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CYCLIC VECTORS FOR INVARIANT SUBSPACES IN SOME CLASSES OF ANALYTIC FUNCTIONS

BY

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1. Let ψ be a positive increasing function on $(0, \infty)$ such that

$$\lim_{t \downarrow 0} \psi(t) = 0, \psi(t) = \psi(1) \text{ for } t > 1 \quad \text{and} \quad \int_0^1 \frac{1}{\psi(t)}\, dt < \infty.$$

Define

$$M_\psi = \left\{ f \in H^\infty | M_\infty(f', r) = o\left(\frac{\psi(1-r)}{1-r} \right) \right\}$$

and

$$L_\psi = \left\{ f \in H^\infty \left| \int_0^1 \frac{M_\infty(f', r)}{\psi(1-r)}\, dr < \infty \right. \right\},$$

where H^∞ is the space of bounded analytic functions on the unit disk, and

$$M_\infty(g, r) = \sup_{|z|=r} |g(z)| = \sup_{|z| \le r} |g(z)|.$$

Each of the spaces M_ψ and L_ψ becomes a Banach algebra under the norms

$$\|f\|_{M_\psi} = \|f\|_\infty + \sup_{0<r<1} \frac{(1-r) M_\infty(f', r)}{\psi(1-r)},$$

$$\|f\|_{L_\psi} = \|f\|_\infty + \int_0^1 \frac{M_\infty(f', r)}{\psi(1-r)}\, dr$$

respectively. Since $M_\infty(f', r)$ is increasing and $\psi(1-r)$ is decreasing, it is

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easy to see that $L_\psi \subset M_\psi$. Indeed,

$$(1-r)\frac{M_\infty(f',r)}{\psi(1-r)} \le \int_r^1 \frac{M_\infty(f',\rho)}{\psi(1-\rho)}\,d\rho,$$

and, for $f \in L_\psi$, the latter quantity tends to 0 as $r \to 1$. It follows immediately from the closed graph theorem that the inclusion is continuous.

In case the function ψ is sufficiently regular, it is possible to characterize the classes L_ψ and M_ψ in terms of moduli of continuity. The regularity condition is as follows: there are positive constants α, and β such that

$$\delta\int_\delta^\infty \frac{\psi(t)}{t^2}\,dt \le \alpha\psi(\delta)$$

and

$$\int_0^\delta \frac{\psi(t)}{t}\,dt \le \beta\psi(\delta)$$

for all $\delta > 0$. If $\omega(f,\delta)$ denotes the modulus of continuity of f on ∂D and the regularity condition holds, then it is shown in [3] that $\omega(f,\delta) = O(\psi(\delta))$ if and only if

$$M_\infty(f',r) = O\left(\frac{\psi(1-r)}{1-r}\right).$$

The proof is easily modified to show that $f \in M_\psi$ if and only if $\omega(f,\delta) = o(\psi(\delta))$. Also, using the techniques of [3], it can be shown that $f \in L_\psi$ if and only if

$$\int_0^1 \frac{\omega(f,t)}{t\psi(t)}\,dt < \infty.$$

In [3], this is proved for $\psi(t) = t^\alpha$, $0 < \alpha < 1$, but essentially the same proof works for an arbitrary regular ψ. In the following, ψ will always be regular and continuous.

A final consequence of the regularity conditions is the existence of a number s, $0 < s < 1$ such that $t^s \le \psi(t)$ for all small t. This can be proved as follows. Define the function $\phi(t)$ by the equation

$$\phi(t) = t\int_t^\infty \frac{\psi(s)}{s^2}\,ds.$$

Note that $\psi(t) \leq \phi(t) \leq \alpha\psi(t)$, and $\alpha > 1$, since otherwise, differentiating

$$\frac{\psi(t)}{t} = \int_t^\infty \frac{\psi(s)}{s^2}\, ds$$

leads to $\psi'(t) = 0$. Differentiating the equation defining ϕ gives

$$\phi'(t) = \frac{\phi(t)}{t} - \frac{\psi(t)}{t},$$

which leads to

$$\frac{\phi'}{\phi} = \frac{1}{t}\left(1 - \frac{\psi}{\phi}\right).$$

Since $\phi \leq \alpha\psi$, it follows that

$$\frac{\phi'}{\phi} \leq \frac{1}{t}\left(1 - \frac{1}{\alpha}\right).$$

If $s = (1 - 1/\alpha)$ and $\gamma = \phi(1)$, then integrating from t to 1 gives

$$\log \frac{\gamma}{\phi(t)} \leq -s \log t,$$

or

$$\gamma t^s \leq \phi(t) \leq \alpha\psi(t).$$

By choosing a slightly larger s the assertion follows.

2. A closed subspace I of L_ψ or M_ψ is invariant if $zI \subset I$. Since the polynomials are dense in L_ψ and M_ψ, the closed invariant subspaces coincide with the closed ideals. In either case the closed invariant subspaces can be described explicitly in terms of the Riesz factorization. Given I, let

$$E = E(I) = \{e^{i\theta} | f(e^{i\theta}) = 0 \text{ for all } f \in I\}$$

and let u be the greatest common divisor of the inner factors of the functions in I. Then a function f belongs to I if and only if f vanishes on E and f is divisible in H^∞ by u. This was proved for $\psi(t) = t^\alpha$ and M_ψ in [1] and independently, by Shamoyan [4]. The same techniques apply for an arbitrary ψ and for the spaces L_ψ. This is also a consequence of the more general results of Shirokov [5], [6].

In each case the technique of proof involves the approximation of functions vanishing on E by functions vanishing on E to high order. To be specific, let $d(z, E)$ denote the distance from z to E, and for $\alpha > 0$ let

$$J^\alpha(E) = \{f \in X| \, |f(z)| \leq Cd^\alpha(z, E)\}$$

where X is L_ψ or M_ψ. It turns out that if $I(E)$ denotes the closed invariant subspace of functions vanishing on E, then for each $\alpha > 0$, $J^\alpha(E)$ is dense in $I(E)$. The known proofs of this are purely constructive (e.g. [2], [5, 6]).

At this point the usual procedure is to consider the linear functionals annihilating the given ideal and, after some preliminary analysis, to apply the Hahn-Banach theorem.

The purpose of this paper is to show that, when the inner factor u associated with the ideal I is trivial, then I is generated by any outer function f which vanishes precisely on $E(I)$. The proof below is purely constructive, and consequently the characterization $I = I(E(I))$ can be established without recourse of the Hahn-Banach theorem. This will be a consequence of the following theorem, in which X denotes L_ψ or M_ψ.

THEOREM. *If f is an outer function in X with boundary zero set E, and if $g \in J^\alpha(E)$ for sufficiently large α, then there is a sequence of functions $\{g_n\}_{n=1}^\infty$ such that*
(i) *$g_n f \in I(f)$ for each n and*
(ii) *$g_n f \to g$ in X.*

3. The construction of the g_n proceeds as follows. Let $\{I_n\}_{n=1}^\infty$ denote the sequence of complementary intervals to E, and let $B_n = \bigcup_{k=n+1}^\infty I_k$, and $E_n = E \cap \overline{B}_n$. Let

$$F_n(z) = \exp\left\{\frac{1}{2\pi}\int_{B_n} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|f(e^{i\theta})| \, d\theta\right\}.$$

Then F_n is outer, $|F_n| = |f|$ on B_n and $|F| = 1$ on the complement of B_n. Define

$$g_n = \frac{g}{f} F_n,$$

so that $g_n f = gF_n$. It is easy to see that $g_n f \to g$ uniformly on compact subsets of D. The theorem will be proved in three steps. First, it will be shown that $g_n f \in X$, and then that $g_n f \to g$ in X. Finally, it will be shown that $g_n f \in J(E)$.

Since $(g_n f)' = F_n g' + F_n' g$, it will be necessary to estimate $F_n' g$. The appropriate estimate will be provided in Lemma 2. A preliminary estimate is given in Lemma 1. If $f \in H^\infty$ and Γ is a measurable subset of the unit circle, define

$$f_\Gamma(z) = \exp\left\{\frac{1}{2\pi}\int_\Gamma \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|\,d\theta\right\}.$$

LEMMA 1. *Let Γ be a measurable set on the unit circle, let $f \in H^\infty$ with $\|f\|_\infty < 1$ and let $0 < \eta \le \frac{1}{2}$. Then there is a constant C such that if $z = re^{it}$ satisfies $e^{it} \notin \Gamma$ and $d(e^{it}, \Gamma) \ge (1-r)^\eta$, then*

$$\log|f_\Gamma(z)| \ge C(1-r)^{1-2\eta}\log|f(0)|.$$

Proof. First note that if $e^{i\theta} \in \Gamma$ and $|\theta - t| \le \pi$, then

$$|e^{i\theta} - re^{it}|^2 \ge (1-r)^2 + \frac{8}{\pi^2}r|e^{i\theta} - e^{it}|^2.$$

It follows that

$$|e^{i\theta} - re^{it}| \ge \frac{1}{4}|e^{i\theta} - e^{it}|.$$

But

$$|e^{i\theta} - e^{it}| \ge d(e^{it}, \Gamma) \ge (1-r)^\eta,$$

and it follows that

$$\frac{1-r^2}{|e^{i\theta}-z|^2} \le 16\frac{1-r^2}{(1-r)^{2\eta}} \le 32(1-r)^{1-2\eta}.$$

Hence

$$\begin{aligned}\log|f_\Gamma(z)| &= \frac{1}{2\pi}\int_1 \frac{1-r^2}{|e^{i\theta}-z|^2}\log|f(e^{i\theta})|\,d\theta \\ &\ge C(1-r)^{1-2\eta}\int_\Gamma \log|f(e^{i\theta})|\,d\theta \\ &\ge C(1-r)^{1-2\eta}\log|f(0)|.\end{aligned}$$

LEMMA 2. *With g and F_n as above, if $X = M_\psi$ then*

$$|g(z)F_n'(z)| = o\left(\frac{\psi(1-r)}{1-r}\right),$$

while if $X = L_\psi$ then

$$\int_0^1 \frac{M_\infty(gF_n', r)}{\psi(1-r)}\, dr < \infty.$$

Proof. Let

$$G_1 = \left\{z = re^{it} \middle| d\left(e^{it}, B_n\right) \le (1-r)^{1/2}\right\},$$
$$G_2 = \left\{z = re^{iz} \middle| d\left(e^{it}, B_n\right) < (1-r)^{1/2}, e^{it} \notin B_n\right\},$$
$$G_3 = \left\{z = re^{iz} \middle| d\left(e^{it}, B_n\right) < (1-r)^{1/2}, e^{it} \in B_n\right\}.$$

For $z = re^{it} \in G_1$, there exists $e^{i\theta} \in E_n$ such that

$$\begin{aligned} d^2(z, E_n) &= |z - e^{i\theta}|^2 \\ &= (1-r)^2 + rd^2\left(e^{it}, E_n\right) \\ &\le (1-r). \end{aligned}$$

By Cauchy's estimate $|F_n'(z)| \le (1-r)^{-1}$, so, using the estimate $t^s \le \psi(t)$,

$$|g(z)F_n'(z)| \le (1-r)^{\alpha/2-1} = o\left(\frac{\psi(1-r)}{1-r}\right),$$

if α is large enough.

If $z = re^{it} \in G_2$ and $e^{i\theta} \notin B_n$, then

$$\begin{aligned} |e^{i\theta} - z|^2 &= (1-r)^2 + 4r\sin^2\left(\frac{\theta - t}{2}\right) \\ &\ge (1-r)^2 + rd^2\left(e^{it}, E_n\right) \\ &\ge (1-r)^2 + r(1-r) \\ &= 1 - r, \end{aligned}$$

and

$$\begin{aligned} d^2(z, E_n) &= (1-r)^2 + rd^2\left(e^{it}, E_n\right) \\ &\le 2d\left(e^{it}, E_n\right). \end{aligned}$$

Hence

$$\begin{aligned} |g(z)| &\le Cd^\alpha(z, E_n) \\ &\le Cd^{\alpha/2}\left(e^{it}, E_n\right). \end{aligned}$$

Now if $a > 0$, $b > 0$, $a + b = 2$, then

$$\begin{aligned}|e^{i\theta} - z|^2 &= |e^{i\theta} - z|^a |e^{i\theta} - z|^b \\ &\geq (1 - r)^{a/2} d^b(e^{it}, E_n),\end{aligned}$$

and this, combined with Lemma 1, with $\eta = \frac{1}{2}$, leads to the estimate

$$|F_n'(z)| \leq C|\log|F(0)||(1 - r)^{-a/2} d^{-b}(e^{it}, E_n),$$

so that

$$|g(z)F_n'(z)| \leq Cd^{(\alpha/2)-b}(e^{it}, E_n)(1 - r)^{-a/2}.$$

Choosing α so that $\alpha/2 - b \geq 0$, this is

$$\begin{aligned}|g(z)F_n'(z)| &\leq C(1 - r)^{-a/2} \\ &= o\left(\frac{\psi(1 - r)}{1 - r}\right),\end{aligned}$$

if a is small enough.

For $z \in G_3$,

$$F_n'(z) = F_n(z)F^{-1}(z)F'(z) - F_n(z) \cdot \frac{1}{\pi} \int_{CB_n} \frac{e^{i\theta}}{(e^{i\theta} - z)^2} \log|F(e^{i\theta})| \, d\theta$$

The second term is bounded as in G_2. But Lemma 1 with $\eta = \frac{1}{2}$ yields

$$|F_n(z)F^{-1}(z)| \leq c|F(0)|^{-2},$$

so the first term is bounded by

$$C|F'(z)| = o\left(\frac{\psi(1 - r)}{1 - r}\right).$$

That completes the proof.

Since $g \in X$ it follows from Lemma 2 and the fact that $F_n \in H^\infty$ that $g_n f = gF_n \in X$. Since $F_n \to 1$ uniformly on compact subsets of D; it follows that $F_n' \to 0$ uniformly on compact subsets, and so to show that $g_n f \to g$ in X it is enough to estimate

$$\frac{M_\infty(((g_n(1-f))')}{\psi(1-r)}$$

for $R < r < 1$, where R is close to 1. But $g_n f - g = g(F_n - 1)$, so the derivative is $g'(F_n - 1) + gF_n'$. The first term is dominated by $2M_\infty(g', r)$ while the estimate of Lemma 2 takes care of the second term. It follows that $g_n f \to f$ in X.

4. To show that $g_n f \in I(f)$ requires a bit more effort. Suppose that $I_k = (e^{i\alpha_k}, e^{i\beta_k})$ for $k = 1, 2, \ldots, n$. For $\delta > 0$ let

$$\psi_\delta(z) = \frac{z-1}{z-1-\delta}.$$

Let

$$\Phi_\delta(z) = \prod_{k=1}^{n} \psi_\delta^2(ze^{-i\alpha_k})\psi_\delta^2(ze^{-ik\beta_k}).$$

It is easy to show that $g_n f\Phi_\delta \to g_n f$ in X as $\delta \to 0$, so it will suffice to show that $g_n F\Phi_\delta = gf_n\Phi_\delta \in I(f)$. To this end, let

$$D_\varepsilon = \bigcup_{k=1}^{n} [\alpha_k + \varepsilon, \beta_k - \varepsilon]$$

for small ε, and let

$$F_\varepsilon(z) = \exp\left(\frac{1}{2\pi}\int_{D_\varepsilon} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|f(e^{i\theta})|\,d\theta\right),$$

and

$$\Phi_{\delta,\varepsilon}(z) = \prod_{k=1}^{n} \psi^2(ze^{-i(\alpha_k+\varepsilon)})(ze^{-i(\beta_k-\varepsilon)}).$$

It follows from Lemma 2 that $\Phi_{\delta,\varepsilon}F_\varepsilon^{-1} \in X$, and so $gf\Phi_{\delta,\varepsilon}F_\varepsilon^{-1} \in I(f)$, and by standard arguments that $gf\Phi_{\delta,\varepsilon}F_\varepsilon^{-1} \to gF_n\Phi_\delta$ as $\varepsilon \to 0$. Hence $gF_n = g_n f \in I(f)$ and the proof of the Theorem is complete.

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