}$ of M.

Let W be a G_*-invariant complement to Ke_i in M, and let $W^{(k)}$ be the copy of W in $M^{(k)}$. Let e $(= e_i)$ be the neutral element of M and $e^{(k)}$ that of $M^{(k)}$.

Lemma B. *$\sum W^{(k)} \subset T$ is a G_*-invariant complement to $\sum Ke^{(k)}$ in T.*

Proof. For $g \in G$, let t be the automorphism of $M \oplus T$ given by the $(*)$-action of G on A. Let $u = t \mid M$. By the Skolem-Noether theorem, u is conjugation by some invertible element B of M. Extend u to an automorphism $\bar{u}$ of $M \oplus T$ by: $\bar{u} \mid M^{(k)} =$ conjugation by B.

$t \circ \bar{u}: \sum M^{(k)} \to \sum M^{(k)}$ is readily checked to be an $M \otimes M^\circ$-module automorphism of T. Therefore, $t \circ \bar{u}$ is described by a matrix (N_{ij}) where N_{ij}: $M^{(i)} \to M^{(j)}$ is an $M \otimes M^\circ$-module morphism. Therefore, by linear algebra, N_{ij} is a scalar multiplication. Hence, $t \circ \bar{u}$ leaves $\sum W^{(k)}$ invariant.

Since $\bar{u}$ leaves W-invariant, $\bar{u}$ leaves $\sum W^{(k)}$ invariant. Therefore, $t = t \circ \bar{u} \circ \bar{u}$ leaves $\sum W^{(k)}$ invariant. This completes the proof of Lemma B.

By Lemma A, f_1: $M \otimes T \to \operatorname{Hom}(M, T)$ is a G-module isomorphism. Under f_1, Der (M, T) and

$$\left\{ e \otimes \sum N^{(k)} - \sum_k (N_k \otimes e^{(k)}) \;\middle|\; N_k \in W;\ N^{(k)} \text{ the copy of } N_k \text{ in } W^{(k)} \right\}$$

correspond.

A G_*-invariant complement to the latter space in $M \otimes T$ is

$$W \otimes T \oplus \left(K \cdot e \otimes \sum_k K \cdot e^{(k)} \right).$$

This completes the proof of (2), and of Proposition 4.

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