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COMPONENT TYPE THEOREMS FOR FINITE GROUPS IN CHARACTERISTIC 2

BY

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I. Introduction

In recent years Aschbacher blocks or constrained components have entered the limelight in the theory of finite simple groups, not only in their connection with pushing up theorems but also as a possible direction for revising some of the classification program. In this paper the basic foundations are laid for a theory of blocks closely analogous to that for ordinary components in M. Aschbacher's fundamental work [3]. Since the development and present status of the theory of blocks is described in detail in the survey article [17] which serves as an introduction to this paper, only the technical essentials are repeated here together with some comments about the proofs.

DEFINITIONS. A subgroup J of a finite group H is called a block of H if and only if (i) $J \trianglelefteq \trianglelefteq H$, (ii) $J = J'$, (iii) $J/O_2(J)$ is quasisimple and (iv) J has a unique non-central 2-chief factor; if $H = J$, we simply say J is a block. For a block J let

$$U(J) = [O_2(J), J] \quad \text{and} \quad \tilde{U}(J) = U(J)/U(J) \cap Z(J).$$

For any finite group G let

$$\mathscr{B}(G) = \{J \mid J \text{ is a block of } N_G(S) \text{ where } S \in Syl_2(C_G(J/O_2(J)))\}.$$

If J_1, J_2 are blocks which are subgroups of a group G, write $J_1 \to J_2$ if and only if $J_1 \subseteq J_2$ with $U(J_1) = [O_2(J_2), J_1]$ and for some 2-subgroup T of $N_G(J_2)$, $\bar{J}_1$ is a component of $C_{\bar{J}_2}(T)$, where an overbar denotes the natural projection of J_2 onto $J_2/O_2(J_2)$. Extend $\to$ via chains to a partial order on $\mathscr{B}(G)$ and let $\mathscr{B}^*(G)$ be the maximal elements under this order.

Say a block J is of $L_2(2^m)$-type if and only if $J/O_2(J) \cong L_2(2^m)$ and $\tilde{U}(J)$ is the natural 2-dimensional $\mathbf{F}_{2^m}L_2(2^m)$-module for $J/O_2(J)$ viewed as a module over $\mathbf{F}_2$. Finally, say a subgroup J of a finite group H is an $\Omega_4^+(2^m)$-block if and only if (i) $J \trianglelefteq \trianglelefteq H$, (ii) $J = J'$, (iii) $J/O_2(J) \cong \Omega_4^+(2^m) = L_2(2^m) \times L_2(2^m)$, and (iv) J has a unique non-central 2-chief factor which is the natural 4-dimensional $\mathbf{F}_{2^m}\Omega_4^+(2^m)$-module for $J/O_2(J)$ viewed as a module over $\mathbf{F}_2$.

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The main theorems can now be stated:

THEOREM A. *Let J be a block and let x be an involution in* Aut (J); *then x centralizes a 2-element in $J - Z(J)$ and, moreover, if $U(J)$ is abelian, x centralizes an involution in $J - Z(J)$.*

THEOREM B. *Let x be an involution in the finite group G, J a block of $C_G(x)$, K a block of G and assume the outer automorphism group of $K/O_2(K)$ is solvable; then one of the following holds:*
(1) $J \subseteq K$ *with* $U(J) \subseteq U(K)$;
(2) $K \neq K^x$ *and* $J = C_{KK^x}(x)'$;
(3) $[J, K] = 1$.

THEOREM C. *Let G be a finite group with a maximal 2-local subgroup M and block J of M such that M is the unique maximal 2-local subgroup of G containing J; then either $M = G$ or $J \trianglelefteq M$.*

THEOREM D. *If G is a finite group of characteristic* 2 *type and J is a block in some maximal 2-local subgroup M of G, then $J \in \mathscr{B}(G)$.*

THEOREM E. *If G is a finite group of characteristic* 2 *type and $J \in \mathscr{B}^*(G)$ with J not of $L_2(2^m)$-type for any m, then J is a block of some maximal 2-local subgroup M of G and M is the unique maximal 2-local subgroup of G containing J.*

THEOREM F. *If G is a finite group of characteristic* 2 *type and $J \in \mathscr{B}^*(G)$ with J* of *$L_2(2^m)$-type, then either*
(1) *J is a block of a maximal 2-local subgroup M of G and M is the unique maximal 2-local subgroup of G containing J,*
or
(2) *$J \subseteq K$ where K is an $\Omega_4^+(2^m)$-block of some maximal 2-local subgroup M of G and M is the unique maximal 2-local subgroup of G containing K.*

THEOREM G. *If J_1, J_2 are distinct blocks with $J_1 \to J_2$, then one of the following holds:*
(1) $\bar{J}_1 \cong A_n$, $\bar{J}_2 \cong A_{n+2k}$, *and $\tilde{U}(J_i)$ is the irreducible constituent of the natural permutation module for J_i over $\mathbf{F}_2$, $i = 1, 2$;*
(2) $\bar{J}_1 \cong Sp_{2n}(q)'$, $\bar{J}_2 \cong \Omega_{2n+2}^{\pm}(q)$, *$q$ a power of 2, $n \geq 1$, and $\tilde{U}(J_i)$ is the natural $\mathbf{F}_q\bar{J}_i$-module viewed as a module over $\mathbf{F}_2$, $i = 1, 2$;*
(3) $\bar{J}_1 \cong U_4(2)$, $\bar{J}_2 \cong Z_3 \cdot U_4(3)$, *and* $\dim_{\mathbf{F}_2} \tilde{U}(J_2) = 12$, $\dim_{\mathbf{F}_2} \tilde{U}(J_1) = 8$.

In the literature the blocks J with $U(J)$ abelian seem to be of primary interest so in Theorems A and C where the arguments handle the cases $U(J)$ abelian, $U(J)$ non-abelian, the former is treated first for those who wish to skip the latter case; indeed, Theorem A is trivial when $U(J)' = 1$ but since it tidies up the proofs of Theorems B and C, it may be worth the inordinately large effort required to complete the non-abelian case.

Theorem B was proven by M. Aschbacher, K. Harada and the author in 1977 at the ongoing conference at Caltech that spring. Using an approach of R. Gilman, Harada has proved this theorem without recourse to Theorem A.

The proof of Theorem C follows the argument of Aschbacher's Standard Form Theorem (specifically, Theorem 5 of [3]) although the endgame is different. The presence of Theorem A makes matters smoother than the original, especially when $U(J)$ is abelian.

The remaining arguments generalize results of Aschbacher in [7] and [9] and in some cases in our more general setting the arguments are easier. The main technical difficulty is in the discussions related to the proof of Theorem G where cores and standard form problems cause the grief. This could be swept under the carpet by invoking the Unbalanced Theorem and complete solutions to certain standard form problems but it seems clearer to maintain independence from these Gargantuan tools.

I am especially indebted to Michael Aschbacher for some helpful conversations and correspondence and, in particular, for the crucial observation, Lemma 6.1.

II. Preliminary lemmas

Throughout the paper we make constant use of the immediate consequence of the 3-subgroups lemma: if X is perfect and $[X, A, A] = 1$, then $[X, A] = 1$. Using this one verifies that for a block or $\Omega_4^+(2^m)$-block J and normal subgroup A of J either $A \subseteq Z(J)$ or $U(J) \subseteq A$.

LEMMA 2.1. *If J, K are distinct blocks or $\Omega_4^+(2^m)$-blocks of G then* $[J, K] = 1$.

Proof. Let $H = O_2(G)$, $\bar{G} = G/H$. By subnormality of blocks, $\bar{J}$, $\bar{K}$ are semisimple subnormal subgroups of G and $J = (JH)^{(\infty)}$, $K = (KH)^{(\infty)}$, so $\bar{J} \neq \bar{K}$. Since J normalizes KH, J normalizes K and so acts on $\tilde{U}(K)$. Since K acts irreducibly on $\tilde{U}(K)$, H centralizes $\tilde{U}(K)$.

Suppose $[\bar{J}, \bar{K}] = 1$. Then $\bar{J}$ commutes with the irreducible action of $\bar{K}$ on $\tilde{U}(K)$ so $\bar{J}$ centralizes $\tilde{U}(K)$. Since $[J, K] \subseteq O_2(K)$ and $K/U(K)$ is semisimple, $[K, J] \subseteq U(K)$. Thus $[K, J, J] \subseteq Z(K)$ so by the 3-subgroups lemma applied to $K/Z(K)$, $[K, J] \subseteq Z(K)$. Thus $[J, K, K] = 1$, so $[J, K] = 1$ as claimed.

If $[\bar{J}, \bar{K}] \neq \bar{1}$, then at least one of $\bar{J}$, $\bar{K}$ is isomorphic to

$$\Omega_4^+(2^m) \cong L_2(2^m) \times L_2(2^m)$$

and, interchanging J, K if necessary, we may assume there exists $\bar{J}_1$, a component of $\bar{J}$ of type $L_2(2^m)$, with $[\bar{J}_1, \bar{K}] = \bar{1}$ and $\bar{J} = \bar{J}_1 \times \bar{J}_2$ with $\bar{J}_2 \subseteq \bar{K}$. Pick the preimage J_1 with $J_1 = (J_1 H)^{(\infty)}$. By the argument of the preceding paragraph applied to J_1 in place of J, $[J_1, K] = 1$. Because $\tilde{U}(J)$ is the direct sum of 2 natural $F_{2^m} L_2(2^m)$-modules for $\bar{J}_1$, $U(J) = [O_2(J), J_1] \subseteq J_1$. But then $\bar{J}_2$ centralizes $\tilde{U}(J)$, a contradiction.

LEMMA 2.2. *Let J be a block, $V = [O_2(J), J]$.*

(a) *If V is abelian, $V \subseteq \Omega_1(Z(O_2(J)))$,*

(b) *If V is non-abelian, $V' = \phi(V)$ is elementary abelian, $C_J(V) = Z(J)$.*

Proof. (a) Note that because J acts irreducibly on $V/V \cap Z(J)$, $[O_2(J), V] \subseteq Z(J)$. Let $\bar{J} = J/V$ so $\bar{J}$ is quasisimple and acts on V. For $\bar{j} \in O_2(\bar{J})$, $v \in V$, $[v, \bar{j}] = z \in Z(J)$; thus for all $x \in J$, $[v^x, \bar{j}] = z$ so $[vv^x, \bar{j}] = 1$. Since $V = \langle vv^x | v \in V, x \in J \rangle$, $V \subseteq Z(O_2(J))$, as desired.

(b) If $V' \neq 1$, since $J = O^2(J)$ acts non-trivially on V/V', J acts non-trivially on $\Omega_1(V/V')$ so the non-central 2-chief factor of J lies in $\Omega_1(V/V')$, whence $V/V' = \Omega_1(V/V')$. Since $V' \subseteq Z(J)$, V' is elementary abelian.

Finally, $[J, C_J(V)] \subseteq C_J(V)$ and as $V' \neq 1$, $V \nsubseteq C_J(V)$. Since $O(J) \subseteq Z(J)$ it follows that J centralizes $C_J(V)$ as claimed.

LEMMA 2.3. *Let K be a block of G, x an involution in G, J a block of $C_G(x)$, W a subgroup of $N_G(K)$ of order 4.*

(a) *If $K \neq K^x$, then $K_0 = C_{KK^x}(x)'$ is a block of $C_G(x)$ isomorphic to a central quotient of K and the map $k \to kk^x$, for all $k \in K$, is a homomorphism of K onto K_0; either $J = K_0$ or $[J, K] = 1$.*

(b) *$\Gamma_{1,W}(K)$ contains a fourgroup and if w is an involution in $N_G(K)$, $|C_K(w)|_2 \geq 8$ or $m(C_K(w)) \geq 2$.*

Proof. (a) Suppose $K \neq K^x$ so, by Lemma 2.1 $[K, K^x] = 1$. Let

$$\overline{KK^x} = KK^x/K \cap K^x,$$

so $\overline{KK^x} = \bar{K} \times \bar{K}^x$ and $\bar{L} = C_{\overline{KK^x}}(x) \cong \bar{K}$. Let L be the complete preimage of $\bar{L}$ in KK^x so, because $\bar{L}$ is perfect and $K \cap K^x \subseteq Z(KK^x)$, $L' = L_0$ is also perfect; moreover, clearly $K_0 \subseteq L_0$. However, $[x, L_0] \subseteq Z(KK^x)$ so $[x, L_0, L_0] = 1$, whence $L_0 \subseteq C_{KK^x}(x)' = K_0$. It is also clear that $k \to kk^x$ is a homomorphism of K into K_0 whose image covers $\bar{L}$. Since this image and K_0 are both perfect and agree modulo a central subgroup, equality holds as claimed. Finally, suppose $J \neq K_0$ so, by Lemma 2.1, $[J, K_0] = 1$. Let y be an odd order element of J. As y permutes the blocks of G but centralizes K_0, y normalizes KK^x, and, since $|y|$ is odd, y normalizes both K and K^x. For $k \in K$,

$$1 = [kk^x, y] = [k, y][k^x, y]$$

so $[k, y] \in K \cap K^x \subseteq Z(K)$. Thus $[y, K, K] = 1$ so $[y, K] = 1$ which proves $J = O^2(J)$ centralizes K as claimed.

(b) Let $V = U(K)$ so $N_G(K)$ acts on V and $m(V/V') \geq 3$. If V is abelian, for every involution $w \in N_G(K)$, $m(C_V(w)) \geq 2$ so all parts of (b) follow in this case. Thus we may assume $V' \neq 1$ and since V' is elementary abelian, by similar reasoning $m(V') \leq 2$.

Let $T \in Syl_2(K)$ with T normalized by W. If $\Gamma_{1,W}(T)$ has 2-rank 1 it follows that every characteristic abelian subgroup of T is cyclic. Since T' is cyclic by Theorem 5.4.9 of [20], $[\tilde{U}(J), T]$ has order 2, so some subgroup T_0 of T of index

≤ 2 in T centralizes $\tilde{U}(J)$, contrary to $T/O_2(J)$ acting faithfully on $\tilde{U}(J)$. This proves the first assertion of (b).

Now suppose w is an involution in $N_G(K)$ with $m(C_K(w)) = 1$. If

$$V' = \langle v_1, v_2 \rangle \cong Z_2 \times Z_2,$$

then we may assume $v_1^w = v_1 v_2$, $v_2^w = v_2$, whence $|C_{K/\langle v_2 \rangle}(w)| = |C_K(w)|$. It therefore suffices to assume $V' = \langle v \rangle \cong Z_2$ and prove for any w, $|C_K(w)|_2 \geq 8$. Let

$$C = \{a \in V \mid [a, w] \in \langle v \rangle\}$$

so $C/\langle v \rangle = C_{V/V'}(w)$. If $|C/\langle v \rangle| \geq 8$, it follows that $|C_V(w)| \geq 8$, as desired. Assume $|C/\langle v \rangle| < 8$, so $|V/V'| \leq 16$ whence $V/V' \cong E_{16}$, $K/O_2(K) \cong A_5$, $V \cong Q_8 Y D_8$ and $|C/\langle v \rangle| = 4$. Since $\mathrm{Aut}\,(V) \cong E_{16} \cdot O_4^-(2)$ and w is not a transvection on V, there exists $k \in K\text{-}O_2(K)$ with $k^{-1}w$ centralizing K. Thus $|C_V(w)| = |C_V(k)| = 4$ and $k \in C_K(w) - V$, so $|C_V(w)|_2 \geq 8$, as needed.

LEMMA 2.4. *If $L = A_n$ is a standard component in $G = A_{n+4}$ and V is an irreducible $\mathbf{F}_2 G$-module in which $[V, L]/C_{[V,L]}(L)$ is the natural module for L (i.e. the non-trivial irreducible constituent of the n-dimensional permutation module over $\mathbf{F}_2$), then V is the natural module for G.*

Proof. See [13].

LEMMA 2.5. *Suppose $H = \Sigma_n$, $n \geq 7$, V is a faithful $\mathbf{F}_2 H$-module such that*

$$[V, H']/C_{[V,H']}(H')$$

is the natural module for H' (as in Lemma 2.4) and suppose t_1, t_2 are involutions in $H\text{-}H'$ with t_1 a transposition; then either t_2 is a transposition or

$$\dim_{\mathbf{F}_2} [V, t_1] < \dim_{\mathbf{F}_2} [V, t_2].$$

Proof. Let $V_0 = [V, H']$, $V_1 = C_V(H')$, $\tilde{V} = V_0/V_0 \cap V_1$. By 11.3 of [5], $|H^1(\tilde{V}, H')| = 1$ if n is odd, 2 if n is even. Note that $t_1 \equiv t_2 (\mathrm{mod}\ H')$ implies $[V_1, t_1] = [V_1, t_2]$. If n is odd, since $\tilde{V}$ is self-dual, $V = V_0 \oplus V_1$, and in this case if t_2 is not a transposition, $\dim_{\mathbf{F}_2} [V_0, t_2] > 1 = \dim_{\mathbf{F}_2} [V_0, t_1]$ as desired. If n is even, $|V : V_0 + V_1| \leq 2$ and $|V_0 \cap V_1| \leq 2$. In this case if t_2 is not a transposition, since $n > 7$ $\dim_{\mathbf{F}_2} [\tilde{V}, t_2] \geq 3$, whence as $\dim_{\mathbf{F}_2} [V_0, t_1] = 1$,

$$\dim_{\mathbf{F}_2} [V_0 + V_1, t_2] > \dim_{\mathbf{F}_2} [V_0 + V_1, t_1] + 1,$$

which suffices to establish the lemma.

LEMMA 2.6. *Let G be a group generated by a conjugacy class D of odd transpositions with $O(G) = 1$ and $G/S(G) \cong L_2(q)$ or $Sz(q)$, $q = 2^m > 2$; let $E \subseteq G$ with $E \cong E_q$. Assume $E^\# \subseteq D$, E is tightly embedded in G and if $E_1, \ldots, E_n$ are distinct commuting conjugates of E, $\langle E_1, \ldots, E_n \rangle = E_1 \times \cdots \times E_n$. Then $S(G) = 1$.*

Proof. Note that since D is a class and $E^{\#} \subseteq D$ with $m(E) \geq 2$, $G = G'$. Proceed by induction on $|G|$ and let M be a minimal normal subgroup of G.

First consider $M = \langle z \rangle \cong Z_2$. Let $E \subseteq T \in Syl_2(G)$, $E_1, \ldots, E_n$ the G-conjugates of E in T, so, by the odd transposition property and our assumptions,

$$\langle E_1, \ldots, E_n \rangle = E_1 \times \cdots \times E_n.$$

Note that E covers a Sylow 2-subgroup of $G/S(G)$ and Sylow 2-subgroups of $G/S(G)$ are T.I.-sets, so

$$N_G(T) = N_G(E_1 \times \cdots \times E_n).$$

Since $E^{\#} \subseteq D$ and the fusion of elements of $E^{\#}$ takes place in $N_G(E_1 \times \cdots \times E_n)$, by the T.I. property of E we may pick $h \in N_G(T) \cap N_G(E)$ with $\langle h \rangle S(G)/S(G)$ a Cartan subgroup of $G/S(G)$. Since D is a class of odd transpotions it follows that for all $e \in E$, $e \not\sim_G ez$. Thus for $\overline{G} = G/\langle z \rangle$, $\overline{E}$ is tightly embedded in $\overline{G}$. Suppose $z \in \langle E_1, \ldots, E_n \rangle$: write $z = e_1 \ldots e_n$, $e_i \in E_i$ and without loss of generality $e_1 \in E_1 = E$, $e_1 \neq 1$; h normalizes E and so normalizes $E_2 \times \cdots \times E_n$, whence

$$z = z^h = e_1^h e_2^h \ldots e_n^h \quad \text{where } e_1^h \neq e_1,$$

contrary to z having a unique expression in this direct product. Thus

$$\langle \overline{E}_1, \ldots, \overline{E}_n \rangle = \overline{E}_1 \times \cdots \times \overline{E}_n,$$

so by induction, $S(\overline{G}) = \overline{1}$. Clearly $G \not\cong SL_2(5)$ and since in $\widehat{Sz(8)}$, $e \sim ez$, $e \in E^{\#}$, $\widehat{Sz(8)}$ is not generated by odd transpositions. These are the only possible perfect extensions of $\overline{G}$ by Z_2, so $|M| \neq 2$.

Now M is an irreducible $\mathbf{F}_2 G/O_2(G)$ module and by the proof of 4.1.8 of [28], E acts quadratically on M. By Lemmas 2.1 and 2.5 of [30], $k = \dim_{\mathbf{F}_2} C_M(E) = \frac{1}{2}\dim_{\mathbf{F}_2} M$. By hypothesis therefore $\langle E^M \rangle = E_1 \times \cdots \times E_{2^k}$ which is absurd in view of $\langle E^M \rangle \subseteq EM$, $k \geq 2$. This contradiction completes the proof of the lemma.

LEMMA 2.7. *Let $H = O_{2n}^{\pm}(2^m)$, $n \geq 2$, $m \geq 1$, V the natural $2n$-dimensional $\mathbf{F}_{2^m}$-module for H viewed as a module over $\mathbf{F}_2$, and let $G = \mathrm{Aut}(V) \cong GL_{2nm}(\mathbf{F}_2)$.*

(a) *If $H \not\cong O_6^+(2)$, $H^1(H', V) = 0 = H^1(H', V^*)$, V^* the dual module to V.*

(b) *Let $H \cong O_4^+(2^m)$, $T \in Syl_2(N_G(H))$, $T_0 = T \cap H'$; then*

$$T/T_0 \cong Z_2 \times Z_{2^k}$$

where $2^k \| m$ and there is an element f of T of order 2^k which induces a field automorphism on H.

Proof. (a) The case $n = 2$ is 4.27 of [6] and 2.7 of [7]. Now the same argument as Lemma 2.2 of [29] yields the general result.

(b) By the irreducible action of H' on V, $|C_G(H')|$ is odd. It is clear that since we are considering $\mathbf{F}_2$-automorphisms such an element f exists, and since f acts on H and $|H:H'| = 2$, T/T_0 is at least as big as claimed. Since $H' = L_1 \times L_2$, $L_i \cong L_2(2^m)$, a Sylow 2-subgroup of Out (H') is of type $Z_{2^k} \wr Z_2$. If T/T_0 is not as described it follows that there exists $f_1 \in T$ with f_1 inducing an outer automorphism on L_1, an inner automorphism on L_2 and with $f_1^2 \in T_0$. Replacing f_1 by $f_1 a$, for suitable $a \in L_2$ we may assume f_1 centralizes L_2. Since the coset $f_1 L_1$ contains an involution which is a field automorphism of order 2 on L_1, we may assume f_1 is such an involution. Now $[V, f_1]$ admits L_2 so since V is the direct sum of two natural irreducible modules for L_2, $[V, f_1]$ is an irreducible $\mathbf{F}_2 L_2$-module. Since $C_{L_1}(f_1) \cong L_2(2^{m/2})$ commutes with the action of L_2 on $[V, f_1]$, $C_L(f_1)'$ centralizes $[V, f_1]$ contrary to all odd order elements of $L_1^{\#}$ acting Frobeniusly on V. This contradiction completes the proof of (b).

LEMMA 2.8. *Let $H = Z_3 \cdot U_4(3) \cdot Z_2$ where H has a faithful irreducible 12-dimensional module V over $\mathbf{F}_2$ such that for some involution $t \in H - H'$, $C_H(t)$ has a component $L \cong U_4(2)$ and V has a unique non-trivial irreducible $\mathbf{F}_2 L$-constituent. Let*

$$G = \mathrm{Aut}\ (V) \cong GL_{12}(\mathbf{F}_2), \quad T \in Syl_2(N_G(H')), \quad T_0 = T \cap H', \quad Z = Z(H').$$

(a) *$T/T_0 \cong Z_2$ or $Z_2 \times Z_2$, tT_0 contains exactly two H-classes of involutions and if a, b are representatives of these,*

$$C_{H'}(a) \cong Z_3 \times U_4(2), \quad C_{H'}(b) \cong Z_3 \times (SL_2(3) \wr Z_2/Z(SL_2(3) \wr Z_2));$$

if $T/T_0 \cong Z_2 \times Z_2$, there is a coset uT_0 of order 2 in T/T_0 with

$$uT_0 \neq tT_0 \quad \text{but} \quad \langle u, H' \rangle/Z \cong \langle t, H' \rangle/Z,$$

tuT_0 contains exactly two H-classes of involutions and if c, d are representatives of these, $C_{H'}(c) \cong \Sigma_6$, $C_{H'}(d) \cong U_3(3)$.

(b) *If e is an involution in tH', $\dim_{\mathbf{F}_2} [V, e] = 2$ or 6.*

(c) $\dim_{\mathbf{F}_2} [V, L]/C_{[V,L]}(L) = 8$.

Proof. Note that $SU_6(2)$ contains a subgroup H with the requisite properties so the situation is not vacuous, by Theorem 16.1.12 of [14]. Since Z acts Frobeniusly on V, $C = C_G(Z) \cong GL_6(4)$ and $N_G(Z) = C\langle f \rangle$ where f induces an involutory field automorphism on C.

The claims in part (*a*) are simply assertions about Aut $(U_4(3))$. Since Out $(U_4(3)) \cong D_8$ and only a fourgroup in Out $(U_4(3))$ normalizes a 3-fold cover (and since $|C_G(H')|$ is odd), $T/T_0 \cong Z_2$ or $Z_2 \times Z_2$. Note by the structure of $C_H(t) \trianglerighteq\trianglerighteq L$, t is a reflection in $H/\langle x \rangle \subseteq O_6^-(3)$; moreover, by 15.1 of [14], if $T/T_0 \cong Z_2 \times Z_2$ there is a coset $uT_0 \neq tT_0$ with $\langle u, H' \rangle/Z \cong H/Z$. The classes of involutions in the coset $t(H'/Z)$ are represented by a reflection and a product $a_1 a_2 a_3$ of three distinct commuting reflections so the structure of the centralizers is easily computed. Finally, we may pick d in the coset tuT_0 with matrix representation

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

acting on H'/Z in its usual matrix representation as $U_4(3)$, (e.g. [25]) so $C_{H'/Z}(d) \cong U_3(3)$. Also, $\langle d, H' \rangle/Z \subseteq O_6^-(3)$ so it acts on the natural projective module V. Thus there is one other class of involutions in the coset dH' and if c represents this class, $[V, c]$ has $\mathbf{F}_3$-dimension 4 and Witt index 1, whence $C_{H'}(c) \cong \Sigma_6$.

To prove (b) and (c) we make use of Lemma 6.1 which asserts that L centralizes $[V, t]$ and $V/C_V(t)$, so, in particular, t does not act freely on V. Thus t centralizes Z and so $H \subseteq GL_6(4)$, $\dim_{\mathbf{F}_4} [V, t] = 1$ or 2. Since $\dim V/C_V(t) = \dim [V, t]$ and L acts on $C_V(t)/[V, t]$ we must have $\dim_{\mathbf{F}_4} [V, t] = 1$ and since $L \nsubseteq GL_3(\mathbf{F}_4)$,

$$\dim_{\mathbf{F}_4} [V, L]/C_{[V,L]}(L) = 4.$$

We may pick commuting H-conjugates a_1, a_2, a_3 of t with span $\{[V, a_i] | i = 1, 2, 3,\}$ of dimension 3 over $\mathbf{F}_4$, whence $\dim_{\mathbf{F}_4} [V, a_1a_2a_3] = 3$ and $a_1a_2a_3$ is an involution in tH'. This completes the proof of both (b) and (c).

LEMMA 2.9. *Suppose $H \trianglelefteq G$ with H isomorphic to one of A_n, $Sp_n(q)$, $U_n(q)$, $\Omega_n^{\pm}(q)$, $Sz(q)$, q even, $\Omega_n^{\pm}(q)$, $q = 3$ or 5, F_{22}, F_{23}, F'_{24} or $L_2(q) \wr A_n$, q even ≥ 4, and assume $H = H'$. If $K \subseteq G$ with $E \in Syl_2(K)$, E elementary abelian of rank ≥ 2, $K = O(K)E$, K tightly embedded in G and K acting faithfully on H, then $|O(K)| \leq 3$.*

Proof. Assume $O(K) \neq 1$. Since $m(E) \geq 2$, there exists $e \in E^{\#}$ with $1 \neq O(C_K(e)) \subseteq O(C_G(e))$. Since K acts faithfully on H, by inspection $H \ncong Sp_n(q), U_n(q), Sz(q), F_{22}, F_{23}, F'_{24}, H \neq \Omega_n^{\pm}(q), (n, q) \neq (2, 4), (4, 2), (4, 4)$, q even, $H \ncong L_2(q) \wr A_n$, $(n, q) \neq (1, 4), (2, 4)$. By Theorem 4.9 of [11] $H \ncong \Omega_n^{\pm}(q)$, $q = 3$ or 5. Thus the only possibilities for H are A_n or $A_5 \wr Z_2$ and the result is easily checked in these instances.

LEMMA 2.10. *Let G be a group generated by a conjugacy class D of odd transpositions with G' semisimple, let V be a faithful irreducible $\mathbf{F}_2G$-module and assume, for $e \in D$, $C_G(e)$ has a component L such that V has a unique non-trivial irreducible $\mathbf{F}_2L$-constituent. One of the following holds:*

(1) *$L \cong A_n$, $G \cong \Sigma_{n+2}$, V is the non-trivial irreducible constituent of the natural $(n + 2)$-dimensional permutation $\mathbf{F}_2G$-module, $n \geq 5$;*

(2) *$L \cong Sp_{2n}(2^m)'$, $G \cong O_{2n+2}(2^m)$, V is the natural $(2n + 2)$-dimensional $\mathbf{F}_{2^m}G$-module viewed as a module over $\mathbf{F}_2$, $n \geq 1$, $m \geq 1$;*

(3) $L \cong U_4(2)$, $G \cong Z_3 \cdot U_4(3) \cdot Z_2$, $\dim_{\mathbf{F}_2} V = 12$,

$$\dim_{\mathbf{F}_2} [V, L]/C_{[V,L]}(L) = 8.$$

Proof. By the Main Theorem of [2] we may identify (G, D); the only instances in which the centralizer of an odd tranposition contains a component are when $G/S(G)$ is one of the following: Σ_n, $O^{\pm}_{2n+2}(2^m)$, $O^{\pm}_n(q)$, $q = 3$ or 5, F_{22}, F_{23} or F_{24}; moreover, in each of these cases for $e \in D$, $\langle e \rangle \leq N_G(L)$, $\langle e \rangle \in Syl_2(C_G(L))$ and any proper subgroup of G containing $\langle e \rangle L$ is contained in $N_G(L)$. It is convenient to use Lemma 6.1 to see that L centralizes $[V, e]$. Let $\tilde{G}$ be the semidirect product VG, so from these remarks it follows that $[V, e]\langle e \rangle$ is a T.I.-set in $\tilde{G}$. By 7.11 of [8], L, G are one of the pairs described by conclusions (1)–(3) so it remains to identify the module structure of V.

If $G \cong \Sigma_n$, by 7.10 of [8], V is the $\mathbf{F}_2 G$-module described in conclusion (1).

Next assume $G \cong Z_3 \cdot U_4(3) \cdot Z_2$ so $C_G(e) = \langle e \rangle \times Z \times L$, $L \cong U_4(2)$, and, for each 3-subgroup T of $C_G(e)$, T has a subgroup $\Delta(T) = T \cap L$ of index ≤ 3 in T with $\Delta(T) \cap Z = 1$ and $\Delta(T)$ centralizing $[V, e]$. Let $X = [V, L]$, $Y = C_X(L)$ $W = X/Y$ so W is an irreducible $\mathbf{F}_2 L$-module. Let z be a 2-central involution in L and since z^L the unique class of root involutions in L we may choose $g \in G$ such that $e^g = ez$.

We show that $[W, z]$ is a T.I.-set under the action of L. For suppose $x \in L$ and

$$[W, z] \cap [W, z^x] \neq 0.$$

Let $\bar{w} \in [W, z] \cap [W, z^x]$, $\bar{w} \neq \bar{0}$ so there exists $w \in [X, z] - \{0\}$ and $y \in Y$ with $wy \in [X, z^x]$. By Lemma 6.1 (or because $X = [V, L]$), e centralizes X so $w \in [X, e^g]$, $wy \in [X, e^{gx}]$. Let

$$\Delta_1 = \langle \Delta(T) \mid T \in Syl_3(C_{L \times Z}(e^g)) \rangle,$$

$$\Delta_2 = \langle \Delta(T) \mid T \in Syl_3(C_{L \times Z}(e^{gx})) \rangle,$$

so, since $O^{3'}(C_L(z)) \cong SL_2(3) Y SL_2(3)$, it follows that

$$\Delta_i \cong SL_2(3) Y SL_2(3) \quad \text{or} \quad Z_3 \times (SL_2(3) Y SL_2(3)).$$

Moreover, because Δ_i centralizes Y and is generated by 3-elements, Δ_i centralizes $wY = wyY = \bar{w}$, $i = 1, 2$. Because

$$\langle z \rangle = Z(O_2(\Delta_1)) \quad \text{and} \quad \langle z^x \rangle = Z(O_2(\Delta_2)),$$

if $z \neq z^x$, $H = \langle \Delta_1, \Delta_2 \rangle \not\leq C_L(z)$, so since $C_H(z)$ has an intrinsic component of 2-rank 1, by inspection in $U_4(2)$ (or by Theorem 1 of [16]), necessarily $H = L$. Since $\bar{w} \neq \bar{0}$ and L acts irreducibly on W, $z = z^x$ as needed to prove $[W, z]$ is a T.I.-set.

By Proposition 1.3 of [30] and the irreducible action of L on W,

$$\dim_{\mathbf{F}_2} W = 8 \quad \text{and} \quad \dim_{\mathbf{F}_2} [W, z] = 2.$$

If $v \in Y \cap [X, e^g]$, $v \neq 0$, then by Lemma 6.1, $C_G(v) \supseteq \langle L, e, L^g, e^g \rangle$ which, as noted earlier, forces $L = L^g$, a contradiction. Thus $[X, e^g] \cap Y = 0$ and so

$$\dim_{\mathbf{F}_2} [X, e^g] = 2 = \dim_{\mathbf{F}_2} [X, z].$$

By Lemma 12.1.11 of [14] there are 5 L-conjugates of z which generate L, whence $\dim_{\mathbf{F}_2} X \leq 10$. Since z inverts an element t of order 3 in L, $[V, t] = [X, t]$ has $\mathbf{F}_2$-dimension ≤ 4. Pick $h \in G$ such that $t^h \notin N_G(L)$ so by a previous remark $G = \langle L, e, t^h \rangle$; since $X \subseteq C_V(e)$, G normalizes $C_V(e) + [V, t^h]$, so $C_V(e)$ has codimension ≤ 4 in V. Note also that since Z acts Frobeniusly on V, by Clifford's Theorem applied to $Z\langle e \rangle$, $\dim_{\mathbf{F}_2} [V, e]$ is even.

Since D is a class of 3-transpositions and $L = \langle D_e \rangle$ has 3 orbits on D it follows that we may pick $f \in A_e$ such that $G = \langle L, f \rangle$. Since G normalizes $X + [V, f]$, $\dim_{\mathbf{F}_2} V \leq 14$. If $\dim_{\mathbf{F}_2} [V, e] = 4$, then as $X \subseteq C_V(e)$ and $Y \subseteq [V, e]$, $\dim_{\mathbf{F}_2} X/Y \leq 6$ which is not true. Thus $\dim_{\mathbf{F}_2} [V, e] = 2$, $\dim_{\mathbf{F}_2} V \leq 12$ and, as above, since $\dim_{\mathbf{F}_2} X/Y = 8$, $\dim_{\mathbf{F}_2} V = 12$, as needed.

Finally, suppose $G \cong O^+_{2n+2}(2^m)$, $n \geq 1$, so $C_G(e) \cong Z_2 \times Sp_{2n}(2^m)$, and first consider the case $n = 1$. Since $L \cong L_2(2^m)$, by 7.7 of [8] applied in $C_G(e)$ to the T.I.-set $[V, e^g]\langle e^g \rangle$, where $e^g \in (\langle e \rangle \times L) - \{e\}$, for suitably $g \in G$, $W = [V, L]/C_{[V,L]}(L)$ is the natural $\mathbf{F}_{2^m}L_2(2^m)$-module for L viewed over $\mathbf{F}_2$. Since $|H^1(L, W)| = 2^m$, $\dim_{\mathbf{F}_2} [V, L] \leq 3m$. Moreover, if t is an element of $L^\#$ of odd order, $\dim_{\mathbf{F}_2} [V, t] = 2m$. Let $h \in G$ with $t^h \notin N_G(L)$; by inspection L is maximal in G' so $\langle L, t^h \rangle = G'$. Thus G' normalizes $[V, L] + [V, t]$, and since V has a unique non-trivial irreducible $\mathbf{F}_2L$-constituent, G normalizes this space as well. This proves $\dim_{\mathbf{F}_2} V \leq 5m$. If $G \cong O^-_4(2^m) \cong L_2(2^{2m})\langle e \rangle$ where e induces a field automorphism, by Lemma 2.6 of [30] $\dim_{\mathbf{F}_2} V = 4m$ and V is either the natural $\mathbf{F}_{2^{2m}}L_2(2^{2m})$-module or the natural $\mathbf{F}_{2^m}\Omega^-_4(2^m)$-module for G'; in the first instance, however, V would be a free $\mathbf{F}_2\langle e \rangle$-module and L would have two non-trivial irreducible constituents, a contradiction. If

$$G \cong O^+_4(2^m) \cong L_2(2^m) \wr Z_2,$$

let G_1, G_2 be the components of G interchanged by e. For each i, V is the sum of (more than one) isomorphic irreducible $\mathbf{F}_2G_i$-modules, whence by Lemma 2.6 of [30] each of these is either the natural $\mathbf{F}_{2^m}L_2(2^m)$-module or the natural $\mathbf{F}_{2^k}\Omega^-_4(2^k)$-module, $2k = m$, so $\dim_{\mathbf{F}_2} V = 4m$. For $E \in Syl_2(G_1)$, $[V, E]$ and $C_V(E)$ admit G_2, whence the only possibility is $\dim_{\mathbf{F}_2}[V, E] = 2m = \dim_{\mathbf{F}_2} C_V(E)$. Since E acts quadratically, V is the sum of natural $\mathbf{F}_{2^m}L_2(2^m)$ modules for G_1 and for $G_1^e = G_2$. Thus if W is such a natural module over $\mathbf{F}_{2^m}$, $V \cong W \otimes_{\mathbf{F}_2} W$ as an $\mathbf{F}_2G$-module, which is the natural module for $O^+_4(2^m)$, as desired.

We have already treated the case $G \cong O^+_6(2) \cong \Sigma_8$. Consider the case $G \cong O^-_6(2)$; so $C_G(e) = \langle e \rangle \times L^*$ where $L^* \cong \Sigma_6$ and we may choose $g \in G$ such that $L^* = L\langle e^g \rangle$. By 7.10 of [8] applied to L^*, $\dim_{\mathbf{F}_2}[V, L]/C_{[V,L]}(L) = 4$ so by 11.3 of [5], $\dim_{\mathbf{F}_2} [V, L] \leq 5$. For a 3-cycle t in L, $\dim_{\mathbf{F}_2} [V, t] = 2$. Let $h \in G$

with $t^h \notin N_G(L)$, whence $G' = \langle L, t^h \rangle$ so G', and hence also G, normalizes $[V, L] + [V, t^h]$. This proves $\dim_{\mathbf{F}_2} V \leq 7$ and since L centralizes $[V, e]$ and $V/C_V(e)$ it follows that e induces an $\mathbf{F}_2$-transvection on V. Thus $\dim_{\mathbf{F}_2} V = 6$ and V is the natural $\mathbf{F}_2 O_6^-(2)$-module for G.

Let $G \cong O_{2n+2}^{\pm}(2^m) \not\cong O_6^{\pm}(2)$, $n \geq 2$, and proceed by induction. Let H be the centralizer in G of some hyperbolic plane chosen so that $e \in H$ and $H \not\cong O_6^{+}(2)$. Since $H' \cong \Omega_{2n}^{\pm}(2^m)$, by Lemma 2.7, $V = V_0 \oplus V_1$ where $V_0 = [V, H']$, $V_1 = C_V(H')$, and by induction V_0 is the natural module. If e does not centralize V_1, let $v \in [V_1, e] - \{0\}$; then $C_G(v) \supseteq \langle e, L, H \rangle = G$, a contradiction. Thus

$$\dim_{\mathbf{F}_2} [V, e] = \dim_{\mathbf{F}_2} [V_0, e] = m.$$

Let $h_1, h_2 \in G$ with $G = \langle H, e^{h_1}, e^{h_2} \rangle$, whence $V = V_0 + [V, e^{h_1}] + [V, e^{h_2}]$ has $\mathbf{F}_2$-dimension $\leq (2n + 2)m$. Now $V \otimes_{\mathbf{F}_2} \mathbf{F}_{2^m}$ is isomorphic as an $\mathbf{F}_2 G$-module to a direct sum of m copies of V; also $V_0 \otimes_{\mathbf{F}_2} \mathbf{F}_{2^m}$ is the direct sum of m natural $\mathbf{F}_{2^m} O_{2n}^{\pm}(2^m)$-modules for H. Thus if $V \otimes_{\mathbf{F}_2} \mathbf{F}_{2^m} = U_1 \oplus \cdots \oplus U_m$ is a Krull-Schmidt $\mathbf{F}_{2^m} G$-module decomposition, since H acts non-trivially on each U_i, $U_i|_H = W_i \oplus T_i$ where W_i is the natural $\mathbf{F}_{2^m} O_{2n}^{\pm}(2^m)$-module for H and T_i is a trivial module. Because e centralizes V_1, e centralizes T_i and so e induces a $\mathbf{F}_{2^m}$-transvection on U_i, $1 \leq i \leq m$. As an $\mathbf{F}_2$-module, therefore, each U_i is the natural module for $O_{2n+2}^{\pm}(2^m)$, as desired.

III. The Proof of Theorem A

Throughout this section let J, x be as given by the hypothesis of Theorem A, let $V = U(J)$, $Z = V \cap Z(J)$ and let $\overline{J\langle x \rangle} = J\langle x \rangle / O_2(J\langle x \rangle)$. We may clearly assume $O(J) = 1$.

We first dispose of the case when V is abelian, that is, (by Lemma 2.2) when

$$V \subseteq \Omega_1(Z(O_2(J))).$$

Let P be a subgroup of J of odd prime order with $\overline{P}$ normalized by $\bar{x}$, where the Baer-Suzuki Theorem [1] is used if $\bar{x} \neq \bar{1}$. Thus x normalizes $[V, P]$ and so has a non-trivial fixed point therein. Since $[V, P] \cap Z(J) = 1$ and V is elementary abelian, the result holds in this case. Henceforth it is assumed that $V' \neq 1$.

The following lemma due to J. G. Thompson facilitates the proof of Theorem A.

LEMMA 3.1. *If t is an involution acting on a solvable group S with*

$$C_S(t) \subseteq O_2(Z(S)),$$

then t inverts a $2'$-Hall subgroup of S.

Proof. First note that if u is an involution acting on a solvable group H with $H = O_{2,2'}(H)$ and u inverting $H/O_2(H)$, then an easy induction on $|H|$ shows u normalizes (hence inverts) a $2'$-Hall subgroup of H.

Now let $G = S\langle t\rangle$ be a counterexample to Lemma 3.1 of minimal order and let $\overline{G} = G/O_2(G)$. If $C_{\overline{G}}(\bar{t})$ is a 2-group, $\bar{t}$ inverts $O(\overline{G})$ and since $\overline{G}/O(\overline{G})$ acts faithfully on $O(\overline{G})$, $\overline{G}$ has a normal 2-complement (which is inverted by $\bar{t}$); in this situation, by the initial paragraph G is not a counterexample. Thus there is a subgroup P of G of odd prime order with $\overline{P} \subseteq C_{\overline{G}}(\bar{t})$, whence

$$G = O_2(G)P\langle t\rangle = O_2(G)P.$$

Moreover, $\langle t, P\rangle$ is also a counterexample so $G = \langle t, P\rangle$. In particular, if $H = O_2(G)$, $H = \langle t^P\rangle$ so $H' = \phi(H)$ and H/H' is a cyclic $\mathbf{F}_2 P$-module. Let $K = [H, P]$ so $K/\phi(K)$ is a direct sum of non-isomorphic $\mathbf{F}_2 P$-modules. Thus if $f \in C_H(P)$ with $f \equiv t \pmod K$, $H = K\langle f\rangle$ and $[K, f] \subseteq \phi(K)$. Since t and P commute in their action on $Z(K)$ and $[Z(K), P] \cap Z(G) = 1$, P centralizes $Z(K)$.

We now prove K has class 2. For suppose A is a characteristic abelian subgroup of K and let $W = [\Omega_1(A), P]$. Since $W \cap Z(G) = 1$, $WW^t = W \times W^t$ and since t centralizes $D = \{ww^t \mid w \in W\}$, $D \subseteq Z(G)$. Thus $WW^t = WD$ admits $\langle t, P\rangle$ and so is normal in G. However t is conjugate in $\langle W, t\rangle$ to every involution in $(WW^t)t$, so G/WW^t is also a counterexample to the lemma. By minimality of G, $W = 1$, i.e. P centralizes every characteristic abelian subgroup of K. By Lemma 5.17 of [27], K is special. Let $\phi_f\colon K/K' \to K'$ by $\phi_f(k) = [k, f]$. It follows that ϕ_f is an $\mathbf{F}_2 P$-module homomorphism. Since K/K' is a Frobenius $\mathbf{F}_2 P$-module and K' is a trivial module, $\mathrm{Hom}_{\mathbf{F}_2 P}(K/K', K') = 0$, whence $[K, f] = 1$. However, $t = kf$, for some $k \in K$ and since t centralizes k, $k \in Z(G)$; but then t centralizes K, the desired contradiction.

Continuing the proof of Theorem A, we proceed by induction and assume $J\langle x\rangle$ is a counterexample of minimal order. It will be necessary to establish a number of properties of $J\langle x\rangle$ before utilizing Thompson's lemma in a setup where a contradiction can be reached.

First observe that x centralizes $Z(J)$. For otherwise there exist $z_1, z_2 \in Z(J)^{\#}$ such that $z_1^{-1}xz_1 = xz_2$ with $[x, z_2] = 1$. Putting $\hat{J} = J/\langle z_2\rangle$, the minimality of J forces the existence of a 2-element $t \in J$ with $\hat{t} \notin Z(\hat{J})$ and $[t, x] \in \langle z_2\rangle$. Since either t or tz_1 centralizes x and neither lies in $Z(J)$, we have the desired contradiction.

Next suppose for some subgroup P of odd prime order in J, $\bar{x}$ centralizes $\overline{P}$. Then $V_0 = [V\langle x\rangle, P]$ admits P and x. Let $x_1 \in V_0 x$ with $[x_1, P] = 1$ and let $Z_0 = V_0 \cap Z$, so x and P commute in their action on V_0/Z_0. Let $Q \supseteq Z_0$ with $Q/Z_0 = C_{V_0/Z_0}(x)$, so Q admits P with $Q/Z_0 = [Q/Z_0, P] \neq Z_0/Z_0$. Since $x \equiv x_1 \pmod{V_0}$, $[Q, x_1] \subseteq Z_0$. As in the proof of Lemma 3.1 the map

$$\phi_{x_1}\colon Q/Z_0 \to Z_0, \qquad \phi_{x_1}(q) = [q, x_1],$$

is an $\mathbf{F}_2 P$-module homomorphism and since $\mathrm{Hom}_{\mathbf{F}_2 P}(Q/Z_0, Z_0) = 0$, $[Q, x_1] = 1$. Now let $v \in V_0$ with $x = vx_1$. Note that $v^2, x_1^2 \in Z_0$, whence

$$1 = x^2 = v^2x_1^2[v, x_1],$$

and so $[v, x_1] \in Z_0$, that is, $v \in Q$. Since x_1 centralizes Q, $[x, v] = 1$, so $v \in Z_0$. But then x centralizes Q, contrary to $Q \nsubseteq Z_0$. This proves $C_J(\bar{x})$ is a 2-group.

Let P be any subgroup of J of odd prime order p inverted by x (such subgroups exist by the Baer-Suzuki Theorem), $V_0 = [V, P]$, $V_1 = C_V(P)$. By arguing as in the previous paragraph with $x_1 \in V_1$ and $q \in V_0$, we obtain $[V_0, V_1] = 1$. Let

$$V_0/V'_0 \cong E_{2^{2n}}, \quad V'_0 \cong E_{2^m}$$

and let Q be the complete preimage in V_0 of $C_{V_0/V'_0}(x)$, so as x is free on V_0/V'_0, $Q/V'_0 \cong E_{2^n}$ and $Q/V'_0 = [V_0/V'_0, x]$. We show Q is abelian. If $a \in Q$, $v \in V_0$ and $z = [a, v]$, then

$$z = z^x = [a^x, v^x] = [az_1, v^x] \quad \text{where } z_1 = [x, a^{-1}] \in Z;$$

so $[a, v] = [a, v^x]$, whence $[a, vv^x] = 1$. Since $Q = \langle Z \cap V_0, vv^x | v \in V_0\rangle$, a centralizes Q, for all $a \in Q$, as desired. Now for $a \in Q\text{-}V'_0$, $\langle Q, V'_0, V_1\rangle \subseteq C_V(a)$ and so

$$| V : C_V(a) | \leq 2^n.$$

This means $A = [V, a]$ has order at most 2^n. Let $\hat{J} = J/A$, so $\hat{J}$ is a block and since $\hat{P}$ acts non-trivially on $\hat{a} \in Z(O_2(\hat{J}))$, the non-central 2-chief factor for $\hat{J}$, namely $\hat{V}/\hat{Z}$, lies in $Z(O_2(\hat{J}))$, whence $V' \subseteq A$ by Lemma 2.2(a). In particular, $V'_0 = V' = A$ so $m \leq n$. However, $[Q, x] \subseteq V'_0$ so $2^m \geq |[Q, x]| = |Q : C_Q(x)|$, and as $C_Q(x) = V'_0$, $|Q : C_Q(x)| = 2^n$, whence $n \leq m$. This proves $m = n$ and since

$$| V_0 : C_{V_0}(x) | = 2^{2n},$$

x is conjugate in $V_0\langle x\rangle$ to every involution in $V_0 \cdot x$. Thus every element of Qx is an involution, x inverts Q and $Q \cong Z_4 \times \cdots \times Z_4$ (n copies).

By considering $\hat{J} = J/V'_0$ as above we obtain $V' \subseteq V'_0$ and so $V'_0 = V' = \phi(V)$.

We next show x centralizes V_1/Z. If $v_1 \in V_1$ and $v_1^x \not\equiv v_1 \pmod Z$, it follows that $u = v_1 v_1^x$ has order 4 and is inverted by x. However, $u^2 \in V'$ so there exists $v \in V_0$ such that $v^{-1}xv = xu^2$, and therefore $uv \in C_J(x)$, contrary to $uv \notin Z(J)$.[1]

Now let $N = N_{J\langle x\rangle}(P)$ and note that N acts on both V_0 and V_1. Let

$$x \in S \in Syl_2(N) \quad \text{and} \quad R = S \cap C_{J\langle x\rangle}(P).$$

We first show R centralizes V_0. If not, pick $r \in R - C(V_0)$ with r^2, $[r, x] \in C(V_0)$. For $q \in Q$,

$$[q, r]^x = [q^x, r^x] = [qz, r[r, x]],$$

where $z = q^2 \in Z$; so $[q, r] = [q, r]^x$, proving $[q, r] \in Z$. Thus $[Q, r] \subseteq Z$ and so

$$[V_0, r] = [Q^P, r] \subseteq Z.$$

[1] See the remarks at the end of this proof.

As usual, r induces an $\mathbf{F}_2 P$-module homomorphism from V_0/V'_0 to Z, whence $[V_0, r] = 1$, as claimed. Note that if y is an odd order element of N centralizing $V_0 Z/Z$, then $[x, y]$ centralizes V/Z so $[x, y] \in O_2(J)$, whence $y = 1$ in view of $C_{\bar{J}}(\bar{x})$ being a 2-group. Now if Sylow p-subgroups of J are not cyclic, there exists $X \subseteq N$ with $P \subseteq X$ and $X \cong Z_p \times Z_p$. Since X is faithful on V_0/V'_0, by Schur's lemma there exists $y \in X$ such that $2n > \dim_{\mathbf{F}_2} [V_0/V'_0, y] \geq n$. Let $v \in C_{V_0}(y) - Z$, so as usual v centralizes $[V_0, y]$. But

$$C_{V_0}(v) \supseteq \langle v, [V_0, y], V' \rangle$$

and the latter group has order exceeding 2^{2n}, contrary to $|[V_0, v]| = |V'| = 2^{2n}$. This proves that Sylow p-subgroups of J are cyclic, and, in particular, N contains a Sylow p-subgroup of J. Next we show $S = R\langle x \rangle$. If this is not true, since S/R is cyclic, there exists $s \in S$ with $s^2 \equiv x \pmod R$ and, of course, $[s, x]$ centralizing V_0. Then for all $q \in Q$, $[q, s]^x = [q, s]$, so $[Q, s] \subseteq V_0 \cap Z = V'_0$. Also, for $q \in Q - Z$ and $v \in V_0$ if $z = [q, v]$, then $z = [q, v^s]$ so $[q, vv^s] = 1$ which gives

$$[V_0, s] \subseteq C_{V_0}(q) = Q.$$

But then s acts as an involution on V_0/V'_0 contrary to s^2 acting identically to x on V_0/V'_0. Now let $\overline{M} = C_{\overline{N}}(V_1/Z)$. Since R centralizes V_0, $\overline{R}$ acts faithfully on V_1/Z, so $\overline{R} \cap \overline{M} = \overline{1}$. Thus $\langle \bar{x} \rangle$ is a Sylow 2-subgroup of $\overline{M}$. Note that $C_{\overline{N}}(\bar{x})$ therefore covers $\overline{N}/\overline{M}$ so $O^2(\overline{N}) \subseteq \overline{M}$ and $\overline{M}$ has a Hall $2'$-subgroup which is inverted by $\bar{x}$. Since P was arbitrary subject to being inverted by x (and by properties of involutions x inverts an element of order p_1 for each odd prime divisor p_1 of $|M|$), applying these results to each odd prime divisor of $|M|$ gives that M has a cyclic $2'$-Hall subgroup P^* inverted by x and P^* is a Hall subgroup of J. Let $1 \neq P_1 \subseteq P^*$ so $[V, P_1] \subseteq V_0$. By arguing with P_1 in place of P, x acts trivially on $C_{V/Z}(P_1)$ so $[V, P_1] = V_0$. Note that $[R, x]$ centralizes V/Z so $[R, x] \subseteq O_2(J)$. Also, $[R, P_1]$ centralizes V/Z so $[R, P_1] \subseteq O_2(J) \cap N = V_1 Z(J)$, hence

$$[R, P_1] = [R, P_1, P_1] = 1.$$

Since P_1 was arbitrary, $R \in Syl_2(C_{J\langle x \rangle}(P_1))$, for all $1 \neq P_1 \subseteq P^*$. Finally, if $x^g \in S$, for some $g \in J$, then as $C_{\bar{J}}(\bar{x})$ is a 2-group, $\overline{x^g}$ inverts $\overline{P}$, so $x^g = xr$, for some $r \in R$; but then x^g is free on V_0/V'_0 and since $\dim_{\mathbf{F}_2} [V/Z, x] = n$, x^g centralizes V_1/Z, so $\bar{r} = \overline{1}$, i.e. $\bar{x} = \overline{x^g}$.

In summary, J, x satisfy the following:

(1) x is an involution acting on J with $C_J(x) \subseteq O_2(Z(J))$;

(2) if P is a subgroup of J of odd prime order p inverted by x, then

(a) Sylow p-subgroups of J are cyclic,

(b) $N_{J\langle x \rangle}(P) = (R \times P^*)\langle x \rangle$, where P^* is a cyclic Hall subgroup of J inverted by x and $R \in Syl_2(C_{J\langle x \rangle}(P_1))$, for all $1 \neq P_1 \subseteq P^*$,

(c) $[R, x] \subseteq O_2(J)$,

(d) if $x^g \in R\langle x\rangle$, for some $g \in J$, $xx^g \in O_2(J)$.

Although one would expect an easy contradiction at this point it seems that a considerable amount of elementary argument is yet required and that the best course is to consider all groups J satisfying (1) and (2) (not just for J a block). The final contradiction will be immediate once we have established:

(*) If J, x are any pair satisfying (1) and (2), then $J\langle x\rangle = O_2(J)H\langle x\rangle$, where H is a cyclic $2'$-Hall subgroup of J inverted by x.

To prove (*) we proceed by induction and let J be a counterexample of minimal order. Note that every proper subgroup of $J\langle x\rangle$ containing x satisfies (1) and (2) so these are described by the conclusion of (*). By Lemma 3.1, J is not solvable so by minimality of J, $J = J'$ and $J/S(J)$ is simple. Moreover,

$$S(J)\langle x\rangle = O_2(J)H\langle x\rangle,$$

where H is cyclic and inverted by x. Let $\overline{J\langle x\rangle} = J\langle x\rangle/O_2(J\langle x\rangle)$ so $\bar{J}$ is quasisimple. Since $N_{J\langle x\rangle}(H)$ covers $\bar{J}$ and contains x, by minimality of $J\langle x\rangle$, $H \le J\langle x\rangle$. By Frobenius' normal p-complement theorem together with property (2a), $H = 1$, so $\bar{J}$ is simple. Note that $C_{\bar{J}}(\bar{x})$ is necessarily a 2-group so we may pick $T \in Syl_2(J\langle x\rangle)$ with $C_{\overline{J\langle x\rangle}}(\bar{x}) \subseteq \bar{T}$.

Let $\bar{t}$ be any involution in $Z(\bar{T})$ and let $\bar{M}$ be a maximal subgroup of $\overline{J\langle x\rangle}$ containing $C_{\overline{J\langle x\rangle}}(\bar{t}) = \bar{M}_0$. If $\bar{M}_0 = \bar{T}$, by a result of Baumann [12], $\bar{J} \cong L_2(q)$, $U_3(q)$, $Sz(q)$, $L_3(q)$, $S_4(q)$, $q = 2^n$ or $L_2(q)$, $q = 2^n \pm 1$. As $C_{\bar{J}}(\bar{x})$ is a 2-group, by Lemma 2.10 of [12], $\bar{x}$ induces inner automorphisms on $\bar{J}$; but then in every case x lies in a proper subgroup of $J\langle x\rangle$ which does not satisfy the conclusion of (*). Thus $\bar{M}_0 \neq \bar{T}$ so we may write $M = O_2(M)H\langle x\rangle$ where H is a cyclic $2'$-Hall subgroup of M inverted by x and $H_0 = H \cap M_0$ is a $2'$-Hall subgroup of M_0. Since $C_{J\langle x\rangle}(H_0)$ covers $C_{\overline{J\langle x\rangle}}(\bar{H}_0)$ we may assume $[t, H_0] = 1$, whence by (2b), $[t, H] = 1$, H is a Hall subgroup of J, $H = H_0$ and $M = M_0$. For any $1 \neq P \subseteq H$, $N_{J\langle x\rangle}(P)$ covers $N_{\overline{J\langle x\rangle}}(\bar{P})$ so by (2b), (2c) and the fact that

$$C_{\overline{J\langle x\rangle}}(\bar{x}) \subseteq \bar{T} \quad \text{and} \quad C_{\overline{J\langle x\rangle}}(\bar{t}) = \bar{M},$$

$N_{\overline{J\langle x\rangle}}(\bar{P}) \subseteq \bar{M}$. Finally, if $x^g \in M$, for some $g \in J$, $\overline{x^g}$ inverts $\bar{H}$ so by properties of involutions there exists $m \in O_2(J\langle x\rangle)$ such that x^{gm} inverts H; then there exists $h \in H$ such that $x^{gmh} \in T$. By property (2d) $\overline{x^{gmh}} = \bar{x}$ so $\overline{gmh} \in C_{\overline{J\langle x\rangle}}(\bar{x}) \subseteq \bar{M}$, whence $\bar{g} \in \bar{M}$. Thus $x^g \in M \Leftrightarrow g \in M$ and, by the structure of M, $x^g \in T \Leftrightarrow g \in T$.

Now if N is any proper subgroup of $J\langle x\rangle$ containing T, then $\bar{N}$ is 2-constrained: for otherwise some odd prime order subgroup P of N inverted by x would have $[O_2(\bar{N}), \bar{P}] = \bar{1}$; but then since $C_{J\langle x\rangle}(P)$ covers $C_{\overline{J\langle x\rangle}}(\bar{P})$, (2c) forces $\bar{x}$ to centralize $O_2(\bar{N})$ and since $\bar{T} = O_2(\bar{N})\langle\bar{x}\rangle$, a previous argument applied to $\bar{t} = \bar{x}$ gives a contradiction. Secondly, if $\bar{N}$ is any 2-local subgroup of $\overline{J\langle x\rangle}$ containing $\bar{x}$, then either $\bar{N} \subseteq \bar{M}$ or $(|\bar{N}|, |\bar{M}|) = 2^a$, for some a. For as x lies in a unique Sylow 2-subgroup of $J\langle x\rangle$, $\bar{N} \cap \bar{T} \in Syl_2(\bar{N})$; suppose some odd prime p divides $|\overline{N}|$ and $|\overline{M}|$ and let P be a Sylow p-subgroup of N inverted by x. Since Sylow p-subgroups of J are cyclic, there exists $g \in J$ such that $P^g \subseteq M$; and finally, as $x^g \in N_{J\langle x\rangle}(P^g) \subseteq M$, $g \in M$ so $N = (N \cap T)N_N(P) \subseteq M$, as claimed.

Now suppose there is an involution $\bar{t}_1$ in $Z(\overline{T})$ with $C_{\overline{J\langle x\rangle}}(\bar{t}_1) \nsubseteq M$. Let

$$\overline{M}_1 = C_{\overline{J\langle x\rangle}}(\bar{t}_1), \quad \overline{M}_2 = C_{\overline{J\langle x\rangle}}(\overline{tt_1}).$$

The arguments using $\bar{t}$ also apply to show that $\overline{M}_1, \overline{M}_2$ are maximal subgroups of $\overline{J\langle x\rangle}$ and, by the previous paragraphs, $\overline{M}, \overline{M}_1, \overline{M}_2$ are 2-constrained 2-locals with $|\overline{M}|_{2'}, |\overline{M}_1|_{2'}, |\overline{M}_2|_{2'}$ pairwise coprime. Thus for two of these subgroups, say $\overline{M}, \overline{M}_1$, $3 \nmid |\overline{M}|, |\overline{M}_1|$. Since $Z(\overline{T}) \ntrianglelefteq \overline{M}$ or $\overline{M}_1$, by the Thompson factorization Lemma 5.54 of [27] one sees that $J(\overline{T}) \trianglelefteq \overline{M}$ and $\overline{M}_1$, contrary to $\overline{M}, \overline{M}_1$ being distinct maximal subgroups. This shows $\overline{M} = C_{\overline{J\langle x\rangle}}(\Omega_1(Z(\overline{T})))$.

As before, if $\bar{x}$ inverts some subgroup $\overline{P}$ of odd prime power order p^α and $p \mid |M|$, then $\overline{P} \subseteq \overline{M}$. If $\langle u\rangle$ is a subgroup of 2-power order inverted by x, then $u \in M$: for let u be of minimal order with respect to $u \notin M$; then $u^2 \in M$ and $xu^2 = x^u \in M$ so $u \in M$, a contradiction. Now for all $g \in J$, x inverts $[g, x]$ so since $J = [J, x] \nsubseteq M$, there exists Q of odd prime order q inverted by x and $q \nmid |M|$. Let $N_0 = N_{J\langle x\rangle}(Q)$, $S \in Syl_2(C_{J\langle x\rangle}(Q))$. If $\overline{S} = \bar{1}$, then $\langle \bar{x}\rangle$ is a Sylow 2-subgroup of $N_{\overline{J\langle x\rangle}}(\overline{Q})$. Since Sylow q-subgroups of J are cyclic but J does not have a normal q complement, $\langle\bar{x}\rangle \in Syl_2(N_{\bar{J}}(\overline{Q}))$, that is, $\bar{x} \in \bar{J}$. However, $\overline{J\langle x\rangle}$ cannot be simple, otherwise by Thompson's transfer lemma [5.38 of 27] there exists $g \in J$ such that $\overline{x^g} \in O_2(\overline{M})$, whereas no such $g \in M$ exists. This argument proves $\overline{S} \neq \bar{1}$ so let $\overline{N}$ be a maximal (2-local) subgroup of $\overline{J\langle x\rangle}$ containing $N_{\overline{J\langle x\rangle}}(\overline{S})$.

We first show $\overline{N}$ contains $\overline{T}$. In any case since T is the unique Sylow 2-subgroup of $J\langle x\rangle$ containing x, $T_0 = T \cap N \in Syl_2(N)$. Let Q^* be a $2'$-Hall subgroup of N inverted by x. Assume $T_0 \neq T$ and let $a \in N_T(T_0) - T_0$ with $a^2 \in T_0$. If $\overline{N}$ is not 2-constrained, by (2b), $\overline{N} = (\overline{Q}^* \times O_2(\overline{N}))\langle\bar{x}\rangle$, and by maximality of $\overline{N}$ and the fact that $\bar{x}$ centralizes $O_2(\overline{N})$ by (2c),

$$O_2(\overline{N}) \cap O_2(\overline{N})^{\bar{a}} = \bar{1}.$$

This forces $\overline{S} = O_2(\overline{N}) \cong Z_2$. Since $N_{\overline{T}}(\overline{S}) = \overline{S} \times \langle\bar{x}\rangle$ has order 4, $Z(\overline{T}) = \langle\bar{t}\rangle$ has order 2 and $\bar{t} \in \{\bar{x}, \bar{x}\bar{s}\}$ where $\overline{S} = \langle\bar{s}\rangle$. As noted before, $\bar{x}$ is not central in $\overline{T}$ so $\bar{t} = \bar{x}\bar{s}$ and $\bar{x}^{\bar{a}} = \bar{x}\bar{t}$. This, however, contradicts property (2d) applied in $N(H)$ and so proves that $\overline{N}$ is 2-constrained. Since $Z(\overline{T}) \subseteq Z(O_2(\overline{N}))$ but $\overline{N} \nsubseteq \overline{M}$, $\overline{Q}^*$ acts faithfully on $Z(O_2(\overline{N}))$. If $|Q^*| > 3$ it follows that $J(\overline{T}_0) \trianglelefteq \overline{N}$ and so $N(J(T_0)) \supsetneq \overline{N}$, a contradiction. It remains to treat the case when $|Q^*| = 3$ and no non-trivial characteristic subgroup of T_0 is normal in $\overline{N}$. By a result of Glauberman [9], $\overline{N}$ has exactly one non-central 2-chief factor which lies in $\Omega_1(Z(O_2(\overline{N})))$, hence equals $\overline{W} = [\Omega_1(Z(O_2(\overline{N}))), \overline{Q}^*]$. Since $\overline{Q}^*$ acts non-trivially on the Frattini quotient of $O_2(\overline{N})$, $\overline{W} \nsubseteq \phi(O_2(\overline{N}))$ whence

$$O_2(\overline{N}) = \overline{W} \times \overline{S}, \quad \overline{S} = C_{T_0}(Q^*), \quad \overline{W} \cong Z_2 \times Z_2,$$

$$\overline{W\langle x\rangle} \cong D_8 \quad \text{and} \quad [\overline{S}, \bar{x}] = \bar{1}.$$

Since $\bar{S} \cap \bar{S}^a = \bar{1}$, $|\bar{S}| \leq 8$, and since $\bar{a}$ normalizes $Z(\bar{T}_0)$, if $\bar{S}$ is abelian, $|\bar{S}| = 2$. Recall that $\bar{t}$ is an involution in $Z(\bar{T}) \cap Z(\bar{T}_0)$ and $\bar{t} \notin \bar{S}$; moreover, by property (2d) applied in $N(H)$, $\bar{x} \nsim \bar{x}\bar{t}$, so $\bar{t} \notin \bar{W}$. Let $Z(\bar{S}) = \langle \bar{s} \rangle$, $\bar{W} = \langle \bar{w}, \bar{u} \rangle$ where $\bar{w} \in Z(T_0)$, so $\bar{x}^{\bar{u}} = \bar{x}\bar{w}$ and $\bar{t} = \bar{s}\bar{w}$. If $|\bar{S}| = 2$, $\bar{T}_0 \cong D_8 \times Z_2$ and necessarily $\bar{s}^{\bar{a}} = \bar{s}\bar{w}$, contrary to $N(\bar{S})$ not containing a Sylow 2-subgroup of $\overline{J\langle x \rangle}$. Thus $|\bar{S}| = 8$ and $\bar{s}^{\bar{a}} = \bar{w}$. Now $\bar{x}^{\bar{a}}$ inverts $\bar{Q}^*$ and so $\bar{x}^{\bar{a}\bar{u}} = \bar{x}^a w$. Thus $\bar{x}^{\bar{a}\bar{u}\bar{a}^{-1}} = \bar{x}\bar{s}$ so with $g = aua^{-1}k$ for suitable k chosen in $O_2(J\langle x \rangle)$ so that x^g normalizes Q^* one sees that property (2d) is violated in $N_{J\langle x \rangle}(Q^*)$. This contradiction proves $\bar{T} \subseteq \bar{N}$.

As decided earlier since $\bar{T} \subseteq \bar{N}$, $\bar{N}$ is 2-constrained so

$$\bar{Z} = \Omega_1(Z(\bar{T})) \subseteq \bar{Z}^* = \Omega_1(Z(O_2(\bar{N}))).$$

Moreover, $\bar{Z} = C_{\bar{Z}^*}(\bar{x})$ and since for every involution $\bar{t}$ in $\bar{Z}$, $C_{\overline{J\langle x \rangle}}(\bar{t}) = \bar{M}$, $\bar{Z}^* = [\bar{Z}^*, \bar{Q}^*]$. Thus $\bar{x}$ acts freely on $\bar{Z}^*$ and so $\bar{x} \sim \bar{x}\bar{t}$, for each $\bar{t} \in \bar{Z}^{\#}$. It follows that (2d) is again violated in $N(H)$. This contradiction completes the proof of ($*$) and hence also of Theorem A.

Remarks. The referee has observed that at the indicated point the following alternate argument shortens the proof of Theorem A.

By the same argument that showed $m \leq n$, $QZ = Z(C_{VZ}(y))$, for all $y \in QZ - Z$, so $QZ/Z = Q^*$ is a TI-set in $VZ/Z = V^*$. Now let $\bar{X} = C_{\bar{G}}(Q^*) \cap C_{\bar{G}}(V^*/Q^*)$ and form the semidirect product, H, of $\bar{G}$ with V^*; let $W = XQ^* \subseteq H$. From the TI property of Q^* it follows that W is an elementary abelian TI-set in H and since $Q^* = [V^*, x]$, $\bar{x} \in \bar{X}^{\#}$. By [8] the members of $\bar{X}^{\#}$ are root involutions in $\bar{G}$ and $\bar{G}$ is described in [8] or [28], whereas by [8] or [28] $\bar{x} \in O_2(\bar{Y})$, for some $\bar{Y} \subseteq \bar{G}$ with $\bar{Y}$ not a 2-group, contrary to Lemma 3.1.

The author has listed his longer but more elementary proof in order to avoid using the deep results of [8] and [28]. Since the principal application for Theorem A is in the proof of Theorem C it seems desirable to maintain such independence, for, as noted in [17], if one uses the classification of characteristic 2 type groups in which a maximal normal elementary abelian 2-subgroup of some maximal 2-local is a TI-set (which relies ultimately on [28]), the proof of Theorem C in characteristic 2 reduces immediately to the "easy" case when $U(J)$ is abelian.

IV. The proof of Theorem B

Throughout this section, G is a minimal counterexample to the assertion of Theorem B, so $G = \langle K, J, x \rangle$. Let $L = \langle K^G \rangle = K_1 Y K_2 Y \cdots Y K_n$ with $K = K_1$. The proof proceeds in a series of steps.

(4.1) $O(G) = 1$.

For it is easily seen that $G/O(G)$ is also a counterexample.

(4.2) $Z(G) = 1$.

For if z is an involution in $Z(G)$, let $\overline{G} = G/\langle z\rangle$; then $|C_{\overline{G}}(\bar{x}): \overline{C_G(x)}| \leq 2$ so $\overline{J}$ is a block of $C_{\overline{G}}(\bar{x})$. It follows that $\overline{G}$ is also a counterexample so by minimality we must have $Z(G) = 1$.

(4.3) $n > 1$.

Suppose to the contrary $n = 1$ and let $V = U(K)$, $\overline{G} = G/V$. If J centralizes V,

$$[J, K] \subseteq K \cap C_G(V) \subseteq O_2(K)$$

so $[\overline{J}, \overline{K}, \overline{K}] = \overline{1}$ whence by the 3 subgroups lemma, $[J, K] \subseteq V$. But then $[K, J, J] \subseteq [V, J] = 1$ so the 3 subgroups lemma shows $[K, J] = 1$, a contradiction. Since J does not centralize V and $J = O^2(J)$, by the $P \times Q$ lemma J does not centralizes $C_V(x)$, from which it follows that $U(J) \subseteq V$.

By assumption $\overline{J}$ induces inner automorphisms on $\overline{K}$ and since $\overline{J} \nsubseteq \overline{K}$ there is a perfect normal subgroup E of KJ with $\overline{KJ} = \overline{K}\overline{Y}\overline{E}$. Because E commutes with the irreducible action of $\overline{K}$ on $\tilde{U}(K)$, E centralizes $\tilde{U}(K)$. Thus $[K, E, E] \subseteq Z(K)$ so by the 3 subgroups lemma, $[K, E] \subseteq Z(K)$; another application of this lemma gives $[E, K] = [E, K, K] = 1$. Thus $E \cap K \subseteq Z(K)$ and since $Z(G) = O(G) = 1$ and $G = EK\langle x\rangle$, $Z(K) = 1$, $EK = E \times K$ and E is quasisimple. Now $[KJ, x] \subseteq K$ so $[E, x] \subseteq C_G(K) \cap K = 1$. Since $E \leq G$ and E is quasisimple, $[E, J] = 1$. But then E is centralized by $\langle K, J, x\rangle = G$, which is absurd. This contradiction establishes (4.3).

(4.4) x normalizes each K_i.

For if $K_i^x = K_j$, $i \neq j$ let $K_0 = C_{K_iK_j}(x)'$ so by Lemma 2.3, K_0 is a block of $C_G(x)$. Since G is a counterexample $K_0 \neq J$ so $[K_0, J] = 1$. But by Lemma 2.3 $[K_i, J] = 1$ and so $K \subseteq \langle K_i^{J\langle x\rangle}\rangle \subseteq C_G(J)$, a contradiction.

(4.5) $Z(L) = 1$.

Notice that $Z(L) = Z(K_1)Z(K_2) \cdots Z(K_n)$ and $Z(L)$ is a 2-group. Suppose $Z(L) \neq 1$ and let $\overline{G} = G/Z(L)$, so $\overline{L} = \overline{K}_1 \times \cdots \times \overline{K}_n$. Since $Z(G) = 1$, $J \nsubseteq L$ and J acts non-trivially on $Z(L)$. Because $J = O^2(J)$, by the $P \times Q$ lemma, J acts non-trivially on $C_{Z(L)}(x)$, hence $U(J) \subseteq Z(L)$. By Theorem A there exists t a 2-element in $C_K(x)$ with $\bar{t} \neq \bar{1}$. Since $\overline{J}$ permutes the $\overline{K}_i$ transitively and $n > 1$, $\overline{J}$ does not centralize $\langle \bar{t}^{\overline{J}}\rangle$; so since $\overline{J}$ is quasisimple, $\overline{J} \subseteq [\overline{J}, \bar{t}] \subseteq \overline{L}$, a contradiction. This also proves:

(4.6) $U(K_i)$ is abelian and $L = K_1 \times K_2 \times \cdots \times K_n$.

Now let $V_i = U(K_i)$, $V = V_1 \times \cdots \times V_n$, $A_i = C_{V_i}(x)$, $B_i = [V_i, x]$,

$$A = A_1 \times \cdots \times A_n, \quad B = B_1 \times \cdots \times B_n,$$

so $A \supseteq B$ and $A/B \cong (A_1/B_1) \times \cdots \times (A_n/B_n)$. Since J acts non-trivially on A, $U(J) \subseteq A$. If $B \neq 1$, J acts non-trivially on B and if $A \neq B$, J acts non-trivially on A/B. Since $A \subseteq C_G(x)$, AJ has a unique non-central 2-chief factor so either $B = 1$ or $A = B$. We argue that the former equality holds, so assume to the contrary $A = B$. This means x acts freely on each V_i and so is conjugate in $\langle V_i, x\rangle$ to each involution in $V_i x$; by a Frattini argument there is (a 2 element) $t \in C_K(x)$ with $t \notin V_1$. As before, however, $J \subseteq [J, t] \subseteq L$ contrary to $n > 1$. Indeed, a similar argument shows $C_{K_i}(x) \subseteq V_i$ so in summary we have

(4.7) $U(J) \subseteq V$ and $C_{K_i}(x) = V_i$, $1 \leq i \leq n$.

The latter equality means $[K_i, x] \subseteq K_i \cap C_G(V) = O_2(K_i)$ and so:

(4.8) x centralizes K_i/V_i, $1 \leq i \leq n$.

Now let $H = N_J(K)$, so $|J:H| = n$. By the argument of (4.6) applied with B_1 any non-zero $\mathbf{F}_2 H$-submodule of V_1 and $A_1 = V_1$ and A_i, B_i J-conjugates of these for $i > 1$, one easily sees we must again have $A_1 = B_1$, that is,

(4.9) H acts irreducibly on V_1 (and non-trivially since $|V_1| > 2$).

Let $M = KH$, $\overline{M} = M/C_M(V_1)$, $U = V_1$, $U^* = V_1\langle x\rangle$ and $\overline{E} = C_{\overline{M}}(\overline{K})$.

(4.10) $\overline{E} = \overline{1}$.

For since $\overline{E}$ commutes with the irreducible action of $\overline{K}$ on U, $\overline{E}$ is cyclic of odd order. Suppose $\overline{E} = \langle\bar{y}\rangle \neq \overline{1}$, whence $U = [U, \bar{y}]$. But then $U^* = [U^*, \bar{y}] \times C_{U^*}(\bar{y})$ and since $\bar{y}$ centralizes U^*/U, $|C_{U^*}(\bar{y})| = 2$. Since $\langle\bar{y}\rangle \trianglelefteq \overline{M}$, $\overline{H}$ fixes $C_{U^*}(\bar{y})$. However, H has a unique non-trivial fixed point on U^*, namely x, hence $\langle x\rangle = C_{U^*}(\bar{y})$. But then $[x, K] = 1$, contrary to (4.7).

Now let P be a minimal normal subgroup of $\overline{H}$ chosen, if possible, within $\overline{K}$. By (4.9), $[U, P] = U$ and so $C_{U^*}(P) = \langle x\rangle$. Since $C_K(x) \subseteq U$ and P fixes x, $\overline{P} \not\subseteq \overline{K}$ and so we must have $\overline{H} \cap \overline{K} = \overline{1}$. Since therefore $\overline{H}$ induces outer automorphisms on $\overline{K}$, by assumption $\overline{H}$ is solvable so $\overline{P}$ is a p-group for some odd prime p. If $p \mid |\overline{K}|$, let $\overline{Q} = C_{\overline{K}}(\overline{P}) \neq \overline{1}$, whereas if $p \nmid |\overline{K}|$, let $\overline{Q}$ be a Sylow q-subgroup of $\overline{K}$ normalized by $\overline{P}$, for some odd prime q dividing $|\overline{K}|$. In any case,

$$U^* = [U^*, \overline{Q}] \times C_{U^*}(\overline{Q}),$$

both factors admit $\overline{P}$, and $[U^*, \overline{Q}] \subseteq U$. Since $\overline{P}$ has a unique non-trivial fixed point, x, on U^* and $x \notin U$, we must have $x \in C_{U^*}(\overline{Q})$. This again contradicts (4.7) and so completes the proof of Theorem B.

V. The proof of Theorem C

Throughout this section G, J, M satisfy the hypotheses of Theorem C and set

$$D = \langle J^M\rangle = J_1 Y J_2 Y \cdots Y J_n \quad \text{with } J = J_1.$$

Proceeding by way of contradiction assume $M \neq G$ and $J \ntrianglelefteq M$; thus $O_2(G) = 1$, $n > 1$, and for any 2-element t of $M^{\#}$ centralizing J_i, for some i, $C_G(t) \subseteq M$. Also, since $m(J) > 1$, $O(G) \subseteq M$ and so $[O(G), D] = 1$.

The proof proceeds in a sequence of lemmas, the first of which explores the action of tightly embedded subgroups on blocks.

LEMMA 5.1. *Let K be a block, $1 \neq T$ a 2-group acting on K with $T \cap K = 1$, $T \in Syl_2(P)$ where $P \subseteq TK$ and P is tightly embedded in TK. Let*

$$T \subseteq S \in Syl_2(TK) \quad \text{with } N_S(T) \in Syl_2(N_{TK}(T)),$$

and assume $|T| \geq |N_S(T): T|$. Then $[T, K] = 1$.

Proof. Note that if $[T, K] \subseteq O(K)$, then since $O(K) \subseteq Z(K)$, $[T, K, K] = 1$ whence $[T, K] = 1$; thus we may assume $O(K) = 1$. The hypotheses of the lemma are set up so that Theorem 3 of [3] applies directly. Let W be the weak closure of T in S with respect to K so we conclude $W \trianglelefteq TK$ and one of the following holds: (a) $W = T$; (b) $W = T \times T^x = N_S(T)$, for some $x \in K$ and $W' = 1$ (note that since $|S/W| > 2$, conclusion 5 of Theorem 3 is impossible). Since (a) is the assertion of the lemma, assume by way of contradiction that (b) holds and let $V = U(K)$. Since $W \subseteq O_2(TK)$, $1 \neq [W, K] \subseteq O_2(K)$, so $V \subseteq W$ and hence V is elementary abelian. Let

$$\overline{TK} = TK/W = TK/C_{TK}(W).$$

For any involution $\bar{x}$ in $\overline{K}$, $W = T \times T^{\bar{x}}$ so $W \cap K = [W, \bar{x}] \subseteq V$, whence $V = [W, \bar{x}] = C_W(\bar{x})$. Since $\bar{x}$ centralizes V, $\bar{x} \in O_2(\overline{K}) \subseteq Z(\overline{K})$. Let Q be a subgroup of K of odd prime order, so $[Q, x] \subseteq V$. Thus $W = [W, Q] \times C_W(Q)$, both factors admit $\bar{x}$ and $1 \neq [W, Q] \subseteq V$. But $\bar{x}$ centralizes $[W, Q]$ and

$$\dim_{\mathbf{F}_2} C_W(Q) \cap C_W(\bar{x}) \leq \tfrac{1}{2} \dim_{\mathbf{F}_2} C_W(Q)$$

so $\bar{x}$ cannot act freely on W. This contradicts a previous remark and so establishes the lemma.

LEMMA 5.2. *J is not tightly embedded in M and, in particular, $O_2(Z(J)) \neq 1$.*

Proof. Since $n > 1$ and $M = \mathcal{M}(J_i)$, $1 \leq i \leq n$, J is tightly embedded in M if and only if J is tightly embedded in G. Assuming this to be so, suppose $g \in G - N_G(J)$ and $J^g \cap N_G(J)$ has even order. Let $T \in Syl_2(J^g \cap N_G(J))$, $P = T(J^g \cap J)$ so $T \in Syl_2(P)$ and P is tightly embedded in TJ. From the symmetry between J and J^g it follows that $|T| = |N_{TJ}(T): T|_2$ so the hypotheses of Lemma 5.1 are satisfied. The conclusion gives $[J, J^g] = 1$. Since g was arbitrary in $G - N_G(J)$, Theorem 1 of [4] asserts that either $J \trianglelefteq \trianglelefteq G$, $\langle J^G \rangle$ has a strongly embedded subgroup, or J has abelian Sylow 2-subgroups, all of which are impossible. This proves J is not tightly embedded in G or M and since for $i > 1$, $J \cap J_i \subseteq Z(J)$, $|Z(J)|$ is even, as claimed.

For the remainder of this section let $Z = \Omega_1(O_2(Z(D)))$.

LEMMA 5.3. *If W is a fourgroup in M such that for some $g \in G - M$, $C_H(w) \subseteq M^g$, for all $w \in W^{\#}$, then W normalizes each J_i.*

Proof. Suppose the lemma is false so that with suitable renumbering $J_1^x = J_2$, for some $x \in W$. By the proof of Lemma 2.8 of [3], since $J_1 = O^2(J_1) \nsubseteq M^g$, $N_W(J_1) \neq 1$. Let

$$\langle y \rangle = N_W(J), \quad K = C_{J_1J_2}(x)'.$$

By Theorem A, $C_J(y) \nsubseteq Z(J)$ so $1 \neq [C_J(y), K] \subseteq M^g$. Again $J \nsubseteq M^g$ whence $U(J) = [C_J(y), K]$. Since $U(J)$ acts on Z^g we may pick $z \in Z^{g\#}$ with z centralizing $\langle U(J), x \rangle$, so $z \in C_G(U(J)) \subseteq M$, and $J_i^z = J_i$, $i = 1, 2$. Let $U = [z, K]$, so $U \subseteq Z^g$. Notice also that $[z, J] \subseteq C_J(U(J))$ so if $U(J)$ is non-abelian $C_J(U(J)) = Z(J)$ and it follows that $[z, J] = 1$; since $C_G(z) \subseteq M^g$, this is impossible, i.e. $U(J)' = 1$.

First assume $U \neq 1$ so since K is a block in $C_M(x)$, $U = U(K)$. Let $j \in J - O_2(J)$ with $j^2 \in O_2(J)$ and let $\bar{J} = J/C_J(U(J))$. Since $\bar{j}$ inverts an element of odd prime order in $\bar{J}$, $C_{U(J)}(j) \nsubseteq Z(J)$. By definition of K, $U(J) \subseteq U \cdot U(J_2)$ and $[j, U(J_2)] = 1$, so j must have a non-trivial fixed point a on U. Then $j \in C_G(a) \subseteq M^g$ so $J = [j, K] \subseteq M^g$, again a contradiction.

Thus $[K, z] = 1$, so for every $j \in J$, $1 = [jj^x, z] = [j, z]^{j^x}[j^x, z]$, so

$$[j, z] \in J \cap J^x \subseteq Z(J).$$

It follows that $J \subseteq C_G(z) \subseteq M^g$, the final contradiction.

LEMMA 5.4. $n = 2$.

Proof. Assume to the contrary $n \geq 3$.

Suppose first that for all $g \in G - M$, $|J^g \cap M|$ is odd. Then for x an involution in J, $x^g \in M \Leftrightarrow g \in M$; also if $x^m \in x^G \cap C_G(x)$ and $y = xx^m \neq 1$, then since $n \geq 3$ and $x^m \in J_i$, for some i, y centralizes J_j, $j \in \{2, 3, \ldots, n\} - \{i\}$, so $C_G(y) \subseteq M$. Suppose $y^g \in M \Leftrightarrow g \in M$, for any such product y; then by Theorem 3.3 of [3] (since $\langle J^G \rangle$ is perfect), $\langle x^G \rangle$ has a strongly embedded subgroup, which is easily seen to be false. Thus for suitable $y = xx^m$ and $g \in G - M$, $y \in M^g$. Since $n \geq 3$, y centralizes a fourgroup W of D^g such that $C_G(w) \subseteq M^g$, for all $w \in W^{\#}$. (This follows from Lemma 2.3 and Theorem A although it is easy to verify directly.) By Lemma 5.3, W normalizes J and so clearly $|J \cap M^g|_2 \neq 1$, contrary to assumption.

Pick $g \in G - M$ such that $J^g \cap M$ has even order, let T be a 2-group in $J^g \cap M$ of maximal order subject to $\langle J^T \rangle \neq D$ and let $T \subseteq T^* \in Syl_2(J^g \cap M)$. Note that $n \geq 3$ implies $T \neq 1$. Let Q be a T-invariant Sylow 2-subgroup of $\langle J^T \rangle \cap M^g$. Since $n \geq 3$, $m(Q) \geq 2$ (again, use Lemma 2.3 and Theorem A or direct argument), so by Lemma 5.3 applied to g^{-1}, $\Omega_1(Q)$ normalizes J^g. Thus

$m(T^*) \geq 2$ by Lemma 2.3(c) and so $\Omega_1(T^*)$ normalizes J. Finally, by Theorem A, $T^* \nsubseteq Z(J^g)$, $Q \cap J \nsubseteq Z(J)$.

Let $R = N_Q(T)$; we show $R = Q$. Note that as $Q \cap T = 1$, $TR = T \times R$. If $x \in N_Q(RT) - R$, then $T \subset TT^x \subseteq RT$, so $T_0 = TT^x \cap R \neq 1$. Since $x \in M^g$ and $n \geq 3$, T_0 centralizes J_j^g, for some j, which is incompatible with $T_0 \subseteq Q$ centralizing J_i, for some i. Thus $R = Q$ and since $Q \cap J \nsubseteq Z(J)$, $\langle J^T \rangle = J$. By maximality, $T^* = T$. Let $P = T(J^g \cap J)$. From the symmetry between J and J^g we may assume

$$|T| \geq |N_{TJ}(T): T|_2.$$

By Lemma 5.1, P is not tightly embedded in TJ so there exists $x \in J - N_J(P)$ with $|T \cap T^x|$ even; note that $x \in M^g$. If $T = T^x$, since $T \nsubseteq Z(J^g)$, x would normalize J^g, hence also P, a contradiction. Thus $T^x \subseteq J^{gx} = J_i^g$, for some $i \geq 2$ and so $\langle T, T^x \rangle = TT^x$ is a 2-group properly containing T. By orders, $T_0 = TT^x \cap J \neq 1$, a contradiction as before. This completes the proof of the lemma.

LEMMA 5.5. *There exists $h \in G - M$ such that $J^h \cap N_G(J)$ contains a fourgroup.*

Proof. Let $A_i \in Syl_2(J_i)$, $A_1A_2 \subseteq S \in Syl_2(G)$ with $S \cap M \in Syl_2(M)$. Note that $A_1 \cup A_2$ is strongly closed in $S \cap M$ with respect to M, $m(A_i) > 1$, A_i is neither dihedral nor quasidihedral, $M \neq G$ and $\langle A_i^G \rangle$ does not have a strongly embedded subgroup. By Lemma 3.4 of [3] therefore, there exists $a \in A_i$, $g \in G$ such that

$$a^g \in N_S(A^i) - (A_1 \cup A_2).$$

In fact, if b^g is the involution in $\langle a^g \rangle$, $b^g \notin A_1 \cup A_2$ else $g \in M$, which is false. We may therefore assume $|a| = 2$. Now a^g normalizes J and $C_G(a^g) \subseteq M^g$ so if $m(C_J(a^g)) \geq 2$, the lemma is true for $h = g^{-1}$ by virtue of Lemma 5.3. If, however, $T \in Syl_2(C_J(a^g))$ and $m(T) = 1$, by Lemma 2.3(b), $|T| \geq 8$, whence $|T \cap N_G(J^g)| \geq 4$. This same lemma now shows $m(\Gamma_{1,T \cap N(J^g)}(J^g)) \geq 2$, so $m(J^g \cap M) \geq 2$. Again Lemma 5.3 establishes this lemma for $h = g$.

LEMMA 5.6. *$U(J)$ is non-abelian.*

Proof. Suppose to the contrary $U(J)$ is abelian and put $V = U(J)$ so V is elementary abelian. Note that by Lemma 5.3 if $J^g \cap M$ contains a fourgroup, then every involution in $J^g \cap M$ normalizes J. Over all $g \in G - M$ such that $J^g \cap M$ contains a fourgroup pick g to maximize $|J^g \cap M|_2$. Let

$$T \in Syl_2(J^g \cap N_G(J)), \quad T \subseteq S \in Syl_2(TJ)$$

with $N_S(T) \in Syl_2(N_{TJ}(T))$, so $N_S(T) = T \times Q$, $Q = N_S(T) \cap J$. Now since $m(V) \geq 3$ by Lemma 2.3b, $m(Q) \geq 2$ so $\Omega_1(Q) \subseteq N(J^g)$. By Theorem A each involution in Q centralizes an involution in $J^g - Z(J^g)$, so $T \nsubseteq Z(J^g)$. Since

$Q \subseteq M^g$ centralizes T, Q normalizes J^g. Since we could replace g by g^{-1}, the maximality of $|T|$ forces $|Q| \leq |T|$. Let $P = T(J^g \cap J)$ so, by Lemma 5.1, P is not tightly embedded in TJ. Let $x \in J$ be such that $P \neq P^x$ and $1 \neq T \cap T^x$. Since $x \in M^g$ and $T \nsubseteq Z(J^g)$, TT^x is a 2-group $\neq T$. Finally, since $x^2 \in N(J^g)$ we may assume $x \in S$ and so x normalizes TQ.

Now suppose T centralizes V. Then $T \subseteq O_2(TJ)$ and so $[T, J] \subseteq V$. If P is any odd order subgroup of J, $[TV, P] \subseteq V$; moreover, as $Z \neq 1$,

$$|[TV, P]| = |[V, P]| < |VC_{Z \cap J}(T)| \leq |Q| \leq |T|,$$

and so $T \cap C_{TV}(P) \neq 1$. Thus $J = O^2(J) \subseteq M^g$, a contradiction.

Thus $T \ntrianglelefteq TV$ so there exists $v \in V$ with $T \neq T^v \subseteq N_{TV}(T)$. Since $v^2 = 1$ it follows from Lemma 5.3 that $T \cap T^v = 1$; therefore $TT^v = T \times T^v$ and $T \cong [T, v] \subseteq V$ so T is elementary abelian. Since $|Q| \leq |T|$, $TQ = T \times T^v$. Let $W = TQ$.

Suppose W is weakly closed in S with respect to J. Set $\overline{TJ} = TJ/V$ so $\overline{W} = \overline{T}$ is weakly closed in $\overline{S}$ with respect to $\overline{J}$ and $\overline{T} \cap \overline{J} = \overline{1}$. By Lemma 4.2 of [3], $[\overline{T}, \overline{J}] \subseteq O(\overline{J})$ so $[\overline{T}, \overline{J}] = \overline{1}$. Since $\overline{T}$ commutes with the irreducible action of $\overline{J}$ on $V/V \cap Z$, $[T, V] \subseteq Z$. Since $V = [V, J]$ and $\mathrm{Hom}_{\mathbf{F}_2J}(V/V \cap Z, Z) = 0$, $[T, V] = 1$, contrary to a previous argument. Thus there exists $y \in J$ such that $W^y \subseteq N_S(W)$, $W^y \neq W$. Without loss of generality, $T^y \nsubseteq W$.

First suppose for all $u \in T^{y\#}$, $TT^u = T \times T^u$. In this situation, for each $t \in T^{\#}$, the map $u \mapsto [u, t]$ is a bijection of T^y with Q. Since $T \neq T^x$, there exists $t \in T$ with $tt^x \in Q^{\#}$; by the preceding remark there exists $u \in T^y$ with $tt^u = tt^x$. But $xu \in C_G(t) \subseteq M^g$ and since $x \in M^g$, $u \in M^g$, contrary to $T^u \notin \{T, T^x\}$.

Let $u \in T^{y\#}$ with $T^u \in \{T, T^x\}$. Note that $x^2 \in N_J(J^g) \cap S = Q$ so for all $t \in T^{\#}$, $C_S(t) \subseteq TQ\langle x \rangle$. If $T^u = T$, then $u \in N_S(T) = TQ$. Write $u = tq$, $t \in T^{\#}$, $q \in Q$. Since u centralizes Q, $Q \subseteq M^{gy}$ whence Lemma 5.3 implies $Q \subseteq N_G(J^{gy})$. It follows that $[Q, T^y] = 1$ and so T^y centralizes t. Then $T^y \subseteq M^g$ and as $m(T^y) > 1$, T^y normalizes J^g, contrary to $T^y \nsubseteq W$. Thus $T^u = T^x$, that is, u is an involution in M^g interchanging J_1^g and J_2^g with $C_G(u) \subseteq M^{gy}$. Let $K = C_{J_1^g J_2^g}(u)'$, so by Lemma 2.3(a),

$$\{tt^u \mid t \in T\} = Q_0 \subseteq K;$$

moreover, since $Q \subseteq V$, by symmetry $T \subseteq U(J^g)$ and hence this lemma shows $Q_0 \subseteq U(K)$. Clearly $Q_0 \subseteq Q$ as well. Now u centralizes some involution $z \in Z^g$, so $z \in M^{gy}$ and $[z, K] = 1$. We show $K = (D^g \cap M^{gy})^{(\infty)}$: for otherwise we must have $U(J^g) \subseteq M^{gy}$; by Lemma 5.3, $U(J^g)$ would normalize $U(J^{gy})$ hence $\Gamma_{1,U(J^g)}(U(J^{gy}))$ would be a subgroup of $J^{gy} \cap M^g$ containing a fourgroup whereas $u \in (J^{gy} \cap M^g) - N(J^g)$, violating Lemma 5.3. This proves $K \trianglelefteq C_{M^{gy}}(z)$. Let

$$X = U(J_1^{gy}) \cdot U(J_2^{gy})$$

and argue that $U(K)$ centralizes X: for if not, since X admits $\langle K, z \rangle$ and $K = O^2(K)$, by the $P \times Q$-lemma, K acts non-trivially on $C_X(z)$; but then

$U(K) \subseteq X$ and as $X' = 1$, $[U(K), X] = 1$ contrary to assumption. Thus, in particular, Q_0 centralizes X, so $X \subseteq M$. Since $M \neq M^{gy}$ (as $y \in M$, $g \notin M$) and $U(J^{gy}) \subseteq M$, Lemma 5.3 yields $U(J^{gy}) \subseteq N_G(J)$. By the maximality of $|T|$, $|U(J^{gy})| = |V| \leq |T|$ so it follows that T centralizes V, contrary to a previous argument. This completes the proof of the lemma.

For the remainder of this section let $g \in G - M$ be such that $T \in Syl_2(J^g \cap N_G(J))$ with $m(T) \geq 2$. Let $T \subseteq S \in Syl_2(TJ)$ with $N_S(T) \in Syl_2(N_{TJ}(T))$, $Q_0 = N_S(T) \cap J$ and $V = U(J)$.

LEMMA 5.7. *T is abelian.*

Proof. Since $[T, J] \neq 1$, by Lemma 4.2 of [3], there exists $x \in J$ such that

$$T^x \subseteq S \quad \text{and} \quad [T, T^x] \subseteq T \cap T^x,\ T \neq T^x.$$

If $x \in M^g$, $x \notin N_G(J^g)$ so $T^x \subseteq J_2^g$ and $[T, T^x] = 1$; if $x \notin M^g$, $T \cap T^x = 1$ so again $[T, T^x] = 1$. Now $T^x \subseteq TQ_0 = T \times Q_0$ so as $T^x \cap Q_0 = 1$, $TQ_0 = T^xQ_0$. Since both T^x and Q_0 centralize T, $T \subseteq Z(TQ_0)$, as desired.

Note that Lemma 5.7 implies $V \not\subseteq M^g$, for otherwise as $m(V \cap N_G(J^g)) \geq 2$ and T was arbitrary, $V \cap N_G(J^g)$ would be an abelian subgroup of V of index ≤ 2, which is impossible.

LEMMA 5.8. *There exists $v \in V$ with $\Omega_1(T)\Omega_1(T^v) = \Omega_1(T) \times \Omega_1(T^v) = \Omega_1(TQ_0)$.*

Proof. Let $U = \Omega_1(T)$. Since as noted $V \not\subseteq M^g$, $[U, V] \neq 1$. Thus there exists $x \in V$ with $U^x \neq U$ and $[U, U^x] \subseteq U \cap U^x$. If $U \cap U^x = 1$, take $v = x$; otherwise $x \in M^g - N(J^g)$. In the latter case since J^g is not tightly embedded in M^g, $U \cap U^x \neq 1$, so $|UU^x| < |U|^2$. Thus if $UU^x \leq UV$, then $V \subseteq M^g$ which we have already seen to be impossible. Pick $v \in V$ with U^v normalizing UU^x, $U^v \not\subseteq UU^x$; hence $U^v \cap U = 1$ and $U^v \subseteq M^g$. By Lemma 5.3, U^v normalizes J^g, hence normalizes $J^g \cap N_S(J) = T$. Since $T \subseteq Z(TQ_0)$, U^v centralizes U, as claimed.

To establish the remaining equality let $Q_1 = [U, v] \cong U$ so $Q_1 \subseteq Q_0$ and, by Lemma 5.3, $Q_1 \subseteq N_G(J^g)$ whence $m(J \cap N_G(J^g)) \geq m(J^g \cap N_G(J))$. However, g was arbitrary so we may apply these arguments to g^{-1} and $Q_1^{g^{-1}} \subseteq J^{g^{-1}} \cap N_G(J)$ to get

$$m(J^g \cap N_G(J)) \geq m(J \cap N_G(J^g)),$$

whence (via Lemma 5.3) $Q_1 = \Omega_1(Q_0 \cap N_G(J^g)) = \Omega_1(Q_0)$, as desired.

LEMMA 5.9. *Let $U = \Omega_1(T)$, $W = \Omega_1(TQ_0)$. If $y \in J$ with U^y normalizing W, then $U^y \subseteq W$ or for all $u \in U^{y\#}$, $UU^u = U \times U^u$.*

Proof. Without loss of generality $U^y \subseteq S$. Assume $U^y \not\subseteq W$ and let $Q_1 = \Omega_1(Q_0)$.

First, suppose there exists $u \in U^y \cap N_S(T)^\#$. Then $u \in \Omega_1(TQ_0) = W$ and so $W \subseteq C_G(u) \subseteq M^{gy}$. Since $gy \notin M$, by Lemma 5.3 (applied to M^h, $h^{-1} = gy$), $Q_1 \subseteq N_G(J^{gy})$. Since U^y normalizes $W \cap J = Q_1$, $[U^y, Q_1] \subseteq Q_1 \cap J^{gy} = 1$. Write $u = aq$, $a \in U^\#$, $q \in Q_1$, whence U^y centralizes a. Thus $U^y \subseteq M^g$ and Lemma 5.3 gives $U^y \subseteq N_G(J^g)$, so U^y normalizes $J^g \cap N_S(J) = T$, contrary to $U^y \nsubseteq \Omega_1(N_S(T))$. This proves $U^y \cap N_S(T) = 1$.

Assuming the lemma to be false, let $u \in U^y$ with $1 \neq U \cap U^u$ and with $u \in M^g - N_G(J^g)$. Let $K = C_{J_1^g J_2^g}(u)'$, so $K/Z(K) \cong J/Z(J)$ and

$$K \subseteq C_G(u) \subseteq M^{gy}.$$

Also, M, M^g, M^{gy} are distinct conjugates of M. As in the proof of Lemma 5.6, if u centralizes a fourgroup, Z_0^g, in Z^g, then Z_0^g acts on Z^{gy} and $\Gamma_{1,Z_0^g}(Z^{gy}) = Z^*$ has rank ≥ 2; so $m(Z^*\langle u \rangle) > 1$, $Z^*\langle u \rangle \subseteq M^g$ but $u \notin N_G(J^g)$, contrary to Lemma 5.3 (applied with suitable change of coordinates). Thus $C_{Z^g}(u) = \langle z \rangle$. Similarly, $U(J^g) \nsubseteq M^{gy}$ (use the remark preceding Lemma 5.8) so

$$K = (D^g \cap M^{gy})^{(\infty)} \trianglelefteq C_{M^{gy}}(z).$$

Let $X = U(J_1^{gy})U(J_2^{gy})$.

We next prove $U(K)'$ centralizes X: this is clear if $[K, X] = 1$; if $[K, X] \neq 1$, by the $P \times Q$ lemma, K acts non-trivially on $C_X(z)$, from which it follows that $U(K) \subseteq X$ and the claim is true by virtue of $X' \subseteq Z(X)$. Now $U(K)' \subseteq Z(D^g)$, however, as noted after Lemma 5.7, $U(J^{gy}) \nsubseteq M^g$, so we must have $U(K)' = 1$. But for $a \in U(J^g)'$, by Lemma 2.3, $aa^u \in U(K)'$ whence $aa^u = 1$, i.e. $[U(J^g)', u] = 1$. Since $\langle z \rangle = C_{Z^g}(u)$, $V \cong EYZ(V)$, $E \cong \text{Ex sp } 2^m$.

Now $m(U) \geq 2$ so there exists $a \in U$ such that $b = aa^u \neq 1$. Note that $Q_1 \subseteq V$ and $m(Q_1) = m(U)$ so by Lemma 5.8 applied to g^{-1} we may assert $U \subseteq V^g$. Thus $b \in U(K)$ and also $b = [a, u] \in Q_1$. Again by the $P \times Q$ lemma either $U(K) \subseteq X$ or $[U(K), X] = 1$, so in either case b induces inner automorphisms on $U(J^{gy})$. Let $E_0 = C_{E^{gy}}(b)$, so $E_0 \subseteq M$. By Lemma 5.7, $E_1 = E_0 \cap N_G(J)$ is abelian, hence E has an abelian subgroup E_1 of index ≤ 4. This forces $E \cong \text{Ex sp } 2^5$. Moreover, by Lemma 5.3, $\Omega_1(C_V\, gy(b))$ is abelian and so $E = V$, $E \cong Q_8 Y D_8$, $E_0 \cong Z_2 \times Q_8$, $E_1 \cong Z_2 \times Z_4$ and b induces an automorphism of E corresponding to an involution in E. Furthermore, $J/O_2(J) \cong A_5$, $\bar{U}(J)$ is the "permutation module" of dimension 4. But then for each involution $e \in E$, a Sylow 2-subgroup F of $C_J(e)$ has index 2 in a Sylow 2-subgroup of J containing it, F covers a Sylow 2-subgroup of $J/O_2(J)$ and so F has no abelian subgroup of index 2 (as V/V' is a free $\mathbf{F}_2(F/F \cap O_2(J))$-module). Since $b \in X$, however, b induces such an inner automorphism on J^{gy} and so Lemma 5.7 is violated for J^{gy} in place of J^g. This contradiction establishes Lemma 5.9.

For the remainder of the proof of Theorem C assume g is chosen subject to the above conditions with $|T|$ as large as possible. Let $Q = Q_0 \cap N_G(J^g)$ so

$$Q \in Syl_2(J \cap N_G(J^g))$$

and by maximality of $|T|$, $|Q| \leq |T|$. Let U, W be as in Lemma 5.9.

LEMMA 5.10. (a) $|Q| = |T|$.
(b) *There exists* $x \in S \cap M^g$ *with* $x \notin Q$.
(c) *There exists* $d \in J$ *with* $U^d \subseteq N_S(W)$, $U^d \nsubseteq W$.
(d) *With* x *as in* (b), $[U, x] = 1$.

Proof. We first prove (c). Suppose to the contrary W is weakly closed in TJ and set $\overline{TJ} = TJ/O_2(J)$, so $\overline{U}$ is weakly closed in $\overline{UJ}$ and $\overline{U} \cap \overline{J} = \overline{1}$. By Lemma 4.2 of [3], $[\overline{U}, \overline{J}] = \overline{1}$, whence $[U, J] \subseteq V$. Since U commutes with the irreducible action of $\overline{J}$ on $V/Z(V)$, $[U, V] \subseteq Z(V)$. Since

$$V = [V, J] \quad \text{and} \quad \mathrm{Hom}_{\mathbf{F}_2 J}(V/Z(V), Z(V)/V') = 0,$$

we have $[U, V] \subseteq V'$. Finally, since $[U, V] \subseteq Z(V)$, $[U, V'] = 1$. Now V' is elementary abelian, so $V' \subseteq \Omega_1(Q_0)$, whence $m(U) \geq m(V')$. If, however, $m(U) > m(V')$, since each $w \in V$ induces a homomorphism

$$[\ \ , w]: U \to V',$$

we would have $V \subseteq \Gamma_{1,U}(V) \subseteq M^g$, a contradiction as usual. Thus $m(U) = m(V')$ and so $W = U \times V'$. But then UV'/V' is weakly closed in UJ/V', so by Lemma 4.2 of [3], $[U, J] \subseteq V' \subseteq Z(J)$, whence $[U, J] = 1$, a contradiction. This proves (c).

To prove (a) suppose $|Q| < |T|$, let $Q^* = S \cap J \cap M^g$ and let U^d be as given by (c). For $u \in U^{d\#}$, by Lemma 5.9, $T \cap T^u = 1$. However, T^u centralizes U^u and $W \cap J$ and so centralizes $U \subseteq U^u \times (W \cap J) = W$, that is, $T^u \subseteq TQ^* = S \cap M^g$. Since $T \cap T^u = 1$, $|Q^*| \leq |T|$ and $[T, u] \subseteq Q^*$, we must have $Q^* = \{[t, u] \mid t \in T\}$ and so Q^* is an abelian group inverted by u and $Q^* \cong T$. Since u was arbitrary and $m(U^d) \geq 2$, Q^* is elementary. But then by Lemma 5.3 (and symmetry), $Q^* \subseteq Q$, a contradiction.

In part (b), if no such x exists it follows that $T(J^g \cap J)$ is tightly embedded in TJ and Lemma 5.1 is violated in view of $[T, J] \neq 1$.

Finally, to prove (d) let x be as in (b) and assume there exists $t \in U^\#$ with $[x, t] \neq 1$. Since $\Omega_1(Q_0) \cong U \cong U^d$, by Lemma 5.9 there exists $u \in U^{d\#}$ with $tt^x = tt^u$, where U^d is as given by part (c). Then $xu \in C_G(t) \subseteq M^g$ and so $u \in M^g$. Since J^g is not tightly embedded in M^g, $U \cap U^u \neq 1$, contrary to Lemma 5.9.

We are now in a position to complete the proof of Theorem C. Notice that by part (a) of Lemma 5.10 we are entitled to continue to apply results for T, J to Q, J^g (using g^{-1} in place of g). In particular, since the element x described in (b) normalizes TQ, TQ contains a Sylow 2-subgroup of $D^g \cap N_G(J)$. By this symmetry TQ also contains a Sylow 2-subgroup of $D \cap N_G(J^g)$, whence

$$TQ \in Syl_2((D \cap N_G(J^g))(D^g \cap N_G(J))),$$

$$TQ\langle x\rangle \in Syl_2((D \cap M^g)(D^g \cap N_G(J)).$$

Let $TQ\langle x\rangle \subseteq A \in Syl_2((D \cap M^g)(D^g \cap M))$, so $|A: TQ\langle x\rangle| \leq 2$. By Lemma

5.10(b) (applied to Q, J^g), $A \not\subseteq N_G(J)$ and there exists $s \in N_{D^g}(J)$ with s normalizing TQ, $s^2 \in TQ$, so $A = TQ\langle s, x\rangle$. Moreover, s centralizes $Q_1 = \Omega_1(Q)$ by Lemma 5.10(d), so $Q_1 \subseteq Z(D)$ and hence $[Q_1, x] = 1$. Similarly, $[U, s] = 1$, and so $W \subseteq Z(A)$.

Note that $C_G(W) \subseteq M \cap M^g$. Let U^d be as described by Lemma 5.10(c), so $U^d \subseteq AD$. Since $A \in Syl_2(C_{AD}(W))$ we may pick $d_1 \in D$ such that

$$U^{d_1} \subseteq N_G(W) \cap N_G(A) \quad \text{with } U^{d_1} \not\subseteq W$$

and for all $u \in U^{d_1^\#}$, $[U, u] = Q_1$. By Lemma 5.10(c) applied to Q (since $A \in Syl_2(C_{AD^g}(W))$) there exists $d_2 \in D^g$ such that $Q_1^{d_2} \subseteq N_G(A) \cap N_G(W)$ and for all $v \in Q_1^{d_2}$, $[Q_1, v] = U = \Omega_1(T)$. Let

$$N = \langle A, U^{d_1}, Q_1^{d_2}\rangle, \quad \tilde{N} = N/C_N(W)$$

and note that N is transitive on $W^\#$ so $O_2(\tilde{N}) = \tilde{1}$. It follows from Theorem 2 of [18] that $\tilde{N} \cong L_2(q)$, $q = |U|$ and W is the natural module for $\tilde{N}$.

First note that A is non-abelian, for $T \neq T^x$ and $\langle T, x\rangle \subseteq A$.

Next recall that $U^d \subseteq TJ$ so U^d normalizes $A \cap TJ = TQ\langle x\rangle$. If for all $u \in U^{d\#}$, $T^u \subseteq TQ$, then $Q = [T, u]$ is abelian and inverted by each $u \in U^{d\#}$. Since $m(U) \geq 2$, Q is elementary and hence so is $T \cong [T, u]$. Thus $TQ = W$ is a central subgroup of A of index 4 and so A' is cyclic. Since $\tilde{N}$ is transitive on $W^\#$, $A' = 1$, a contradiction. This proves there exists $u \in U^{d\#}$ such that $(TQ)^u \subseteq TQ\langle x\rangle$ and $(TQ)^u \neq TQ$. Symmetrically (or because we could now choose g to be an involution) there exists $v \in N$ such that

$$(TQ)^v \subseteq TQ\langle s\rangle \quad \text{and} \quad (TQ)^v \neq TQ.$$

Thus

$$|TQ: (TQ)^u \cap (TQ)^v| \leq 4 \quad \text{and} \quad A = \langle TQ, (TQ)^u, (TQ)^v\rangle,$$

whereupon as TQ is abelian $|A: Z(A)| \leq 16$.

Next we decide $Z(A) \subseteq TQ$. Suppose this is not the case and let $z \in Z(A) - TQ$. Since $A' \neq 1$, $|A: Z(A)TQ| = 2$ so either $z \notin TQ\langle x\rangle$ or $z \notin TQ\langle s\rangle$. Without loss of generality $z \notin TQ\langle x\rangle$, so $A = \langle TQ, x, z\rangle$. Since $Q\langle x\rangle = TQ\langle x\rangle \cap J \trianglelefteq TQ\langle x\rangle$, it follows that $(Q\langle x\rangle)^s = Q\langle x\rangle$, whereas $J \neq J^s$ and, by Theorem A, $Q\langle x\rangle \not\subseteq Z(J)$, a contradiction.

Now $\tilde{N}$ acts on $A/Z(A)$ and if $\tilde{N}$ centralizes $A/Z(A)$, then $\tilde{N}$ normalizes TQ; but then U^{d_1} normalizes TQ and a previous argument leads to a contradiction. Since $\tilde{N} \cong L_2(q)$, $q \geq 4$, the only possibility is $A/Z(A) \cong E_{16} \cong W$. But W is the natural $\mathbf{F}_2 L_2(4)$-module and the map $(a_1, a_2) \to [a_1, a_2]$ induces a non-trivial $\mathbf{F}_2\tilde{N}$-module homomorphism from $(A/Z(A)) \otimes_{\mathbf{F}_2} (A/Z(A))$ to W, whereas for either of the two possible module structures for $A/Z(A)$ no such homomorphism exists (see, for example, Lemma 2.2 of [26]). This contradiction completes the proof of Theorem C.

VI. The proofs of Theorems D, E, F and G

We first study the following setup:

(6a) G is a finite group, S a 2-subgroup of G, L a product of components of $C_G(S)$;
(6b) V is a faithful $\mathbf{F}_2 G$-module;
(6c) as an $\mathbf{F}_2 L$-module V has a unique non-trivial irreducible composition factor.

Under these hypotheses, for every subgroup H of G let $\tilde{V}(H) = [V, H]/C_{[V,H]}(H)$, so $\tilde{V}(L)$ is a non-trivial irreducible $\mathbf{F}_2 L$-module.

LEMMA 6.1. *If $s \in S$, L centralizes $[V, s]$ and $V/C_V(s)$.*

Proof. By induction on $|s|$, L centralizes $[V, s^2]$. Let $\overline{V} = V/[V, s^2]$ and so s induces an automorphism of order 1 or 2 on $\overline{V}$. The map $\overline{V} \to \overline{V}$ by

$$\bar{v} \to [\bar{v}, s]$$

is an $\mathbf{F}_2 L$-module homomorphism and so $\overline{V}/C_{\overline{V}}(s)$ and $[\overline{V}, s]$ are isomorphic $\mathbf{F}_2 L$-modules. Since $[\overline{V}, s] \subseteq C_{\overline{V}}(s)$, property (6c) forces them to be trivial $\mathbf{F}_2 L$-modules. Thus

$$[[V, s], L] \subseteq [V, s^2]$$

so since $[[V, s], L, L] = 1$, the 3 subgroups lemma forces $[[V, s], L] = 1$.

Similarly, by induction, $[V, L] \subseteq C_V(s^2)$ so the above argument applied to $C_V(s^2)$ in place of $\overline{V}$ gives $[V, L] \subseteq C_V(s)$, as claimed.

LEMMA 6.2. *If SL normalizes an odd order subgroup K of G, then L centralizes K.*

Proof. By induction a minimal counterexample G satisfies $G = SLK$, K (being solvable) is either an elementary abelian or special p-group of exponent p, for some prime p, SL acts irreducibly on $K/\phi(K)$ and $[L, \phi(K)] = 1$.

Let $V_0 = [V, K]$, $\overline{G} = G/C_G(V_0)$. Note that if S centralizes V_0, since G is faithful on V and $[S, K]$ centralizes V_0 and $C_V(K)$, $[S, K] = 1$; but then $[L, K] = 1$ as $L \trianglelefteq C_G(S)$, a contradiction. This proves $\overline{S} \neq \overline{1}$. Since K is faithful on V_0, if J is a component of $C_L(V_0)$, then, by (6c), $[L, V_0] = 1$, so $\overline{K} = [\overline{K}, \overline{L}] \subseteq C_G(V_0)$, a contradiction. Thus $C_G(V_0) \subseteq SZ(L)$ so $\overline{L} \trianglelefteq C_{\overline{G}}(\overline{S})$ and $\overline{L}$ acts non-trivially on $\overline{K}$. By minimality of G, $C_G(V_0) = 1$ and we may assume $V = V_0$.

Now let s be an involution in $Z(S)$. By the irreducible action of SL on $K/\phi(K)$, s either inverts or centralizes $K/\phi(K)$. Assume the latter happens so that s centralizes K and so $s \in Z(G)$. But then $[V, s]$ is a non-trivial $\mathbf{F}_2 G$-submodule which, by Lemma 6.1, is centralized by L (hence also by K), contrary to $V = [V, K]$. Thus s inverts $K/\phi(K)$, so $C_G(s) \subseteq SL\phi(K) \subseteq N_G(L)$. By minimality of G, $S = \langle s \rangle$.

If K is abelian, since s inverts K, s acts freely on V i.e. $[V, s] = C_V(s)$ and it follows from Lemma 6.1 and the 3 subgroups lemma that L centralizes V, a contradiction. Thus K is special and since s inverts $K/\phi(K)$, s centralizes $\phi(K)$.

Let $D = \phi(K)$ and argue that $|D| = p$, $V = [V, D]$. For let V_0 be an irreducible $\mathbf{F}_2 G$-submodule of $[V, D]$, $D_0 = C_D(V_0)$ and $V_1 = C_V(D_0) \cap [V, D]$. Thus $|D: D_0| = p$, $V_1 = [V_1, D/D_0]$ and since $D_0 \subseteq Z(G)$, V_1 admits G. If $\overline{G} = G/C_G(V_1)$, then $\overline{S} \cong S$, $\overline{K} \cong K/D_0$ and $\overline{L}$ is a central quotient of L, whence $\overline{G}$, V_1 is also a counterexample. Thus $\overline{G} = G$ and we may assume $V = V_1$, as desired.

Now if e is an element of K of order p inverted by s, argue that s centralizes $C_V(e)$. For otherwise $C_V(e) \cap [V, s] \neq 0$ and so by Lemma 6.1,

$$C_V(e) \cap C_V(L) \neq 0;$$

but since $e \in K - \phi(K)$ and L acts irreducibly on $K/\phi(K)$, $K = \langle e^L \rangle$, contrary to $V = [V, K]$. In particular, if $|K| > p^3$, since K is extra-special of exponent p, there exists $E \cong Z_p \times Z_p$, with $E \subseteq K$ and E inverted by s; since E acts faithfully on V it follows easily by Schur's lemma that for some $e \in E^{\#}$, s does not centralize $C_V(e)$, contrary to the previous argument. This reduces to the case $|K| = p^3$ and so $L = L_0 L_1$, $L_1 = C_L(K)$, $L_0 \cong SL_2(p)$, $p > 3$.

Let $E \subseteq K$ with $E \cong Z_p \times Z_p$, let $\mathscr{E}_1(E) = \{E_1, E_2, \ldots, E_p, D\}$ with E_1 inverted by s, let $V_i = C_V(E_i)$, $1 \leq i \leq p$ and let $W = [V, E_1]$, so $V = V_1 \oplus W$. Since K acts transitively by conjugation on $\{E_1, \ldots, E_p\}$, $\dim_{\mathbf{F}_2} V_i = \dim_{\mathbf{F}_2} V_1$, $2 \leq i \leq p$. Because D is fixed point free on V, $V_i \cap V_j = 0$, $i \neq j$, and since each V_i admits E_1, $V_i \subseteq W$, $2 \leq i \leq p$. Thus since $V_2 \oplus V_3 \subseteq W$, $\dim W > \dim V_1$, whence $\dim W > \frac{1}{2} \dim V$. Now let q be a prime divisor of $|L_0|$ with $q \neq 2, p$ and let x be an element of L_0 of order q. As noted earlier s centralizes V_1, so

$$[V, s] = [W, s] \quad \text{and} \quad V/C_V(s) \cong W/C_W(s);$$

moreover, s inverts E_1 so $C_W(s) = [W, s]$. By Lemma 6.1, x centralizes $[V, s]$ and $V/C_V(s)$ so it follows that $\dim C_V(x) \geq \dim W > \frac{1}{2} \dim V$. Since x is not a scalar transformation on $K/\phi(K)$, there exists $k \in K - \phi(K)$ with $\langle k, k^x \rangle$ covering $K/\phi(K)$, whence $K \subseteq \langle x, xk \rangle$. Because $\langle K, x \rangle/\phi(K)$ is a Frobenius group and $xk\phi(K)$ is not in the Frobenius kernel, $|xk\phi(K)| = q$. Thus if x_1 is an element of order q in the coset $xk\phi(K)$, $K \subseteq \langle x, x_1 \rangle$. Moreover, $\langle x \rangle$ is conjugate in $K\langle x \rangle$ to $\langle x_1 \rangle$ so by dimension counting $\langle x, x_1 \rangle$ has a non-zero fixed point on V, contrary to $C_V(K) = 0$. This completes the proof of the lemma.

We now list some additional hypotheses we will be working under:

(6d) V is an irreducible $\mathbf{F}_2 G$-module;

(6e) $G = \langle L^{E(G)} \rangle S$.

Note that by the ordinary L-balance theorem for components, Theorem 3.1 of [23], and by Lemma 6.2, $G = E(G) \cdot S$.

LEMMA 6.3. *If H_0 is a subgroup of G containing LS, $H = \langle L^{E(H_0)} \rangle S$, and W is a non-trivial irreducible $\mathbf{F}_2 H$-constituent of V, let $\overline{H} = H/C_H(W)$; then $(\overline{H}, \overline{L}, \overline{S}, W)$ satisfy* (6a)–(6e) *in place of* (G, L, S, V) *and $E(\overline{H})$ is isomorphic to a central quotient of $E(H)$.*

Proof. Clearly only (6a) needs verifying to confirm the first assertion. Again by the L-balance theorem $L \subseteq E(H)$ so if $K_1, \ldots, K_n$ are the components of $E(H)$,

$$K_1 \cdots K_n = \langle L^{E(H)} \rangle = \langle L^{E(H)S} \rangle.$$

If $C_H(W) \cap E(H) \nsubseteq Z(E(H))$, there exists i such that $K_i \subseteq C_H(W)$; but then $\langle K_i^H \rangle$ contains some component J of L so as $J \subseteq C_H(W)$, by (6c), $L \subseteq C_H(W)$, whence

$$E(H) = \langle L^H \rangle \subseteq C_H(W),$$

contrary to W being a non-trivial $\mathbf{F}_2 H$-constituent. This proves $E(\overline{H})$ is a central quotient of $E(H)$. It remains to show $\overline{L}$ is subnormal in $C_{\overline{H}}(S)$, for which it suffices to show $L \trianglelefteq \trianglelefteq N_H(SC_H(W))$. But

$$[C_H(W), L] \subseteq Z(E(H))$$

so as $[C_H(W), L, L] = 1$, $[C_H(W), L] = 1$, hence $L \trianglelefteq \trianglelefteq C_H(S \cdot C_H(W)) \subseteq C_H(S)$, as needed.

LEMMA 6.4. *Let D be a semisimple subgroup of G. Assume $C_G(D)$ is tightly embedded in G with $N_G(C_G(D)) = N_G(D)$ and for all $g \in G - N_G(D)$, $[D, D^g] \nsubseteq D \cap D^g$. For any involution $x \in C_G(D)$ assume D centralizes $[V, x]$ and that $D \trianglelefteq C_G(v)$, for $v \in [V, x] - \{0\}$. Let z_1, z_2 be involutions in $C_G(D)$, $h \in G$; then the following hold*:

(1) *if $\langle z_1, z_2^h \rangle$ is a 2-group, either $z_2^h \in N_G(D)$ or $z_1 \in N_G(D^h)$;*

(2) *if $\langle z_1, z_2^h \rangle \cong D_{4k}$, k odd > 1, $z_2^h \in C_G(D)$.*

Proof. To prove (1) suppose $\langle z_1, z_2^h \rangle$ is a 2-group $z_2^h \notin N_G(D)$, and $|\langle z_1, z_2^h \rangle|$ is minimal subject to these conditions. Set $a = z_1, b = z_2^h, t = ab$, so $t \notin N_G(D)$ but $t^2 \in N_G(D)$; moreover, as $C_G(a) \subseteq N_G(D)$, $|t| > 2$. Let $U = C_V(t^2)$; since $\langle t \rangle$ acts faithfully on V, by looking at this representation of t in Jordan canonical form one sees that t acts non-trivially on U, hence one of a, b does also. Since $\langle a, b \rangle^{h^{-1}}$ is also a minimal counterexample, we may replace a, b by $a^{h^{-1}}, b^{h^{-1}}$ if necessary to assume a acts non-trivially on U. Then $[U, a] = [U, at^2] = [U, a^t]$, so for $v \in [U, a] - \{0\}$, $D, D^t \trianglelefteq C_G(v)$. By hypothesis therefore $D = D^t$, a contradiction.

To prove (2) suppose

$$\langle z_1, z_2^h \rangle \cong D_{4k}, \quad k \text{ odd} > 1$$

and let $a = z_1$, $b = z_2^h$, $\langle x\rangle = O(\langle a, b\rangle)$, and let $\langle t\rangle = Z(\langle a, b\rangle)$. Let

$$U = C_V(t) \cap [V, x]$$

so $U \neq 0$ by the $P \times Q$ lemma and $\langle a, b\rangle$ acts on U. Since x acts Frobeniusly on U and a inverts x, $[U, a] \neq 0$. Thus $[U, a] = [U, at] \neq 0$ and $at = b^{x_1} = z_2^{hx_1}$, for some $x_1 \in \langle x\rangle$. As before, $hx_1 \in N_G(D)$, so $t \in C_G(D)$. Thus

$$\langle a, b\rangle \subseteq C_G(t) \subseteq N_G(D),$$

so $x_1 \in N_G(D)$ and therefore $h \in N_G(D)$, as desired.

THEOREM 6.5. *Assume* (6a)–(6e) *hold and also that* $L/Z(L) \cong \Omega_4^+(2^n)$ *and* $\tilde{V}(L)$ *is the natural* 4*-dimensional module for* $L/Z(L)$ *viewed as a module over* $\mathbf{F}_2$*; then* $G = L$.

Proof. Note that by (6b) and (6d), $O_2(G) = 1$ so (6e) implies

$$L = G \Leftrightarrow L \trianglelefteq\trianglelefteq G \Leftrightarrow S = 1.$$

Assume G is a minimal counterexample and let $V_0 = [V, L]$, $V_1 = C_V(L)$. Since

$$\Omega_4^+(2^n) \cong L_2(2^n) \times L_2(2^n)$$

let L_1, L_2 be the components of L, so $Z(L)$ is a 2-group. By Lemma 2.7, $V = V_0 \oplus V_1$ where $V_0 \cong \tilde{V}(L)$ as $\mathbf{F}_2 L/Z(L)$-modules. Since $Z(L)$ centralizes V_0 and V_1, by (6b), $Z(L) = 1$ and so $L = L_1 \times L_2$. Also, since S centralizes V_0, $V_0 \neq V$ so $N_G(V_i) \subset G$, $i = 0, 1$.

Let s be an involution in $Z(S)$, $H_0 = C_G(s)$, $H = \langle L^{E(H_0)}\rangle S$. Since $H \subset G$, by minimality of G, Lemma 6.3 forces $L = H \trianglelefteq\trianglelefteq H_0$, so by the arbitrary nature of S we may assume $S = \langle s\rangle$. Let $S \subseteq S^* \in Syl_2(C_G(L))$, s^* an involution in $Z(S^*)$. The same argument shows $L \trianglelefteq\trianglelefteq C_G(s^*)$, whence we may assume $s = s^*$. Now applying this argument to any involution t in S^* we obtain $L \trianglelefteq\trianglelefteq C_G(t)$, so $L \trianglelefteq\trianglelefteq C_G(t_1)$, for all involutions $t_1 \in C_G(L)$. Finally, this argument shows that if H is any proper subgroup of G containing L with $|C_H(L)|$ even, then $L \trianglelefteq\trianglelefteq H$. In particular, $L \trianglelefteq\trianglelefteq N_G(V_1)$ and $L \trianglelefteq N_G(V_0)$.

Next suppose L^g normalizes L, for some $g \in G$; we prove either $L = L^g$ or $[L, L^g] = 1$. Notice that V_0^g is the unique non-trivial irreducible constituent of L^g on V^g and L^g acts on V_0, V_1, so either $V_0^g \subseteq V_0$ or $V_0^g \subseteq V_1$. In the latter case L^g must centralize V_0 so since L, L^g commute in their action on V, by (6b), $[L, L^g] = 1$. If $V_0^g \subseteq V_0$, $V_0^g = V_0$ and so as $L \trianglelefteq N_G(V_0)$, $L = L^g$. This establishes the initial claim of the paragraph.

Thus L acts like a single component so if $A_1, \ldots, A_r$ are a maximal set of pairwise commuting conjugates of L with $L = A_1$ and $D = A_1 \cdots A_r$, the argument of Theorem 9.7 of [3] verifies the hypotheses of Theorem 5 of [3]. Since $m(L) > 1$ and $O_2(G) = 1$ Theorem 5 of [3] gives that one of the following holds:

(1) $D = G$;

(2) $C_G(L)$ is tightly embedded in G with $N_G(C_G(L)) = N_G(L)$ and for all $g \in G - N_G(L)$, $[L, L^g] \not\subseteq L \cap L^g$.

If $D = G$, $L \trianglelefteq\trianglelefteq G$ which we have seen means G is not a counterexample; thus (2) holds and so Lemma 6.4 applies, via Lemma 6.1, to $D = L$.

If $C_G(L)$ has 2-rank 1, let z be an involution in $C_G(L)$. It follows from Lemma 6.4 that z^G is a class of odd transpositions in G, and, by (6e), we may assume

$$G = E(G)\langle z\rangle.$$

Since $C_G(z)$ has a "standard subgroup" of type $L_2(2^n) \times L_2(2^n)$ and $G/S(G)$ is described by the Main Theorem of [2], (and the components of the centralizers of the odd transpositions are described in this paper) the only candidate is

$$G/S(G) \cong O_5(5), \quad L/Z(L) \cong L_2(4) \times L_2(4).$$

However, since $O_2(G) = 1$ we would have $G \cong O_5(5)$ and since one easily sees that $O_5(5)$ contains no subgroup isomorphic to $L_2(4) \times L_2(4)$ we must have

$$L \cong SL_2(5)YSL_2(5)$$

contrary to $Z(L) = 1$. This argument proves $C_G(L)$ has 2-rank > 1.

It follows from Theorem 3 of [4] that Sylow 2-subgroups of $C_G(L)$ are not non-abelian dihedral groups nor are they weakly closed fourgroups. Thus by Theorems 2 and 3 of [3] there exists $g \in G - N_G(L)$ such that $C_G(L)^g \cap N_G(L)$ contains a fourgroup, W. Since

$$L \nsubseteq N_G(L^g) \quad \text{but} \quad \Gamma_{1,W}(L) \subseteq N_G(L^g),$$

by Lemma 2.8 of [3], W normalizes L_1 and L_2; moreover, the argument of Lemma 3.5 of [3] is easily modified to show that if some $w \in W$ induces an outer automorphism on L_i, for some i, then

$$L_i \subseteq \Gamma_{1,W}(L_i) \subseteq N_G(L^g).$$

Suppose say $L_1 \subseteq N_G(L^g)$. Since W centralizes an involution, a, in $C_G(L)$, L_1 is a component of $C_G(a) \cap N_G(L^g)$ so by the L-balance theorem $L_1 \subseteq L(N_G(L^g))$; more precisely, by Lemma 2.7(2) of [3] either

$$[L^g, L_1] = 1, \quad L_1 \in \{L_1^g, L_2^g\} \quad \text{or} \quad L_1^{ga} = L_2^g \text{ with } L_1 = C_{L_1^g L_2^g}(a).$$

In the first two instances $L^g \subseteq N_G(L_1) \subseteq N_G([V, L_1]) = N_G(V_0) = N_G(L)$, a contradiction. If L_1 lies on the diagonal of L^g, note that in fact W centralizes a fourgroup U in $C_G(L)$ which we may assume contains a, so again by Lemma 2.8 of [3],

$$L^g = \Gamma_{1,U}(L^g) \subseteq N_G(L),$$

a contradiction. Similarly $L_2 \nsubseteq N_G(L^g)$ and so each involution in W induces a non-trivial inner automorphism on each L_i.

Now L contains a diagonal subgroup, L_0, the centralizer of a transvection in $O_4^+(2^n)$, satisfying:

(i) $L_0 \cong L_2(2^n)$;
(ii) $C_{V_0}(L_0) \subseteq [V_0, L_0]$;

(iii) $\dim_{\mathbf{F}_2}[V_0, L_0] = 3n$, $\dim C_{V_0}(L_0) = n$;

(iv) V_0 is an indecomposable $\mathbf{F}_2 L_0$-module with $[V_0, L_0]/C_{V_0}(L_0)$ the standard $\mathbf{F}_2 L_2(2^n)$-module for L_0.

Since the diagonal involutions in L are all conjugate and each $w \in W^\#$ induces an inner automorphism on L corresponding to a diagonal involution we may replace $C_G(L)^g$ by an L-conjugate so that for some $w \in W^\#$, $w \in L_0 C_G(L)$. Since all non-trivial odd order elements of L_1 act Frobeniusly on V_0 and w inverts one of these,

$$\dim [V_0, w] = 2n,$$

and, of course, $[V_0, w] \subseteq [V_0, L_0]$. Thus it follows from (iii) and (iv) that

$$C_{V_0}(L_0) \subseteq [V_0, w].$$

Let $v \in C_{V_0}(L_0) - \{0\}$. By Lemma 6.2, $L^g\langle w\rangle \subseteq C_G(v)$ so as previously noted

$$C_G(v) \subseteq N_G(L^g).$$

However, $C_L(w)$ contains a Sylow 2-subgroup T of L, whence $\langle L_0, T\rangle \subseteq N_G(L^g)$. One easily checks that $L = \langle L_0, T\rangle$ (see, for example, Lemma 2.5 (3) of [3]) so a previous result (applied to g^{-1}) gives $L = L^g$ or $[L, L^g] = 1$. Both equalities are impossible and this contradiction completes the proof.

THEOREM 6.6. *Assume that* (6a)–(6e) *hold and that L is quasisimple. One of the following holds:*

(1) $G = L$;

(2) $E(G) \cong A_{n+2k}$, $L \cong A_n$, *V is the non-trivial irreducible constituent of the natural $(n + 2k)$-dimensional permutation module for $E(G)$ over $\mathbf{F}_2$, $n \geq 5$;*

(3) $E(G) \cong \Omega^{\pm}_{2n+2}(2^m)$, $L \cong Sp_{2n}(2^m)'$, *V is the natural $(2n + 2)$-dimensional $\mathbf{F}_{2^m}E(G)$-module viewed as a module over $\mathbf{F}_2$, $n \geq 1$, $m \geq 1$;*

(4) $E(G) \cong Z_3 \cdot U_4(3)$, $L \cong U_4(2)$, $\dim_{\mathbf{F}_2} V = 12$, $\dim_{\mathbf{F}_2} \tilde{V}(L) = 8$.

Proof. Let G be a minimal counterexample. Since $O_2(G) = 1$ and $G \neq L$, $S \neq 1$. Also for all $v \in V - \{0\}$, $C_G(v) \subset G$ and for all involutions $t \in G$, $C_G(t) \subset G$. Notice that Lemmas 6.1–6.4 apply to arbitrary L, S which satisfy (6a)–(6e) for we will have occasion to change both L and S in the proof.

(6.6.1) We may assume $|S| = 2$.

To prove this let s be an involution in $Z(S)$, $H_0 = C_G(s)$, $H = \langle L^{E(H_0)}\rangle S$ and W a non-trivial irreducible $\mathbf{F}_2 H$-constituent of V. By Lemma 6.3 and the minimality of G,

$$E(H)/Z(E(H)) \cong L/Z(L),\ A_{n+2k},\ \Omega^{\pm}_{2n+2}(2^m) \text{ or } Z_3 \cdot U_4(3)$$

and in all but the first instance we may identify $L/Z(L)$ and W as well. In any case since $L \subseteq E(H)$, V has a unique non-trivial irreducible $\mathbf{F}_2 E(H)$-constituent so without loss of generality $W = \tilde{V}(E(H))$. One easily checks that $G_0 = \langle E(H)^G \rangle \langle s \rangle$, $L_0 = E(H)$, $S_0 = \langle s \rangle$, $V_0 = V$ satisfy (6a)–(6e) in place of G, L, S, V resp. If L_0 is not quasisimple, $L_0/Z(L_0) \cong \Omega_4^+(2^m)$ and $\tilde{V}(L_0)$ is the natural module. By Theorem 6.5, $G_0 = L_0$ whence $s \in O_2(G_0)$, contrary to $O_2(G_0) = 1$. Thus L_0 is quasisimple and the hypotheses of this theorem are satisfied by G_0, L_0, S_0, V_0 with $L_0 \neq G_0$ and $E(G) = E(G_0)$. Suppose one of the conclusions of Theorem 6.6 holds for the new quadruple. In this situation, if $L_0 = L$, G is not a counterexample so we must have

$$L_0/Z(L_0) \cong A_{n+2k},\ \Omega_{2n+2}^{\pm}(2^m) \text{ or } Z_3 \cdot U_4(3).$$

Since L_0 must be one of the groups described in conclusions (2)–(4), the only possibilities are $L_0 \cong A_{n+2k}$ or $U_4(2)$ ($\cong \Omega_6^-(2)$) (note that $\Omega_6^+(2) = A_8$ and the 6-dimensional $\mathbf{F}_2$-modules are the same for these groups). If $L_0 \cong \Omega_6^-(2)$, we previously identified $W = \tilde{V}(L_0)$ as the natural 6-dimensional module over $\mathbf{F}_2$ whereas conclusion (4) of Theorem 6.5 asserts that if L_0 has this isomorphism type, the constituent $\tilde{V}(L_0)$ must have dimension 8, a contradiction. If $L_0 \cong A_{n+2k}$, $n + 2k \geq 7$ and $E(G_0) \cong A_{n+2k'}$, V is the non-trivial irreducible constituent of the natural permutation module, whence it follows that L is an alternating group and G is not a counterexample. This argument proves that G_0, L_0, S_0, V_0 do not satisfy the conclusions of the theorem so without loss of generality $G = G_0$, $L = L_0$, $S = \langle s \rangle$ as claimed in (6.6.7).

By a similar argument we get the following two results:

(6.6.2) L is maximal in the component ordering of [3];
(6.6.3) L is a component of $C_G(t)$, for all involutions $t \in C_G(L)$.

By using (6.6.3) and the fact $G = E(G)\langle s \rangle$ we may replace s by another involution in $C_G(L)$ and decide via the L-balance theorem (Lemma 2.7 (3) of [3]) that

(6.6.4) if $|C_G(L)|_2 > 2$, G is quasisimple.

Now let $A_1, \ldots, A_r$ be a maximal set of pairwise commuting conjugates of L with $L = A_1$ and let $D = A_1 \cdots A_r$. The proof of Theorem 9.7 of [3] shows that the hypotheses of Theorem 5 of [3] are satisfied so since $O_2(G) = 1$, one of the following holds:

(1) $D \trianglelefteq G$;
(2) $D = A_1 A_2$, $m(A_1) = 1$, $|A_1 \cap A_2|$ is even, $[L, L^g] = 1 \Leftrightarrow L^g = A_2$, for all $g \in G$, and $C_G(D)$ is tightly embedded in G with $N_G(C_G(D)) = N_G(D)$;
(3) $D = A_1$, $[L, L^g] \neq 1$, for all $g \in G$, and $C_G(D)$ is tightly embedded in G with $N_G(C_G(D)) = N_G(D)$.

Let $N = N_G(D)$ and $C = O^{2'}(C_G(D))$.

If (1) holds, $L \trianglelefteq \trianglelefteq G$ which forces $L = G$, a contradiction; thus (2) or (3) holds. Note that if (2) holds, since A_1 and A_2 are conjugate in G and $x \in G$ with

$A_1^x = A_2$, then $x \in C_G(O_2(D)) \subseteq N_G(D)$; moreover, in this case, by Theorem 3.1 of [15], $|C|_2 = 2$. (The B-conjecture is not needed for the proof of Theorem 3.1 of [15]).

We first handle the situation when $m(C) = 1$. Let z be an involution in C and for each element y of G, let $V_y = [V, y]$. By Lemma 6.1, L centralizes V_z and in case (2) if $A_1^x = A_2$, x centralizes $\langle z \rangle = O_2(A_1) = O_2(A_2)$ so D centralizes V_z in this instance as well. Suppose for all $v \in V_z - \{0\}$, $D \trianglelefteq C_G(v)$: it follows from Lemma 6.4 that z^G is a class of odd transpositions in G whence Lemma 2.10 and (6e) assert that G is not a counterexample. This proves there exists $v \in V_z - \{0\}$ such that $D \ntrianglelefteq C_G(v)$, so by (2), (3), $L \trianglelefteq\trianglelefteq C_G(v)$. Let

$$H_0 = C_G(v), \quad H_1 = \langle L^{E(H_0)} \rangle \langle z \rangle, \quad W = \tilde{V}(H_1),$$

and note that as $L \trianglelefteq\trianglelefteq H_1$, $O_2(H_1) = 1$ and as $L \subseteq H_1$, W is a non-trivial irreducible $\mathbf{F}_2 H_1$-module. Let $\overline{H}_1 = H_1/C_{H_1}(W)$ so by Lemma 6.3 and the minimality of G we may identify $\overline{H}_1$, $\overline{L}$ and W; furthermore, the odd order group $C_{H_1}(W)$ stabilizes the chain

$$V \supseteq [V, H_1] \supseteq C_{[V, H_1]}(H_1) \supseteq 0$$

whence centralizes V, so $C_{H_1}(W) = 1$. Let $H = E(H_1)$ so (we now know $m(L) > 1$) since L is in standard form in H_1, one of the following holds:

(i) $L \cong A_n$, $H \cong A_{n+2}$, $H\langle z \rangle \cong \sum_{n+2}$;
(ii) $L \cong Sp_{2n}(2^m)$, $H \cong \Omega_{2n+2}^{\pm}(2^m)$, $H\langle z \rangle \cong O_{2n+2}^{\pm}(2^m)$;
(iii) $L \cong U_4(2)$, $H \cong Z_3 \cdot U_4(3)$;

and in all cases z^H is a class of odd transpositions in H_1 and $\tilde{V}(H)$ is the natural module for H (described by conclusions (2)–(4) of this theorem). We may therefore always pick $g \in H_1 - N_G(L)$ such that $[z, z^g] = 1$ and $H = \langle L, L^g \rangle$. Note that

$$v \in V_z \cap V_{z^g} \neq 0.$$

We now include discussion which circumvents using the full weight of the solutions to the various standard form problems we are faced with—this seems desirable not only for reasons of independence but also to avoid invoking the Unbalance Theorem on which some of these solutions rest.

Case $L \cong A_n$, $H \cong A_{n+2}$, $n \geq 5$. Let

$$\langle z, z^g \rangle \subseteq T \in Syl_2(N_G(H));$$

since L is in standard form, $C_G(H)$ has odd order so T is isomorphic to a Sylow 2-subgroup of $\sum_{n+2}$. Let $\mathscr{T} = z^H \cap T$, $U = V_z \cap V_{z^g}$. Since $N_{H\langle z \rangle}(\mathscr{T})$ is doubly transitive on $\mathscr{T}$ and $H\langle z \rangle$ centralizes U,

$$U = \bigcap_{a \in \mathscr{T}} V_a, \; N_G(U) \subseteq N_G(H) \quad \text{and} \quad T \in Syl_2(N_G(\mathscr{T})).$$

Let $\langle z, z^g \rangle \subseteq P \in Syl_2(N)$, $S = P \cap C$; we prove $S = \langle z \rangle$. Choose a permutation representation of $H\langle z \rangle$ so that $z = (12)$; since H is doubly transitive on $\mathscr{T}$, without loss of generality $z^g = (34)$. Assume $|S| > 2$ so as S centralizes $L \subseteq H$,

$$S \nsubseteq N_G(H) = N_G(U);$$

therefore z^g does not centralize S. Since $m(S) = 1$, there exists $s \in S$ such that

$$z^{gs} = zz^g = (12)(34).$$

Now let $h \in G$ be chosen with $z^h = (34)(56) \in L \cap P$, so z^h centralizes S. If $n \geq 8$, let

$$B = O^2(C_L(z^h)) \cong A_{n-4}$$

so $B \subseteq N^h$; moreover, since $m(C) = 1$, $SB \cap C^h = 1$ so S acts faithfully on $L^hC^h/C^h \cong A_n$ and centralizes its subgroup $BC^h/C^h \cong A_{n-4}$. Thus if $n \geq 8$, $S = \langle s \rangle$ and s induces a 4-cycle on L^hC^h/C^h, hence also on L^h. Note that if $n = 5$, 6, or 7, since $m(S) = 1$ and S acts faithfully on L^h, $S = \langle s \rangle \cong Z_4$ and s is either a 4-cycle or the product of a 4-cycle and a transposition on L^h. In any case, let S_0 be a Sylow 2-subgroup of C^h normalized by S. Thus $S_0 \subseteq C_G(z) \subseteq N$, so $[S, S_0] = 1$. By the action of z^g on S, no 2-element of $C_G(S)$ induces an outer automorphism on L, whence in N^h, S cannot induce a 4-cycle on L^h. The only remaining possibilities are $n = 6$ or 7 and s the product of a 4-cycle and a transposition on L^h. Let $P_1 = S\langle z^g \rangle$, $P_2 = P \cap L$ so

$$P = P_1 \times P_2 \cong D_8 \times D_8, \quad Z(P)^{\#} = \{z, z^h, zz^h\},$$

so by orders $P \in Syl_2(C_G(z^h))$ as well. It follows therefore that if $P \subseteq P^*$ with $|P^*: P| = 2$, then $P = J(P^*)$, $\langle z, z^h \rangle$ char P^* and hence

$$P^* \in Syl_2(G).$$

Since $H\langle z^g \rangle \cong \Sigma_8$ or Σ_9 (i.e. $|T| = 2^7$), such P^* exists so by orders $T \in Syl_2(G)$. Since $T \cap H \subseteq G'$ and $z^G \cap H \neq \phi$, G is perfect whence G is quasisimple with Sylow 2-subgroups of type A_{10}. By [21], however, no involution centralizer has a component of type A_6 or A_7 centralized by a Z_4 subgroup. This contradiction proves $S = \langle z \rangle$. Note that as $L\langle z, z^g \rangle$ contains a Sylow 2-subgroup of N, z is not rooted in G.

Next we prove $T \in Syl_2(G)$; for otherwise let $T \subseteq T^*$ with $|T^*: T| = 2$ and let $t \in T^* - T$. By the initial paragraph of this case there exists $y \in H$ such that $z^y \in \mathscr{T}$ but $z^{yt} \notin \mathscr{T}$. Since every involution in H is rooted in $H\langle z^g \rangle$, $z^{yt} \notin H$. Now Lemma 2.5 asserts $z \not\sim_G z^{yt}$, a contradiction.

Again, since z is not rooted in G, $z^G \cap H = \phi$. By Thompson's transfer lemma [5.38 of 27], $z \notin G'$ and since $H \subseteq G'$, $T_0 = T \cap H \in Syl_2(G')$. If $n \leq 9$, [21] applied to G' and the fact that L is standard in G gives $G = H\langle z \rangle$, again a contradiction. Thus we may assume $n > 9$. Let $u = zz^g$, $K = C_G(u)$, so $(K \cap H)^{(\infty)} = B \cong A_{n-2}$ and $\tilde{V}(B)$ is the natural module for B. Since B is a component of $C_K(z)$ which has property (6c), as usual by Lemma 6.3 and induction there exists L^* a component of K with $L^* \cong A_{n-2+k}$ and $\tilde{V}(L^*)$ the natural module. But now $\langle L^*, u\rangle \subseteq G' \subset G$ and V is an irreducible $\mathbf{F}_2 G'$-module so by minimality of G, $G' \cong A_{n-2+k'}$. By inspection, since L is standard in G with $m(C_G(L)) = 1$, $G \cong \sum_{n+2}$, again contradicting $G \neq H\langle z\rangle$.

Case $L \cong L_2(2^m)$, $H \cong \Omega_4^-(2^m)$, $m \geq 2$. Note that $H \cong L_2(2^{2m})$ and z is a field automorphism of H, so all involutions of $H\langle z\rangle - H$ are conjugate under H to z. Also, $m(C) = 1$ implies $m(C_G(z)) = m + 1$ so as $m \geq 2$, $z^G \cap H = \phi$. Since

$$C_{H\langle z\rangle}(z) = \langle z\rangle \times L$$

it follows that if

$$z \in T \in Syl_2(N_G(H)),$$

then $N_{H\langle z\rangle}(\mathscr{T})$ is doubly transitive on $\mathscr{T} = z^H \cap T$, and as in the previous case, $T \in Syl_2(G)$. Let $T_0 = T \cap H$ so T/T_0 is cyclic. By Thompson's transfer Lemma, $z \notin G'$ so since $|G: G'| \leq 2$ and $H \subseteq G'$, $T_0 \in Syl_2(G')$. Since Sylow 2-subgroups of G' are elementary abelian and L is standard in G, $H\langle z\rangle = G$, a contradiction.

Case $L \cong L_2(2^m)$, $H \cong \Omega_4^+(2^m)$, $m \geq 2$. Let $H = H_1 \times H_2$, $H_i \cong L_2(2^m)$, $H_1^z = H_2$ and let $V_0 = [V, H]$, $V_1 = C_V(H)$, so, by lemma 2.7, $V = V_0 \oplus V_1$ and for each i, V_0 is the direct sum of two natural modules for H_i. For $L = C_H(z)$, V_0 is an indecomposable module with $\dim_{\mathbf{F}_2} [V_0, L] = 3m$, and $C_{V_0}(L) = [V_0, z]$ of dimension m. Let

$$\langle z, z^g\rangle \subseteq T \in Syl_2(N_G(H)), \quad E = T \cap H.$$

By Lemma 2.7, $T/E \cong Z_2 \times Z_{2^k}$ where $2^k | m$ and there exists an automorphism f_1 of H whose coset generates the second cyclic factor and with f_1 a field automorphism on H_i, $i = 1, 2$. Frattini's argument, since z acts freely on E, shows that $C_T(z)$ covers T/E so we may pick $f_2 \in C_T(z)$ with $f_2 \equiv f_1 (\text{mod } E)$; set $f = f_2^{2^{k-1}}$, if $k \geq 1$, and $f = 1$ otherwise. Note that if $f \neq 1$, f induces an outer involutory automorphism on each H_i, hence is a field automorphism on each H_i, so f acts freely on both E and V_0, and

$$C_{H\langle z\rangle}(f) \cong L_2(2^a) \wr Z_2, \quad 2a = m.$$

Since $m(C) = 1$, it follows easily that if $f \neq 1$, $z^G \cap fE = \phi$. Furthermore, since $m(C_G(z)) = m + 1$ and $m(E) = 2m$, $z^G \cap E = \phi$. Suppose $f \neq 1$ and $z^G \cap zfE \neq \phi$. Since zf interchanges H_1 and H_2, zf is conjugate in $\langle zf, E\rangle$ to every involution in zfE, whence there exists $h \in G$ such that $z^h = zf$. Since f induces a field automorphism on L and $[V_0, L]$ is the full cover of the natural $\mathbf{F}_2 L_2(2^m)$-module, f acts non-trivially on

$$[V_0, L] \cap C_{V_0}(L) = [V_0, z].$$

Thus there exists $w \in V_0 \cap V_z \cap V_{z^h}$ with $w \neq 0$. As usual, by induction applied in $C_G(w)$, there exists H^* a component of $C_G(w)$ with

$$H^* \cong \Omega_4^{\pm}(2^m) \quad \text{and} \quad H^* = \langle L, L^h \rangle$$

($H^* \ncong A_7$ since $\tilde{V}(L)$ is not the permutation module for A_5). However,

$$C_H(z^h) \cong L_2(2^m)$$

so as $C_G(z^h)$ has a unique component of this type, $C_H(z^h) = L^h$. This means

$$H^* = \langle L, L^h \rangle = H,$$

contrary to H not centralizing w. This argument proves $z^G \cap zfE = \phi$.

Again we argue $T \in Syl_2(G)$. Let $\mathcal{T} = z^H \cap T$ so by the preceding fusion arguments $z^g \in \mathcal{T}$. Since $N_H(\mathcal{T})$ is doubly transitive on $\mathcal{T}$, $V_z \cap V_{z^g} = \bigcap_{t \in \mathcal{T}} V_t$ and since

$$N_G(\mathcal{T}) \subseteq N_G(H) \quad \text{and} \quad z^G \cap T = \mathcal{T},$$

we have $T \in Syl_2(G)$. Moreover, by Thompson's transfer lemma applied to $\langle f_1, E \rangle$ the previous results on fusion also give $z \notin G'$. Note that $E \subseteq H \subseteq G'$.

If $f = 1$, we must have $E \in Syl_2(G')$ so G' is a product of Goldschmidt groups. Since L is standard in G it follows that $G = H\langle z \rangle$, a contradiction. It remains to consider the case $f \neq 1$. Let $T_0 = T \cap G'$ and note that $E \subseteq T_0$, T_0/E is cyclic and G' is perfect. If $|T_0 : E| = 2$, $T_0 = \langle E, f \rangle$ or $\langle E, zf \rangle$, whence in either case $T_0 \cong E_{2^m} \wr Z_2$, $m \geq 2$. By a result of Harada [the proof of Lemma 18 of 24] G' is not perfect, a contradiction. Thus $T_0/E \cong Z_{2^r}$, $r > 1$. Since f acts freely on E and $\langle fE \rangle = \Omega_1(T_0/E)$, $E = J(T_0)$ char T_0 and so $E \leq N_{G'}(T_0)$. Since T_0/E is cyclic,

$$T_0 \cap N_{G'}(T_0)' \subseteq E.$$

Also, for each Sylow 2-subgroup Q of G', Q' is elementary so $Q' \cap T_0 \subseteq \langle E, f \rangle$. By Grun's Theorem [7.4.2 of 20], $T_0 \cap (G')' \subseteq \langle f, E \rangle$ again contrary to G' being perfect. This completes the proof of the case.

Case $L \cong A_6 \cong Sp_4(2)'$, $L^* \cong \Omega_6^{\pm}(2)$. We have already considered when $H\langle z \rangle \cong O_6^+(2) \cong \Sigma_8$ (note that the corresponding modules are the same for the two isomorphism types), and when $H\langle z \rangle \cong O_6^-(2) \cong$ Weyl (E_6) the arguments are similar—we sketch the details.

Let $V_0 = [V, H]$, $V_1 = C_V(H)$ so by Lemma 2.7a, $V = V_0 \oplus V_1$; and since

$$\text{Aut}\,(\Omega_6^-(2)) = O_6^-(2),$$

if $\langle z, z^g \rangle \subseteq T \in Syl_2(H\langle z \rangle)$, then $T \in Syl_2(N_G(H))$. Moreover, H is doubly transitive on $\mathcal{T} = z^H \cap T$ so $T \in Syl_2(N_G(\mathcal{T}))$ as usual.

Let $\hat{V}_0 = Q_8 Y Q_8 Y Q_8$ be extraspecial so that Out $(\hat{V}_0) \cong O_6^-(2)$ and $\hat{V}_0/Z(\hat{V}_0)$ is the natural module for $O_6^-(2)$. From this representation it is easily deduced that $H\langle z\rangle$ has exactly 4 classes of involutions, and representatives z, t, t_1, t_2 have the properties: $t \in Hz$, $t_1, t_2 \in H$,

$$\dim_{\mathbf{F}_2}[V_0, z] = 1, \quad \dim_{\mathbf{F}_2}[V_0, t] = 3, \quad \dim_{\mathbf{F}_2}[V_0, t_i] = 2,\ i = 1, 2,$$

$$C_H(t_1) \cong (SL_2(3) \wr Z_2)/Z(SL_2(3) \wr Z_2),$$

$$C_H(t_2) \cong Z_2/(E_4 \times A_4), \quad C_H(z) \cong \Sigma_6$$

and every involution in $O^2(C_H(t_2))$ is H-conjugate to t_2.

Suppose $z^h \in H$, for some $h \in G$. By the structure of $C_H(t_1)$ $z \not\sim_G t_1$ so we may assume $z^h = t_2$. Let $\langle z, z^g\rangle \subseteq P \in Syl_2(N)$, $P_1 = P \cap C\langle z^g\rangle$, $P_2 = P \cap L$ and $\langle x\rangle = Z(P_2)$ so by the last remark of the preceding paragraph $x \in z^G$, whence

$$P \in Syl_2(C_G(x))$$

as well. Since $N_G(H) = N_G(V_z \cap V_{z^g})$ and z is not rooted in $N_G(H)$, $\langle z\rangle = C_{P_1}(z^g)$, so $P = P_1 \times P_2$ with P_1 dihedral or quasidihedral. By Sylow's Theorem x is conjugate to z in $N_G(P)$ so $P_1 \cong P_2 \cong D_8$ and, as in the $O_6^+(2)$ case, because

$$|G|_2 \geq |H\langle z\rangle|_2 = 2^7,$$

there exists $P^* \supseteq P$ with $|P^*: P| = 2$. It follows that $\langle z, x\rangle = Z(J(P^*))$ so $P^* \in Syl_2(G)$ and therefore $T \in Syl_2(G)$. Again $H \leq G'$ and $z^G \subseteq G'$ so G is quasisimple with Sylow 2-subgroups of type A_{10}. By [21], G cannot have an involution centralizer with a component of type A_6 centralized by a Z_4 subgroup. This contradiction proves $z^G \cap H = \phi$.

As noted earlier, $\dim_{\mathbf{F}_2}[V, z] = \dim_{\mathbf{F}_2}[V, t] - 2$, whence $z \not\sim_G t$: thus $z^G \cap T = \mathscr{T}$ so because $T \in Syl_2(N_G(\mathscr{T}))$, $T \in Syl_2(G)$. By Thompson's transfer lemma, $z \notin G'$ so

$$T \cap H \in Syl_2(G'),$$

that is, G' has Sylow 2-subgroups of type A_8. It follows from [21] that $G = H\langle z\rangle$, a contradiction.

Case $L \cong U_4(2)$, $H \cong Z_3 \cdot U_4(3)$. In this situation let $\langle x\rangle = Z(H)$, $V_0 = [V, x] = [V, H]$, $V_1 = C_V(x) = C_V(H)$, so $V = V_0 \oplus V_1$. By Lemma 6.1, z is not free on V_0 so $[z, x] = 1$. By Lemma 2.8, zH contains 2 classes of involutions with representatives z, u and

$$\dim_{\mathbf{F}_2}[V_0, z] = 2, \quad \dim_{\mathbf{F}_2}[V_0, u] = 6.$$

Because $[V_1, z] = [V_1, u]$, $z \not\sim_G u$. Let $z \in T \in Syl_2(N_G(H))$, $T_0 = T \cap H$ so by Lemma 2.8 and the fact that $|C_G(H)|$ is odd,

$$T/T_0 \cong Z_2 \text{ or } Z_2 \times Z_2 \quad \text{and} \quad T_0\langle z\rangle = C_T(x).$$

Suppose there exists $h \in G$ such that $z^h \in T - T_0\langle z\rangle$, so z^h inverts $\langle x\rangle$. Let $Q = C_H(z^h)$ so by Lemma 2.8,

$$\frac{\langle x\rangle Q}{\langle x\rangle} \cong U_3(3),\ \Sigma_6,\ U_4(2) \text{ or } SL_2(3) \wr Z_2/Z(SL_2(3) \wr Z_2).$$

In any case, since $C_G(L^h)$ has 2-rank 1 and $|N: LC_G(L)| \leq 2$ it follows that

$$Q_0 = Q \cap L^h \neq 1.$$

Note that since z centralizes $\tilde{V}(L)$ z centralizes $[V, L]$ whence z^h centralizes $[V, Q_0]$; but then z^{hx} centralizes $[V^x, Q_0^x] = [V, Q_0]$ and so

$$[V, Q_0] \subseteq V_0 \cap C_V(z^h) \cap C_V(z^{hx}) = 0,$$

contrary to $Q_0 \neq 1$. This argument proves $z^G \cap T \subseteq T_0\langle z\rangle$.

Finally, suppose $z^h \in T_0$, for some $h \in G$. Since H has one class of involutions we may assume $z^h \in Z(T_0\langle z\rangle)$. Since $T_0\langle z\rangle$ is isomorphic to a Sylow 2-subgroup of $E_{2^5}A_6$, where E_{2^5} is the permutation module modulo the one dimensional submodule, $z^h \in (T_0\langle z\rangle)''$. However, in a Sylow 2-subgroup P of $C_G(z)$, since $m(C) = 1$, $z \notin P''$, a contradiction.

For $\mathscr{T} = z^H \cap T$ H is doubly transitive on $\mathscr{T}$ (z^H is the class of reflections in $O_6^-(3)$) so as usual $T \in Syl_2(G)$ and for any subgroup T_1 of T with $T_0 \subseteq T_1$, $z \notin T_1$ and $|T: T_1| = 2$, by Thompson's transfer lemma applied to T_1, $z \notin G'$. Now let $T_1 \in Syl_2(G')$. Since $\dim_{\mathbf{F}_2}[V_0, z] = 2$, for any involution $a \in H$, a is a product of two H-conjugates of z so $\dim_{\mathbf{F}_2}[V, a] = \dim_{\mathbf{F}_2}[V_0, a] \leq 4$. On the other hand, if d is an involution in $T_1 - T_0$, d inverts x so $\dim_{\mathbf{F}_2}[V, d] \geq \frac{1}{2}\dim_{\mathbf{F}_2} V_0 = 6$. This proves $d^{G'} \cap H = \phi$ so since by Lemma 2.8 each coset of T_0 in T contains involutions, Thompson's transfer lemma applied to the perfect group G' forces $T_1 = T_0$. Because L is standard in G, [22] implies $G = H\langle z\rangle$, a contradiction.

Case $L \cong Sp_{2n}(2^m)$, $H \cong \Omega^{\pm}_{2n+2}(2^m)$, $n \geq 2$. By previous considerations we may also assume $L \ncong Sp_4(2)'$. Let $V_0 = [V, H]$, $V_1 = C_V(H)$ so by Lemma 2.7, $V = V_0 \oplus V_1$. Let w_0 be any non-singular vector in V_0 and let w be an H-conjugate of w_0 with $w \in [V_0, z]$, z being an $\mathbf{F}_{2^m}$orthogonal transvection on V_0. By Lemma 6.1,

$$C_G(w) \supseteq \langle L, z\rangle$$

whence as usual, by induction, there exists M a component of $C_G(w)$ with $L \subseteq M$ and either

$$M \cong \Omega^{\pm}_{2n+2}(2^m) \quad \text{or} \quad M = L.$$

If $M \neq L$, however, by the decomposition of V under $\Omega^{\pm}_{2n+2}(2^m)$ it follows that

$$w \notin [V, L] \subseteq [V, M],$$

whereas one easily sees that V_0 is an indecomposable $\mathbf{F}_2 L$-module with

$$C_{V_0}(L) \subseteq [V_0, L].$$

Thus $L \trianglelefteq \trianglelefteq C_G(w)$ and as L is in standard form, $C_G(w) \subseteq N$. This proves that for each non-singular vector w_0 in V_0, $C_G(w_0)$ has a unique component of type $Sp_{2n}(2^m)$, denoted by L_{w_0}, and $L_{w_0} \subseteq H$.

If u_0 is a non-zero singular vector in V_0, $C_H(u_0) = EK$ where $E \cong E_{2^{2nm}}$, $K \cong \Omega^{\pm}_{2n}(2^m)$ and E is the natural module for K. Since N does not contain a subgroup isomorphic to EK, $u_0 \not\sim_G w_0$.

Now let a be an element of H which is of type a_2 (in the sense of [10] page 16) so $C_{V_0}(a)$ has $\mathbf{F}_2$-codimension $2m$ in V_0 and so $C_V(a)$ also has $\mathbf{F}_2$-codimension $2m$ in V. Since $H \not\leq G' = \langle a^G \rangle$ we may pick a G-conjugate b of a with $V_0^b \neq V_0$. Since $\dim_{\mathbf{F}_2} V_0 = m(2n+2)$, $\dim_{\mathbf{F}_2} V_0 \cap V_0^b \geq 2nm$, from which it follows that $V_0 \cap V_0^b$ contains a vector w_0 which is nonsingular with respect to the form on V_0 (consider the corresponding $\mathbf{F}_2$-quadratic form). Thus for some $h \in H$, $C_G(w_0) \subseteq N^h$. Considering w_0 in the form on V_0^b, by the previous remarks w_0 is also non-singular with respect to this form and $L^h = L_{w_0} \subseteq H^b$. Thus $V_0 \cap V_0^b \supseteq [V, L^h]$. As argued before, $w \in [V, L^h]$ is non-singular in V_0 if and only if w is non-singular in V_0^b. One easily checks, however, that

$$H = \langle L_w | w \text{ is a non-singular vector in } [V, L^h] \rangle.$$

Thus $H^b = \langle L_w | w$ is non-singular in $[V, L^h] \rangle$ by this argument, contrary to $H \neq H^b$.

This completes the treatment of the various standard form problems which have arisen when $m(C) = 1$. This lengthy argument plus (6.6.4) gives:

(6.6.5) L is in standard form in G, $m(C) > 1$ and G is quasisimple.

Next suppose for some proper subgroup H of G with $L \subseteq H$ and $|C \cap H|$ even, $L \ntrianglelefteq H$, whence also $L \trianglelefteq/\!\!\trianglelefteq H$. By Lemma 6.3, and induction $L \cong A_n$, $Sp_{2n}(q)$, or $U_4(2)$, for some $q = 2^m$ and $L \subseteq L^* \subseteq H$ with

$$L^* \cong A_{n+k}, \ \Omega^{\pm}_{2n+2}(q) \text{ or } Z_3 \cdot U_4(3) \text{ resp.},$$

$\tilde{V}(L^*)$ the natural module. By the Main Theorem of [11] we must have $L \cong A_n$, $G \cong A_{n+4}$ or $L \cong A_5$, $G \cong J_2$. In the former case, by Lemma 2.4, V is the natural module for G, contrary to G being a counter-example. In the latter case, since by the 2 local structure J_2 does not contain subgroups of type A_9, $\Omega_4^+(4)$ or $\Omega_4^-(4)$ we must have $L^* \cong A_7$ and for z an involution in $C \cap H$, $\langle z \rangle L^* = \Sigma_7$; then

$$C_{L^*\langle z \rangle}(z) = Z_2 \times \Sigma_5$$

which is incompatible with the structure of $C_{J_2}(z)$. This contradiction proves:

(6.6.6) $L \trianglelefteq H$ whenever $L \subseteq H \subset G$ and $|C \cap H|$ is even.

By lemma 6.4 we obtain:

(6.6.7) If z_1, z_2 are involutions in C and $\langle z_1, z_2^h\rangle$ is a 2-group, for some $h \in G$, then either $z_2^h \in N$ or $z_1 \in N^h$; and if z_1, z_2 are involutions in C and $\langle z_1, z_2^h\rangle \cong D_{4k}$, k odd > 1, for some $h \in G$, then $z_2^h \in C$.

We next prove:

(6.6.8) $O_2(C) = 1$.

For suppose $O_2(C) \neq 1$ and let $Z = \Omega_1(Z(O_2(C)))$, $\mathscr{X} = \{z^g \mid z \in Z^\#, g \in G\}$. It follows from (6.67) that $\mathscr{X}$ is a set of root involutions in G, hence G may be identified by [28]. However, in none of the groups in Timmesfeld's list does the centralizer of a root involution contain a standard component centralized by a fourgroup. This contradiction establishes (6.6.8).

(6.6.9) If $|C^g \cap N|$ is even, for some $g \in G - N$, $[C, C^g] = 1$.

Suppose $|C^g \cap N|$ is even, for some $g \in G - N$ and let $T \in Syl_2(C^g \cap N)$, t be an involution in T. If $t \notin O_2(CT)$, by the Baer-Suzuki Theorem t inverts an element of $C^\#$ of odd order. Note that since $C = O^{2'}(C)$, by Lemma 6.7, for every $x \in C$,

$$[V, x] \subseteq C_V(L).$$

It follows therefore that $[V, t] \cap C_V(L) \neq 0$. But then for some involution $z \in C_C(t)$ and some non-zero $v \in [V, t] \cap C_V(L) \cap C_V(z)$, $C_G(v) \supseteq \langle L, L^g, z, t\rangle$ and (6.6.6) conflicts with (6.6.5). Thus $t \in O_2(CT)$ so (6.6.8) forces $[t, C] = 1$. In particular,

$$C \subseteq C_G(t) \subseteq N^g$$

and since g was arbitrary in $G - N$, this argument applied to g^{-1} gives $C^g \subseteq N$. Since for each $x \in C$, $[V, x] \subseteq C_V(L)$ and $O(G) \subseteq Z(G)$,

$$O(C) \cap O(G) = 1.$$

If $O(C) \cap O(C^g) \neq 1$, by (6.6.6) and (6.6.5), $L = L^g$, a contradiction. This proves

$$[C, C^g] \subseteq O(C) \cap O(C^g) = 1,$$

as claimed.

By Theorem 1 of [4] the Sylow 2-subgroups of C are elementary abelian and by Theorem 4 of [3] C is solvable. This establishes:

(6.6.10) $C/O(C)$ is an elementary abelian 2-group;
(6.6.11) If $|C^g \cap N|$ is even, for some $g \in G - N$, $\langle C, C^g\rangle = C \times C^g$.

Now let $E \in Syl_2(C)$, $E \subseteq S \in Syl_2(G)$ and let $\{E_1, \ldots, E_n\} = E^G \cap S$, $g_i \in G$

such that $E_i \subseteq C^{g_i}$. By (6.6.7), (6.6.10) and (6.6.11) $\langle E_1, \ldots, E_n \rangle$ is elementary abelian and since E_i centralizes $O(C^{g_j})$, for all $i \neq j$ it follows that

(6.6.12) $E^G \cap S = E_1 \times E_2 \times \cdots \times E_n$.

For each set $\mathscr{S}$ of commuting conjugates of C define $M(\mathscr{S}) = \bigcap_{C^g \in \mathscr{S}} N^g$, where $M(\phi) = G$. Over all such sets let $\mathscr{S}^*$ be one of largest cardinality such that there exists $g \in G$ with C^g commuting with all members of $\mathscr{S}^*$ and $C^g \trianglelefteq/\trianglelefteq \mathscr{M}(\mathscr{S}^*)$; since $C \trianglelefteq/\trianglelefteq G$, such $\mathscr{S}^*$ is always available. Replacing $\mathscr{S}^*$ by a G-conjugate if necessary we may assume C commutes with all elements of $\mathscr{S}^*$ and $C \trianglelefteq/\trianglelefteq M(\mathscr{S}^*)$. Set $M = M(\mathscr{S}^*)$.

If z is an involution in C^m, $m \in M$ define

$$\theta(z) = \langle C^h | h \in M \text{ and } |C^h \cap N_M(C^m)| \text{ is even} \rangle.$$

By the maximality of $\mathscr{S}^*$ and (6.6.9), whenever $|C^h \cap N_M(C^m)|$ is even, for some $h \in M - N_M(C^m)$,

$$[C^h, C^m] = 1 \quad \text{and} \quad C^h \trianglelefteq \trianglelefteq N_M(C^m).$$

Thus $\theta(z) \trianglelefteq N_M(C^m)$. Since $C/O(C)$ is an elementary abelian 2-group, by (6.6.12) so is $\theta(z)/O(\theta(z))$. Finally, since $C^m \trianglelefteq \theta(z)$, by construction of M, $\theta(z) \ntrianglelefteq M$.

(6.6.13) If $z_1 \in C$, $z_2 \in C^h$, $h \in M$ and z_1, z_2 are commuting involutions, $\theta(z_1) = \theta(z_2)$.

By symmetry it suffices to show $\theta(z_2) \subseteq \theta(z_1)$: this is clear if $C = C^h$ so we may assume $h \notin N$. By (6.6.9), $[C, C^h] = 1$, so $C \subseteq \theta(z_2)$. Since Sylow 2-subgroups of $\theta(z_2)$ are abelian and $C_M(z_1) \subseteq N_M(C)$, the latter group contains a Sylow 2-subgroup of $\theta(z_2)$. Finally, since $m(E) \geq 2$, $O(\theta(z_2)) = \Gamma_{1,E}(O(\theta(z_2))) \subseteq N_M(C)$, whence $\theta(z_2) \subseteq N_M(C)$ which yields the inclusion $\theta(z_2) \subseteq \theta(z_1)$.

Let D be the involutions in M-conjugates of C. Note that by (6.6.7), D satisfies property $(+)$: for $d, e \in D$, either $Z(\langle d, e \rangle) = 1$ or $Z(\langle d, e \rangle) \cap D \neq \phi$. Let $\mathscr{D}$ be the graph whose vertices are the elements of D and (d, e) an edge if and only if $de = ed \neq 1$.

(6.6.14) $\mathscr{D}$ is disconnected.

For if $\mathscr{D}$ is connected, by (6.6.13), for all z_1, $z_2 \in D$, $\theta(z_1) = \theta(z_2)$, and so $\theta(z_1) \trianglelefteq M$, a contradiction.

Let $H = \langle D \rangle$, $\overline{H} = H/S(H)$. By (6.6.14) and Theorem 4.1 of [28] of the following holds:

(i) $\overline{H} = \overline{1}$;
(ii) $\overline{H}$ is a Bender group;
(iii) $\overline{H} \cong L_2(q) \wr \Sigma_k$, $q = 2^m > 2$, $k = 3$ or 4;

(iv) $\overline{H}' \neq \overline{H}''$ and $\overline{D}$ is a class of odd transpositions in $\overline{H}$.

Let $E \subseteq T \in Syl_2(H)$ and note that if $C^h \cap T \neq 1$, $C^h \subseteq \theta(z)$, for all $z \in E^{\#}$.

Suppose H is non-solvable but $E \cap S(H) \neq 1$. In this situation, by Frattini's argument,

$$N_H(T \cap S(H)) = H_1$$

covers $\overline{H}$ and for $z \in E^{\#} \cap S(H)$, $\theta(z) = \theta(z_1)$, for all $z_1 \in D \cap H_1$. Thus H_1 normalizes $\theta(z)$, so $\theta(z) \subseteq S(H)$ contrary to $E \nsubseteq S(H)$. If H is non-solvable, therefore, $E \cap S(H) = 1$. This means cases (iii) and (iv) cannot hold for in each of these $\overline{H}$ contains no fourgroup all of whose involutions are in $\overline{D}$, whereas $\overline{E}^{\#} \subseteq \overline{D}$ and $m(E) = m(\overline{E}) \geq 2$.

Since the centralizer of each involution in a simple Bender group is a 2-group and $m(E) \geq 2$, in either case (i) or (ii), $O(C) \subseteq S(H)$. As above, T normalizes $\theta(z)$, for each $z \in E^{\#}$ so by properties of solvable groups

$$O(C) \subseteq O(\theta(z)) \subseteq O(TS(H)) \subseteq O(H).$$

Let $\tilde{H} = H/O(H)$ so by (6.6.7) we obtain:

(6.6.15) $\tilde{D}$ is a set of odd transpositions in $\tilde{H}$.

Suppose $\tilde{D}$ is not a single class in $\tilde{H}$ so by properties of odd transpositions $\tilde{D} = \tilde{D}_1 \cup \tilde{D}_2$ where $[\tilde{D}_1, \tilde{D}_2] = 1$ for some non-empty subsets $\tilde{D}_i$ of $\tilde{D}$. If $\tilde{E}^{\#} \subseteq \tilde{D}_1$, let D_1 denote the preimage set of involutions in H i.e. $D_1 = E^{\#\langle D_1\rangle}$. Also, since $D = E^{\# M}$, there exists $h \in M$ such that $D_2 = E^{h\#\langle D_2\rangle}$. It follows from (6.6.9) that $[D_1, D_2] = 1$ which contradicts $\mathscr{D}$ being disconnected. Thus $\tilde{E}^{\#} \nsubseteq \tilde{D}_1$ and similarly $\tilde{E}^{\#} \nsubseteq \tilde{D}_2$. Now $H = \Gamma_{1,E}(H) \subseteq N(C)$, contrary to $\mathscr{D}$ being disconnected. This argument proves $\tilde{D}$ is a single class in $\tilde{H}$. In particular, $\tilde{H} = \tilde{H}'$ so H is not solvable. Since $E \cap S(H) = 1$, it follows that

(6.6.16) $\overline{H} \cong L_2(q), Sz(q), U_3(q), q = |E|$.

Now replace H by a suitable subgroup H_0 containing $O(H)E$ with $\overline{H}_0 \cong L_2(q)$, $Sz(q)$ and $\tilde{D} \cap \tilde{H}_0$ is a single class in $\tilde{D}$ with $\tilde{H}_0 = \langle \tilde{D} \cap \tilde{H}_0\rangle$. We lose no generality in assuming $H = H_0$, i.e. $\overline{H} \cong L_2(q)$ or $Sz(q)$. By Lemma 2.6 and (6.6.12) applied to $\tilde{H}$,

(6.6.17) $\tilde{H} \cong L_2(q)$ or $Sz(q)$.

Let $\tilde{h}$ be an element of $\tilde{H}$ of order $q - 1$ normalizing E. As noted, $O(H) \subseteq N$ so every element of the coset $\tilde{h}$ is in N. Since $\tilde{h}$ is inverted by e^x, for some $e \in E$, $x \in H$ we may pick $h \in H$ in the coset $\tilde{h}$ with h inverted by $f = e^x$. Clearly $f \notin N$ else $H = \langle O(H), E, h, f\rangle \subseteq N$, contrary to C being solvable.

First note that if $a \in E^{\#}$ and $V_a = [V, a]$, $V_a \cap V_a^f = 0$: for otherwise there exists $v \in C_{V_a}(f)$ with $v \neq 0$ and $C_G(v) \supseteq \langle a, f, L\rangle$; by (6.4.6) $C_G(v) \subseteq N$, a contradiction. For any $a \in E^{\#}$, $a \sim_G f$ so dim $V_a = \dim [V, f]$; thus if U is any subspace of V with $U \cap U^f = 0$, $\dim U \leq \dim V_a$. Since $V_a \subseteq C_V(L)$ but by (6.6.8), a does not centralize $C_V(L)$, $V_a \subset C_V(L)$ and so $C_V(L) \cap C_V(L)^f \neq 0$. Let

$$A = N_G(C_V(L) \cap C_V(L)^f) \supseteq \langle L, f, h \rangle.$$

If $|C \cap A|$ is even, by (6.6.6), $A \subseteq N$, again a contradiction. Next suppose $C^x \cap A$ contains a fourgroup F which we may assume contains f. Since $F \subseteq C^x \subseteq H$, by properties of $L_2(q)$, $Sz(q)$, $\langle F, h \rangle$ covers $\tilde{H}$. Thus $\langle F, h \rangle$ contains E_0 with $E_0 O(H) = EO(H)$. But since $O(H) \subseteq N$, $E_0 \subseteq O^{2'}(EO(H)) = C$, contrary to $|C \cap A|$ being odd. This proves $\langle f \rangle \in Syl_2(C^x \cap A)$. It follows from (6.6.7) that f^A is a class of odd transpositions in A. By [2], $f^A = F_1 \cup F_2 \cup \cdots \cup F_r$ where F_i is a non-empty class of odd transpositions in $\langle F_i \rangle$ and $[F_i, F_j] = 1$, for all $i \neq j$. Without loss of generality we assume $f \in F_1$ and set $A_1 = \langle F_1 \rangle$; note that as f inverts $h \in A$, $h \in A_1$.

(6.6.18) $L \subseteq A_1$.

If $L = [L, h]$, then $L \subseteq \langle f^A \rangle$; since $A_1 \trianglelefteq \langle f^A \rangle$, $L \subseteq A_1$ as claimed. If $[L, h] = 1$, since L permutes $\{F_1, \ldots, F_r\}$, L normalizes A_1. Suppose $L \nsubseteq A_1$. Because

$$O(H) \subseteq A \quad \text{and} \quad A_1 \trianglelefteq\trianglelefteq A,$$

we have $[O(H), f] = X \subseteq A_1$. Since $O(H) \subseteq N$ either $[X, L] = L$ or $[X, L] = 1$. Because $L \nsubseteq A_1$, X centralizes L. Since $O(H) \subseteq N^x$, $X \subseteq O(C^x)$ and since $X \cap O(G) = 1$,

$$\langle L, L^x, f \rangle \subseteq N_G(X) \subset G.$$

By (6.6.6), $N_G(X) \subseteq N^x$. Thus L acts on $C_V(L^x)$ and $V/C_V(L^x)$. If L centralizes $C_V(L^x)$, by orders $C_V(L^x) = C_V(L)$ so by (6.6.6) applied to

$$N_G(C_V(L)) \supseteq \langle L, L^x, C, C^x \rangle$$

we obtain $L = L^x$ or $[L, L^x] = 1$, a contradiction. If, however, L does not centralize $C_V(L^x)$, $[L, L^x]$ centralizes $C_V(L^x)$ and $V/C_V(L^x)$ and so is a 2-group. This forces $[L, L^x] = 1$ again contrary to L being in standard form. Thus $L \subseteq A_1$.

Let $A_2 = A_1^{(\infty)}\langle f \rangle$, $B = A_2'$. By [2], $A_2/S(A_2)$ is isomorphic to one of:

(1) Σ_n;
(2) $Sp_{2n}(r)$, $U_n(r)$, $O_n^{\pm}(r)$, $Sz(r)$, r even;
(3) $O_n^{\pm}(r)$, $r = 3$ or 5;
(4) F_{22}, F_{23}, F_{24};
(5) $L_2(r) \wr \Sigma_n$, r even.

Now because G is not a Bender group, by Theorem 2 of [3], there exists $g \in G - N$ such that $|C^g \cap N|$ is even, whence, by (6.6.9), C^g normalizes L. If

L normalizes L^g, since $C \subseteq C_G(C^g) \subseteq N^g$, by (6.6.6), L^g normalizes L, a contradiction. Since $L \nsubseteq N^g$ and $C^g = O^{2'}(C^g)$ it follows from (6.6.6) that C^g centralizes $C_V(L)$, whence $C^g \subseteq A$. Since C^g normalizes L and permutes $\{F_1, \ldots, F_r\}$, C^g normalizes B.

Suppose $X \subseteq O(C^g)$ with $X \neq 1$, $|N_{C^g}(X)|$ even and $[X, L] = 1$. As $X \cap O(G) = 1$, $N_G(X) \subset G$ so by (6.6.6), L normalizes L^g which we have seen to be impossible. By Lemma 5.34 of [27] there exists $K_1 \times K_2 \subseteq C^g$ with $K_i \cong D_{2p_i}$, for some odd primes p_i.

If $LS(B)/S(B) \subseteq E(B/S(B))$ and $B/S(B) \cong L_2(r) \rfloor A_n$, then $L \cong L_2(r_1)$, for some $r_1 \mid r$; in this situation C^g has a normal subgroup X with $|O(C^g): X| \leq 3$ and $[X, L] = 1$. By the preceding paragraph, $X = 1$ and so the statements $|O(C)| \leq 3$, $m(E) \geq 2$ and $O_2(C) = 1$ are incompatible. Thus in case (5),

$$LS(B)/S(B) \nsubseteq E(B/S(B)).$$

From this it follows that $B = [B, d]$, $d \in K_i^\#$, $i = 1, 2$.

Note that $O(B) = \Gamma_{1,E^g}(O(B)) \subseteq N^g$, so

$$[O(B), C^g] \subseteq O(B) \cap C^g = X.$$

Since $[X, L] \subseteq L \cap O(B) \subseteq Z(L)$, $[X, L] = 1$ by the 3 subgroups lemma, whence $X = 1$ by previous results. Thus $B = [B, C^g]$ centralizes $O(B)$. Let d be an involution in K_1, so $C_{O_2(B)}(d) \subseteq N^g$. Therefore

$$[C_{O_2(B)}(d), O(K_2)] \subseteq O_2(B) \cap O(C^g) = 1$$

so, by the $P \times Q$ lemma, $O(K_2)$ centralizes $O_2(B)$, whence so does $[B, O(K_2)] = B$. This argument proves $S(B) \subseteq Z(B)$. Indeed, if $O_2(B) \neq 1$, as L centralizes $O_2(B)$, by (6.6.6), $L \trianglelefteq A$, contrary to $f \notin N$. Thus $S(B) = O(B)$ as well. By Lemma 2.9 applied to $B/O(B)$, C^g has a normal subgroup X with

$$|O(C^g): X| \leq 3 \quad \text{and} \quad [X, B] \subseteq O(B).$$

Again $[X, L] = 1$ so $X = 1$ and the properties $m(C) \geq 2$, $|O(C)| \leq 3$ and $O_2(C) = 1$ are incompatible. This contradiction completes the proof of Theorem 6.6.

Proof of Theorem G. Let J_1, J_2 be distinct blocks with $J_1 \to J_2$ and let

$$V = \mho(J_2), \quad \bar{J}_2 = J_2/O_2(J_2).$$

By definition of "$\to$" there is a 2-group S normalizing J_2 such that $\bar{J}_1$ is a component of $C_{\bar{J}_2}(S)$; moreover, as $\mho(J_1) = [O_2(J_2), J_1]$, $\bar{J}_1$ has a unique nontrivial irreducible constituent in V. Let $\widetilde{J_2 S} = J_2 S/C_{J_2 S}(V)$ so $\tilde{J}_2$ is a central (odd order) quotient of $\bar{J}_2$. If $j \in J_2$ and $[\tilde{S}, \tilde{j}] = \tilde{1}$, then $[S, j] \subseteq C_{J_2}(V) \subseteq O_{2,2'}(J_2)$ so it follows that $C_{\bar{J}_2}(S)$ covers $C_{\tilde{J}_2}(\tilde{S})$. Theorem G is now immediate from Theorem 6.4.

Proof of Theorem E *and* F. Let $J \in \mathscr{B}^*(G)$, for some finite group G of characteristic 2 type, $S \in Syl_2(C_G(J/O_2(J)))$, so by assumption $J \trianglelefteq \trianglelefteq N_G(S)$. Let

$$\mathscr{B}_1(G) = \{K \mid K \text{ is an } \Omega_4^+(2^m)\text{-block, } K \trianglelefteq \trianglelefteq N_G(T),\ T \in Syl_2(C_G(K/O_2(K)))\},$$

so the relation "$\to$" extends mutatis mutantis to $\mathscr{B}(G) \cup \mathscr{B}_1(G)$.

(EF.1) If $K \in \mathscr{B}^*(G) \cup \mathscr{B}_1(G)$, either $K \trianglelefteq \trianglelefteq N$, for every maximal 2-local $N \subseteq K$ or K is a block of $L_2(2^n)$-type and $K \to L \in \mathscr{B}_1(G)$.

To prove this let $K \subseteq Y \subseteq KT$, $T \in Syl_2(C_G(K/O_2(K)))$, with Y maximal subject to $Y \trianglelefteq \not\trianglelefteq N$, for some maximal 2-local N containing Y. Let $Q = O_2(Y)$; we first show $K \trianglelefteq \trianglelefteq N_G(Q)$. This is true by assumption if $Q = T$ so consider when $Q \subset T$. Then $Q^* = N_T(Q) \supset Q$ so as $Y = QK$, by maximality of Y, $K \leq \leq N_G(Q)$ as claimed. Let $H = O_2(N)$. If $QH \supset Q$, then $Q \subset N_{QH}(Q)$ and $N_{QH}(Q) \subseteq C_G(K/O_2(K))$, so by maximality of Y, $K \trianglelefteq \trianglelefteq N$, contrary to assumption. Thus $QH = Q$ and since $H \neq Q$, $H \subset Q$. Since G is of characteristic 2 type $U(K) \subseteq H$. Since KH/H is a component of $C_{N/H}(Q/H)$, by the L-balance Theorem [3.1 of 23], $KH/H \subseteq L(N/H)$. Let

$$H \subseteq X \subseteq N \quad \text{with } X/H = \langle (KH/H)^{L(N/H)} \rangle$$

so X/H is a product of 2-components of N/H and Q normalizes X. Since $H \subseteq Q$, KH has a unique non-central 2-chief factor, whence XQ has a unique non-central 2-chief factor, V. Let an overbar denote passage to $XQ/C_{XQ}(V)$. Since G is of characteristic 2 type every non-trivial odd order element of N acts faithfully on H, so $\overline{X} = X/H$. Thus $\overline{K}$ is a component of $C_{\overline{X}}(\overline{Q})$. By Lemma 6.2, $\overline{K}$ centralizes $O(\overline{X})$ so $\overline{X} = \langle \overline{K}^{E(\overline{XQ})} \rangle$ is semisimple. If $\overline{K}$ is an $\Omega_4^+(2^m)$-block, by Theorem 6.5, $\overline{X} = \overline{K}$, whence $K = X^{(\infty)} \trianglelefteq \trianglelefteq N$, a contradiction. Assume therefore $\overline{K}$ is quasisimple. By Theorem 6.6, $X^{(\infty)}$ is either a block or an $\Omega_4^+(2^m)$-block and $K \to X^{(\infty)}$. Next, over all such $N \supseteq Y$ with $Y \trianglelefteq \trianglelefteq N$ pick N to maximize first $|X^{(\infty)}|$ and, subject to this, to maximize

$$|C_N(X^{(\infty)}/O_2(X^{(\infty)}))|_2.$$

Let $L = X^{(\infty)}$ and $P \in Syl_2(C_N(L/O_2(L)))$ with P normalized by Q. Let M be a maximal 2-local subgroup of G containing $N_G(P)$, so $Y \subseteq M$. Since initially N was arbitrary, there exists L_1, a block or $\Omega_4^+(2^m)$-block of M, with $K \subseteq L_1$. Since $L \subseteq M$ we must have $L \subseteq L_1$ so by maximality of $|L|$, $L = L_1$. Since

$$P \subseteq O_2(N_G(P)) \subseteq C_M(L/O_2(L)),$$

by maximality of $|P|$, $P \in Syl_2(C_G(L/O_2(L)))$. Thus $L \trianglelefteq \trianglelefteq N_G(P)$ implies

$$L \in \mathscr{B}(G) \cup \mathscr{B}_1(G).$$

Since $\overline{K}$ is quasisimple, by hypothesis $K \in \mathscr{B}^*(G)$ and since $K \to L$ and $K \neq L$, $L \notin \mathscr{B}(G)$. It follows therefore by Theorem 6.6 that the second conclusions of (EF.1) holds in this situation.

By (EF.1) to prove both Theorems E and F it suffices to show:

(EF.2) If $K \in \mathscr{B}^*(G) \cup \mathscr{B}_1(G)$ and $K \trianglelefteq \trianglelefteq N$, for every maximal 2-local $N \supseteq K$, then K is contained in a unique maximal 2-local subgroup.

To prove this let $K_1, \ldots, K_r$ be a maximal set of commuting conjugates of K with $K = K_1$ and set

$$D = \langle K_1, \ldots, K_r \rangle, \quad M = N_G(O_2(D)).$$

By the hypothesis of (EF.2), M is a maximal 2-local subgroup and $M = N_G(D)$. Suppose N is any maximal 2-local containing K. Let

$$D_0 = \langle K^g | K^g \subseteq N, g \in G \rangle.$$

By hypothesis $D_0 \trianglelefteq N$. Let L be a block of D_0; we show $L \in \{K_1, \ldots, K_r\}$: this is clear if $L = K$ so assume $[L, K] = 1$. Then $N_G(O_2(K)) \supseteq D_0$, D and by hypothesis $D_0, D \trianglelefteq \trianglelefteq N_G(O_2(K))$ so by Lemma 2.1 distinct blocks in $\langle D, D_0 \rangle$ commute; by maximality of D, $L \in \{K_2, \ldots, K_r\}$ as claimed. Thus $D_0 \leq D$ so $D \subseteq N_G(D_0) = N$, whence $D \subseteq D_0$ by definition. Thus $N = N_G(D) = M$, as needed to complete the proof.

Proof of Theorem D. Let J be a block in some maximal 2-local subgroup M of G with G of characteristic 2 type, let $Q = O_2(M)$, $Q \subseteq F$ with $F/Q = F^*(M/Q)$ and note that as $Q = F^*(M)$, $U(J) \subseteq Q$ so $J \trianglelefteq F$.

(D.1) There is a maximal 2-local M_1 of G with $N_G(J) \subseteq M_1$ and $J \trianglelefteq \trianglelefteq M_1$.

Assume this is not the case so, in particular, $N_G(J) \nsubseteq M$. Let N be a maximal 2-local containing $N_G(J)$, $P = O_2(N)$ and $J^M = \{J_1, J_2, \ldots, J_r\}$, $r \geq 2$, $J = J_1$. For $P_0 = N_P(Q)$, $P_0 \subseteq M$ so $[P_0, F] \subseteq F \cap P \subseteq Q$; thus P_0Q/Q centralizes F/Q so by properties of F^*, $P_0 \subseteq Q$. This proves $P \subseteq Q$ so $P \subset Q$. Since J_i acts non-trivially on P, $U(J_i) \subseteq P$ so J_iP/P is a component of $C_{N/P}(Q/P)$. As in the preceding proof by Theorem 6.6, there exist K_i, blocks or $\Omega_4^+(2^m)$-blocks of N with $J_i \rightarrow K_i$, $1 \leq i \leq r$. Moreover, either $J_i = K_i$ or $K_i/O_2(K_i) \cong A_n$, $\Omega_{2n}^\pm(2^m)$ or $Z_3 \cdot U_4(3)$. From this it follows that $[K_i, K_j] = 1$, for all $i \neq j$. Now let $m_i \in M$ such that $J_i = J^{m_i}$, $1 = 2, 3, \ldots, r$. Since $J^{m_i-1} \neq J$, $[J^{m_i-1}, K_1] = 1$ whence $[J, K_1^{m_i}] = 1$, $i = 2, \ldots, r$. Thus $K_1^{m_i} \subseteq N_G(J) \subseteq N$. Since $J_i \subseteq K_i \cap K^{m_i}$ and $K_i \trianglelefteq \trianglelefteq N$, $K_i = K_1^{m_i}$, $i = 2, \ldots, r$. Suppose $\langle K_1, \ldots, K_r \rangle \nleq N$ so there exists $n \in N$, $j \in \{1, \ldots, r\}$ such that $[K_j^n, K_i] = 1$, $1 \leq i \leq r$. In this situation

$$K = K_j^n \subseteq C_G(\langle J_1, \ldots, J_r \rangle) \leq M.$$

Furthermore, since $K \trianglelefteq \trianglelefteq N \supseteq F$, by properties of F^*, K is a block of M. Let

$$K^* = \langle K^M \rangle \subseteq C_G(\langle J_1, \ldots, J_n \rangle)$$

so $K^* \subseteq F \subseteq N$. Since for each K_i either $K_i = J_i$ or K_i is of known type and since $[K^*, J_1] = 1$ by inspection $[K^*, K_1] = 1$. Thus

$K_1 \subseteq N_G(K^*) \subseteq M$ and since $J_1 \trianglelefteq\trianglelefteq M$, $K_1 = J_1$. But now we may take $M_1 = N$, contrary to assumption. This argument proves

$$\langle K_1, \ldots, K_r \rangle \trianglelefteq N.$$

Since $N_G(J_1) \subseteq N$ and $N_G(J_1)$ normalizes K_1, $\langle K_1, \ldots, K_r \rangle = \langle K_1^M \rangle$ is normalized by M, contrary to $M \neq N$. This establishes (D.1).

Without loss of generality $N_G(J) \subseteq M$ so as an immediate consequence of (D.1) for $R \in Syl_2(C_M(J/O_2(J)))$ we obtain

(D.2) $R \in Syl_2(C_G(J/O_2(J)))$.

It remains to prove $J \trianglelefteq\trianglelefteq N_G(R)$. Let $g \in N_G(R)$; we prove $J \subseteq M^g$. If $U(J)$ is abelian,

$$U(J) \subseteq Z(R) \subseteq Z(Q) \quad \text{and} \quad [J, Z(Q)^g] \subseteq U(J) \subseteq Z(R) = Z(R)^g \subseteq Z(Q)^g,$$

whence

$$J \subseteq N_G(Z(Q)^g) = M^g.$$

If $U(J)$ is non-abelian, since R centralizes $U(J)'$ and $[U(J), R] \subseteq U(J)'$, R' centralizes $U(J)$. Thus $[J, R'] \subseteq U(J) \cap C(U(J)) \subseteq Z(J)$ and so $[J, R'] = 1$ by the 3-subgroups lemma. Since $U(J) \subseteq Q$, $Q' \neq 1$ and $[J, Q'^g] \subseteq [J, R'] = 1$, whence

$$J \subseteq N_G(Q'^g) = M^g.$$

In both cases $J \subseteq M^g$ and, since $J \leq\leq M$,

$$J \trianglelefteq\trianglelefteq \bigcap_{g \in N_G(R)} M^g \cap N_G(R) \trianglelefteq N_G(R).$$

This completes the proof of Theorem D.

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