ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 2, Summer 1980

INTERPOLATION SETS AND EXTENSIONS OF THE GROTHENDIECK INEQUALITY

BY

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The key step in the proof of the Grothendieck inequality is an "integral representation" of the inner product in a Hilbert space (see Theorem 2.3 in [1], for example):

THEOREM. *There is a compact abelian group G, a constant $K > 0$, and a function Φ: $l^2 \to L^\infty(G)$ so that*

(i) *for all $x \in l^2$, $\|\Phi(x)\|_\infty \le K\|x\|_2$,*

and

(ii) *for all $x, y \in l^2$, $(x, y) = (\Phi(x) * \Phi(\bar{y}))(0)$ ($*$ denotes convolution).*

A natural task is to extend the above theorem and design an analogous representation for the dual action between l^p and l^q, where p and q are conjugate exponents. This is what we do in this paper. The present work could be viewed as a postscript to [1], and, indeed, methods here are modifications of those used in [1].

We employ basic notation and facts of commutative harmonic analysis as presented and followed in [5]. Γ, as usual, will be a discrete group and $G = \Gamma^\wedge$ will denote its compact dual group. In the first section, work will be performed in the framework of $\otimes \mathbf{Z}_2 = \Omega$, the (compact) direct product of $\mathbf{Z}_2$, and $\oplus \mathbf{Z}_2 = \hat{\Omega}$, its (discrete) dual group, the direct sum of $\mathbf{Z}_2$. Throughout, $E = \{r_n\}_{n=1}^\infty \subset \hat{\Omega}$ will denote the system of Rademacher functions realized as characters in $\hat{\Omega}$. In Section 1, we extend the notions of $\Lambda(2)$ and Sidon sets: $F \subset \Gamma$ is an $L(p)$ set, $1 < p \le 2$, if for every $\phi \in l^p(F)$ there is $f \in L^\infty(G)$ so that $\hat{f} = \phi$ on F, and $\hat{f} \in l^p$. Analogously, F is an $S(q)$ set, $2 < q \le \infty$, if for every $\phi \in l^q(F)$ there is $\mu \in M(G)$ so that $\hat{\mu} = \phi$ on F and $\hat{\mu} \in l^q$. These notions lead to an integral representation of the dual action between l^p and l^q (Theorem 1.3), and to an extension of the classical Grothendieck inequality (Corollary 1.4). Deserving a study for its own sake, the $L(p)$ property is briefly examined in the second section, where a class of $L(p)$ sets is obtained as a subclass of $\Lambda(q)$ sets of a certain type (Theorem 2.2). We conclude with some questions.

Received March 29, 1978.

[1] The author was partially supported by a National Science Foundation grant.

1. A representation of the dual action between l^p and l^q

Let $1 \le p \le q \le \infty$. We set $A_p(G) = \{f \in L^\infty(G): \hat{f} \in l^p\}$ and norm $A_p(G)$ by

$$\|f\|_{A_p} = \max\{\|f\|_\infty, \|\hat{f}\|_p\}.$$

Similarly, we let $M_q(G) = \{\mu \in M(G): \hat{\mu} \in l^q\}$ and norm $M_q(G)$ by

$$\|\mu\|_{M_q} = \max\{\|\mu\|_M, \|\hat{\mu}\|_q\}.$$

DEFINITION 1.1. (a) Let $1 < p \le 2$. $F \subset \Gamma$ is an *$L(p)$ set* if whenever $\phi \in l^p$ there is $f \in A_p(G)$ so that $\hat{f} = \phi$ on F. F is a *uniformizable $L(p)$ set* if for all $0 < \delta < 1$, there is $\beta_p(F, \delta) = \beta$ so that whenever $\phi \in l^p(F)$ there is a (uniformizing) $f \in A_p(G)$ with the following properties:

(i) $\hat{f} = \phi$ on F;
(ii) $\|f\|_{A_p} \le \beta\|\phi\|_p$;
(iii) $\|\hat{f}\,|_{\sim F}\|_p \le \delta\|\phi\|_p$ ($\hat{f}\,|_{\sim F}$ denotes the restriction of $\hat{f}$ to $\sim F$, where $\sim F$ denotes the complement of F).

(b) Let $2 < q \le \infty$. $F \subset \Gamma$ is an *$S(q)$ set* if whenever $\phi \in l^q(F)$ there is $\mu \in M_q(G)$ so that $\hat{\mu} = \phi$ on F. F is a *uniformizable $S(q)$ set* if for all $0 < \delta < 1$, there is $\lambda_q(F, \delta) = \lambda$ so that whenever $\phi \in l^q(F)$ there is a (uniformizing) $\mu \in M_q(G)$ with the following properties:

(i) $\hat{\mu} = \phi$ on F;
(ii) $\|\mu\|_{M_q} \le \lambda\|\phi\|_q$;
(iii) $\|\hat{\mu}\,|_{\sim F}\|_q \le \delta\|\phi\|_q$.

The $L(2)$ property is, of course, the usual $\Lambda(2)$ property. The $S(\infty)$ property is the usual Sidonicity property, and uniformizable $S(q)$ sets are referred to in [6] as uniformizable p-Sidon sets (p and q are conjugate exponents). The $S(q)$ property, however, seems considerably sharper than p-Sidonicity and we prefer the present terminology ($F \subset \Gamma$ is p-Sidon if

$$l^q(F) \subseteq M(G)^\wedge/\{\hat{\mu}: \hat{\mu} = 0 \text{ on } F\}).$$

The proof of the following lemma is based on a standard use of Riesz products.

LEMMA 1.2. *Let $1 < p \le 2 < q \le \infty$ (not necessarily conjugate exponents). Then $E = \{r_n\}_{n=1}^\infty \subset \hat{\Omega}$ is a uniformizable $L(p)$ and uniformizable $S(q)$ set.*

Proof. (a) Let $1 < p \le 2$ and $\phi \in l^p$ be real valued. Let $0 < \delta < 1$ be arbitrary and define the Riesz product (see p. 211 of [7])

$$h \sim (\|\phi\|_p/i\delta) \prod_{n=1}^{\infty} (1 + (i\delta\phi(n)/\|\phi\|_p)r_n) - \|\phi\|_p/i\delta.$$

A routine estimate yields $\|h\|_\infty \le (\|\phi\|_p/\delta)(\exp(\delta^2/2) + 1)$. Next, the spectral analysis of h yields

$$h \sim \sum_{k=1}^{\infty} (i\delta/\|\phi\|_p)^{k-1} \sum_{j_k > \cdots > j_1 = 1}^{\infty} \phi(j_1) \cdots \phi(j_k) r_{j_1} \cdots r_{j_k},$$

and, therefore, for all $n \ge 1$, $\hat{h}(r_n) = \phi(n)$. We estimate

$$\|\hat{h}|_{\sim E}\|_p^p = \sum_{k=2}^{\infty} \left(\frac{\delta}{\|\phi\|_p}\right)^{p(k-1)} \sum_{j_k > \cdots > j_1 = 1}^{\infty} |\phi(j_1)|^p \cdots |\phi(j_k)|^p$$

as follows: For each $k > 1$,

$$\begin{aligned}\sum_{j_k > \cdots > j_1 = 1}^{\infty} |\phi(j_1)|^p \cdots |\phi(j_k)|^p &\le \frac{1}{k!} \sum_{j_1, \cdots, j_k = 1}^{\infty} |\phi(j_1)|^p \cdots |\phi(j_k)|^p \\ &\le \|\phi\|_p^{kp}/k!.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\hat{h}|_{\sim E}\|_p^p &\le (\|\phi\|_p/\delta)^p \sum_{k=2}^{\infty} \delta^{kp}/k! \\ &= (\|\phi\|_p/\delta)^p[\exp(\delta^p) - \delta^p - 1] \\ &\le \|\phi\|_p^p \delta^p.\end{aligned}$$

This proves that E is a uniformizable $L(p)$ set. To prove that E is a uniformizable $S(q)$ set, we follow a similar route. Let $\phi \in l^q$ be real valued. For $0 < \delta < 1$, the $M_q(\Omega)$ uniformizing measure is given by the Riesz product

$$\mu \sim (\|\phi\|_q/\delta) \prod_{n=1}^{\infty} (1 + (\delta\phi(n)/\|\phi\|_q) r_n) - \|\phi\|_q/\delta. \quad \blacksquare$$

In what follows below $0 < \delta < 1$ is fixed. $\beta_p(E, \delta) = \beta_p$ and $\lambda_q(E, \delta) = \lambda_q$ are $L(p)$ and $S(q)$ uniformizing constants for E, respectively.

THEOREM 1.3. *For any $1 < p \le 2 < q < \infty$ there are functions*

$$\Phi_p\colon l^p \to A_p(\Omega) \quad \textit{and} \quad \Psi_q\colon l^q \to M_q(\Omega)$$

with the following properties:

(i) $\|\Phi_p(x)\|_{A_p} \le [\beta_p/(1-\delta)]\|x\|_p$ and $\|\Psi_q(y)\|_{M_q} \le [\lambda_q/(1-\delta)]\|y\|_q$ *for all* $x \in l^p$ *and* $y \in l^q$.

(ii) *Let $1 < p < 2 < q < \infty$ be conjugate exponents and $x \in l^p$, $y \in l^q$ be arbitrary. Then $(x, y) = (\Phi_p(x) * \Psi_q(\bar{y}))(0)$. Moreover, for any $x, y \in l^2$, $(x, y) = (\Phi_2(x) * \Phi_2(\bar{y}))(0)$ $((\cdot, \cdot)$ denotes the dual action between l^p and $(l^p)^*)$.*

Proof. Let $E_0, \ldots, E_n, \ldots$ be an infinite collection of disjoint infinite subsets of $E = \{r_n\} \subset \hat{\Omega}$ so that $\bigcup_{j=0}^{\infty} E_j = E$. Let $\hat{\Omega}_j$ be the group generated by $E_j \subset \hat{\Omega}$.

From the "independence" of E in $\hat{\Omega}$, it is clear that $\hat{\Omega}_j \cap \hat{\Omega}_{j'} = \{0\}$ for $j \neq j'$. Routine arguments (see Chapter 3 in [5], for example) yield that for each $j = 0, \dots$ there is $\sigma_j \in M(\Omega)$ with the property

$$\hat{\sigma}_j = \begin{cases} 1 & \text{on } \hat{\Omega}_j \\ 0 & \text{on } \hat{\Omega}\backslash\hat{\Omega}_j. \end{cases}$$

Next, we enumerate each $E_j = \{r_n^{(j)}\}_{n=1}^{\infty}$ and agree on a one-one correspondence between E_{j+1} and $\hat{\Omega}_j \backslash E_j$:

$$E_{j+1} \ni r_n^{(j+1)} \leftrightarrow \chi_n^{(j)} \in \hat{\Omega}_j \backslash E_j.$$

Fix $1 < p \leq 2 < q < \infty$ and let $x \in l^p$ and $y \in l^q$ be arbitrary. We proceed to define $\Phi_p(x) \in A_p(\Omega)$ and $\Psi_q(y) \in M_q(\Omega)$. By Lemma 1.2, select $g_0 \in A_p(\Omega)$ and $v_0 \in M_q(\Omega)$ so that:

(i)$_0$ $\hat{g}_0(r_n^{(0)}) = x(n)$, $\hat{v}_0(r_n^{(0)}) = y(n)$ for all n;
(ii)$_0$ $\|g_0\|_{A_p} \leq \beta_p \|x\|_p$, $\|v_0\|_{M_q} \leq \lambda_q \|y\|_q$;
(iii)$_0$ $\|\hat{g}_0|_{\sim E_0}\|_p \leq \delta\|x\|_p$, $\|\hat{v}_0|_{\sim E_0}\|_q \leq \lambda\|y\|_q$.

Set $f_0 = \sigma_0 * g_0$, and $\mu_0 = \sigma_0 * v_0$. We proceed by induction. Let $j > 0$. Select $g_j \in A_p(\Omega)$ and $v_j \in M_q(\Omega)$ so that:

(i)$_j$ $\hat{g}_j(r_n^{(j)}) = \hat{f}_{j-1}(\chi_n^{(j-1)})$, $\hat{v}_j(r_n^{(j)}) = \hat{\mu}_{j-1}(\chi_n^{(j-1)})$ for all n;
(ii)$_j$ $\|g_j\|_{A_p} \leq \beta_p \delta^j \|x\|_p$, $\|v_j\|_{M_q} \leq \lambda_q \delta^j \|y\|_q$;
(iii)$_j$ $\|\hat{g}_j|_{\sim E_j}\|_p \leq \delta^{j+1}\|x\|_p$, $\|\hat{v}_j|_{\sim E_j}\|_q \leq \delta^{j+1}\|y\|_q$.

Set $f_j = \sigma_j * g_j$ and $\mu_j = \sigma_j * v_j$. Finally, define

$$\Phi_p(x) = \sum_{j=0}^{\infty} (i)^j f_j \quad \text{and} \quad \Psi_q(y) = \sum_{j=0}^{\infty} (i)^j \mu_j \qquad (\text{where } i = \sqrt{-1}).$$

Observe that $\Phi_p(x) * \Psi_q(y)(0) = \sum_{j=0}^{\infty} (-1)^j f_j * \mu_j(0)$. We leave to the reader the remaining details of the verification that Φ_p and Ψ_q so defined satisfy the requirements of the theorem. ∎

We are now ready to state the extension of the Grothendieck inequality. In order to recognize the classical inequality as an instance of this extension, we formalize the following simple fact.

LEMMA 1.4. *Let $(a_{mn}) \subset \mathbf{C}$. The following are equivalent:*

(a) $|\sum_{m,n} a_{mn} f_m * \bar{g}_n(0)| \leq \sup_{m,n} (\|f_m\|_\infty \|g_n\|_\infty)$ *for all* $(f_m)_{m=1}^{\infty}$ *and* $(g_n)_{n=1}^{\infty} \subset L^\infty(\Omega)$.

(b) $|\sum_{m,n} a_{mn} s_m \bar{t}_n| \leq \|t\|_\infty \|s\|_\infty$ *for all* $(t_m)_{m=1}^{\infty} = t$ *and* $(s_n)_{n=1}^{\infty} = s$ *in* l^∞.

Proof. (a) $\Rightarrow$ (b). Let $f_m(\omega) = s_m$ and $g_n(\omega) = t_n$ for all $\omega \in \Omega$.
(b) $\Rightarrow$ (a). Integrate the inequality. ∎

COROLLARY 1.5. *Let* $1 \le p \le \infty$ *and* q *be its conjugate exponent. Let* $(a_{mn})_{m,n} \subset \mathbf{C}$ *satisfy*

$$\left|\sum_{m,n} a_{mn} f_m * \bar{\mu}_n(0)\right| \le \sup_{m,n} (\|f_m\|_{A_p} \|\mu_n\|_{M_q}) \tag{1.1}$$

for all $(f_m)_{m=1}^{\infty} \subset A_p$ *and* $(\mu_n)_{n=1}^{\infty} \subset M_q$.

(a) *If* $1 \le p < 2 < q \le \infty$ *then there is* $K_p > 0$ *so that*

$$\left|\sum_{m,n} a_{mn}(x_m, y_n)\right| \le K_p \sup_{m,n} (\|x_m\|_p \|y_n\|_q)$$

for all $(x_m)_{m=1}^{\infty} \subset l^p$ *and* $(y_n)_{n=1}^{\infty} \subset l^q$.

(b) *If* $1 \le q \le 2 \le p \le \infty$, *then* (1.1) *implies that*

$$\left|\sum_{m,n} a_{mn} f_m * \bar{g}_n(0)\right| \le \sup_{m,n} (\|f_m\|_{\infty} \|g_n\|_{\infty}) \tag{1.2}$$

for all $(f_m)_{m=1}^{\infty}$ *and* $(g_n)_{n=1}^{\infty} \subset L^{\infty}(\Omega)$.

Moreover, it follows that there is $K > 0$ *so that*

$$\left|\sum_{m,n} a_{mn}(x_m, y_n)\right| \le K \sup_{m,n} (\|x_m\|_2 \|y_n\|_2) \tag{1.3}$$

for all $(x_m)_{m=1}^{\infty}$ *and* $(y_n)_{n=1}^{\infty} \subset l^2$.

Proof. (a) If $1 < p < 2$ the assertion is an immediate consequence of Theorem 1.3. When $p = 1$, (1.1) is equivalent to

$$\sum_n \left\|\sum_m a_{mn} f_m\right\|_{\infty} \le \sup_m (\|f_m\|_{A(\Omega)}) \quad \text{for all } (f_m)_{m=1}^{\infty} \subset A(\Omega).$$

(Recall that

$$A_1(\Omega) = A(\Omega) = \{f \in L^{\infty}(\Omega) : \hat{f} \in l^1(\hat{\Omega})\}.).$$

Therefore, $\sum_{m,n} |a_{mn}| \le 1$, and the claim follows trivially.

(b) Observe that for all $2 \le p \le \infty$ and $1 \le q \le 2$, $A_p(\Omega) = L^{\infty}(\Omega)$ and $M_q(\Omega) \subseteq L^2(\Omega)$. Therefore, (1.1) $\Rightarrow$ (1.2) is trivial. That (1.2) implies (1.3) is again an immediate consequence of Theorem 1.3. ∎

Remarks. (1) In part (b) of 1.5, when $p = \infty$ and $q = 1$, the implication (1.1) $\Rightarrow$ (1.3) is the classical Grothendieck inequality. For, in this case (1.1) is easily seen to be equivalent to (1.2). On the other end, when $p, q = 2$, the implication (1.1) $\Rightarrow$ (1.3) can be established with the aid of only the classical Khintchin inequality and without the $\Lambda(2)$ uniformizability property of E (exercise).

(2) A consequence of Corollary 1 in Section 4 of [4] is that the inner product of l^2 cannot be replaced in Grothendieck's inequality by the dual action between l^p and l^q, $p \neq 2$: If every array of scalars $(a_{mn})_{m,n}$ that satisfies (a) (or (b)) of Lemma 1.4 also satisfies

$$\left|\sum_{m,n} a_{mn}(x_m, y_n)\right| \leq C \sup_{m,n} (\|x_m\|_p \|y_n\|_q) \tag{1.4}$$
$$\text{for all } (x_m)_{m=1}^{\infty} \subset l^p,\ (y_n)_{n=1}^{\infty} \subset l^q,$$

then $p = 2$. Part (a) of Corollary 1.5 gives the (stronger) condition that should be imposed on $(a_{mn})_{m,n}$ in order that (1.4) hold where $p \neq 2$.

2. $L(p)$ sets

We recall the following.

DEFINITION 2.1. Let $2 < q < \infty$. $F \subset \Gamma$ is a $\Lambda(q)$ set if there is $\alpha > 0$ so that for all $f \in L_F^2(G)$

$$\alpha \|f\|_2 \geq \|f\|_q. \tag{2.1}$$

The "smallest" constant in (2.1) is the $\Lambda(q)$ constant of F and is denoted by $A(q, F)$.

THEOREM 2.2. *Let $1 \leq r \leq 2$ and $F = \{\gamma_j\}_{j=1}^{\infty} \subset \Gamma$ be (a $\Lambda(q)$ set for all q) so that $A(q, F) \leq \alpha_0 q^{1/r}$ for some fixed $\alpha_0 > 0$ and all $2 < q < \infty$. Then, F is a uniformizable $L(p)$ set for $2/r \leq p \leq 2$.*

We require two lemmas.

LEMMA 2.3. *Let $K \geq 1$ be a fixed integer and $F = \{\gamma_j\}_{j=1}^{\infty} \subset \Gamma$ be a $\Lambda(2K)$ set. Let $1 < p < 2$, $\phi \in l^p$, $\|\phi\|_p = 1$, and $h = \sum_n \phi(n)\gamma_n$. Then,*

$$\|(h^K)^\wedge\|_p \leq [A(2K, F)]^{2K/p}.$$

Proof. Define $\psi \in l^2$ by $\psi(n) = |\phi(n)|^{p/2}$ and let $f = (\sum_n \psi(n)\gamma_n)^K$. By the hypothesis,

$$\|f\|_2^2 \leq [A(2K, F)]^{2K}. \tag{2.3.1}$$

Clearly, for any $\gamma \in \Gamma$,

$$\hat{f}(\gamma) = \sum_{\gamma_{n_1} + \cdots + \gamma_{n_K} = \gamma} \psi(n_1) \cdots \psi(n_K)$$

and

$$(h^K)^\wedge(\gamma) = \sum_{\gamma_{n_1} + \cdots + \gamma_{n_K} = \gamma} \phi(n_1) \cdots \phi(n_K).$$

Therefore, we obtain

$$\begin{aligned}|\hat{f}(\gamma)|^2 &= \Bigg|\sum_{\gamma_{n_1}+\cdots+\gamma_{n_K}=\gamma} \psi(n_1)\cdots\psi(n_K)\Bigg|^2\\ &\geq \Bigg(\sum_{\gamma_{n_1}+\cdots+\gamma_{n_K}=\gamma} |\psi(n_1)|^{2/p}\cdots|\psi(n_K)|^{2/p}\Bigg)^p\\ &\geq |(h^K)^\wedge(\gamma)|^p.\end{aligned}$$

Finally, from (2.3.1) we obtain $\|(h^K)^\wedge\|_p \leq \|f\|_2^{2/p} \leq [A(2K, F)]^{2K/p}$. ∎

LEMMA 2.4. *Suppose $F \subset \Gamma$ has the property that for every $0 < \delta < 1$ and $\phi \in l^p(F)$ there is $f \in A_p(G)$ so that*

$$\|\hat{f}|_{\sim F}\|_p \leq \delta\|\phi\|_p, \quad \|\hat{f}|_F - \phi\|_p \leq \delta\|\phi\|_p \quad \text{and} \quad \|f\|_{A_p} \leq \beta\|\phi\|_p$$

where β depends only on δ and F. Then, F is a uniformizable $L(p)$ set.

Proof. Let $\phi_0 = \phi \in l^p$ and $f_0 \in A_p(G)$ be as in the hypothesis of the lemma. Let $\phi_1 = \hat{f}_0|_F - \phi$. We continue by induction. For $j > 0$, let $f_j \in A_p(G)$ be so that

$$\|\hat{f}_j|_F - \phi_j\|_p \leq \delta\|\phi_j\|_p, \ \|\hat{f}_j|_{\sim F}\|_p \leq \delta\|\phi_j\|_p \quad \text{and} \quad \|f_j\|_{A_p} \leq \beta\|\phi_j\|_p.$$

Let $\phi_{j+1} = \hat{f}_j|_F - \phi_j$. Finally, let $f = \sum_{j=0}^{\infty}(-1)^j f_j$. We conclude that

$$\hat{f}|_F = \phi, \ \|\hat{f}|_F\|_p \leq \frac{\delta}{(1-\delta)}\|\phi\|_p \quad \text{and} \quad \|f\|_{A_p} \leq \beta/(1-\delta). \ ∎$$

Proof of Theorem 2.2. First, assume that $F \cap (-F) = \emptyset$; for, otherwise identify $F \subset \Gamma$ with $F \times \{1\}$ in $\Gamma \times \mathbf{Z}$ where $(F \times \{1\}) \cap (-F \times \{-1\}) = \emptyset$ and proceed to work in the latter setting. Further, without loss of generality we assume that $A(q, F \cup (-F)) \leq q^{1/r}$. Let $0 < \delta < 1$ be arbitrary and ϕ be an arbitrary real valued finitely supported element of l^p, $\|\phi\|_p = 1$. Let

$$f_1 \sim \frac{1}{\delta}\sin\left(\delta\sum_n \phi(n)(\gamma_n + \bar{\gamma}_n)\right) \in L^\infty(G) = \sum_n \phi(n)(\gamma_n + \bar{\gamma}_n) + P$$

where $P \sim \sum_{j=1}^{\infty}\delta^{2j}[\sum_n \phi(n)(\gamma_n + \bar{\gamma}_n)]^{2j+1}/(2j+1)!$ By Lemma 2.3 and the hypothesis, we obtain

$$\begin{aligned}(2.2.1)\qquad \|\hat{P}\|_p &\leq \sum_{j=1}^{\infty}\left\|\left(\left(\sum_n \phi(n)(\gamma_n + \bar{\gamma}_n)\right)^{2j+1}\right)^\wedge\right\|_p \delta^{2j}/(2j+1)!\\ &\leq \sum_{j=1}^{\infty}\delta^{2j}(4j+2)^{(4j+2)/pr}/(2j+1)!.\end{aligned}$$

Since $2/r \leq p$, we conclude that (2.2.1) is bounded by 50δ. Similarly, define

$$f_2 \sim \frac{1}{\delta}\sin\left(\delta\sum_n \phi(n)(\gamma_n - \bar{\gamma}_n)/i\right).$$

Finally, let $f = (f_1 - if_2)/2$ and verify that $f \in A_p(G)$ satisfies $\|\hat{f}|_F - \phi\|_p < 50\delta$, and $\|\hat{f}|_{\sim F}\|_p < 50\delta$.

The proof of the theorem is completed by an application of Lemma 2.4. ∎

Questions (a) Theorem 2.2 is the present limit of our knowledge. Let F_1 and F_2 be infinite and mutually disjoint subsets of a lacunary set in $\mathbf{Z}$. Is $F_1 + F_2 \subset \mathbf{Z}$ an $L(p)$ set for some or all $1 < p < 2$ (Recall that $A(q, F_1 + F_2)$ is $O(q)$.)?

(b) By applying Drury's line (see [3], for example), one proves that every Sidon set is a uniformizable $S(q)$ set for every $2 < q \leq \infty$. This, too, is the limit to present knowledge. Are there non-Sidon sets which are $S(q)$ sets for some $2 < q < \infty$? In particular, let $F \subset \Gamma$ be so that $A(q, F)$ is $\mathcal{O}(q^s)$ for $s \geq 1/2$. Is F an $S(q)$ set, e.g., for $2 < q \leq 4s/(1 - 2s)$ (the latter guess is prompted by results in [2])?

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