ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 4, Winter 1992

SOME RESEARCH PROBLEMS ABOUT ALGEBRAIC DIFFERENTIAL EQUATIONS II

BY

LEE A. RUBEL[1]

Dedicated to my son and daughter, Mark and Natasha

In 1983, I published (see [RUB-I]) a collection of research problems about ADE's. Since then, some of the problems have been solved, some have resisted all attacks, and a number of new ones have been generated. In this paper, which I have titled as Part II of the original one, I bring the project as up to date as I can. We refer the reader to the "Explanatory section" of Part I for some of the definitions and notation, as well as for specific problems described there. I also pose a number of new problems. As in Part I, I surround the new problems with descriptive material, including background and partial solutions, where they are available.

I repeat here the closing paragraph of Part I.

"In summary, these are some aspects of the subject as I have expressed them in problems. I hope not too many of them turn out to be embarrassingly easy. At any rate, I have hard going making limited progress on a few. I hope people who find complete or partial solutions will communicate them to me and to others. Finally, I hope I have shown that, just a little off the beaten track in differential equations, there lies a fascinating vein of mathematics."

1. Problems that have been (partially) solved

(Numbers of problems in this section refer to Part I of this paper [RUB-I].)

Of course, Problems 1 and 3 were already solved in Part I. We recall:

PROBLEM 2. *What kind of gaps can the power series of a solution of an* ADE have?

Several partial solutions to this problem were later given in [LIR-I]. The most complete of these results goes as follows. Let $\{n_k\}$ be an increasing

Received January 24, 1991

1991 Mathematics Subject Classification. Primary 34-02.

[1]The author's research was partially supported by a grant from the National Security Agency.

sequence from $\mathbf{N}$. Define

$$\Delta_k = n_{k+1} - n_k,$$

and let $d(k)$ be the largest integer $\leq k$ such that

$$\Delta_k \geq n_{d(k)}$$

if such an integer exists, and 0 otherwise.

THEOREM. *Let $f(z) = \sum_{k=0}^{\infty} f_{n_k} z^{n_k}$ with the $f_{n_k} \in \mathbf{C}$, $f_{n_k} \neq 0$, and suppose that*

$$\lim_{k\to\infty} d(k)/k = 1. \tag{1.1}$$

Then f is not differentially algebraic.

(Note. We abandon the term "hypotranscendental" for the more usual term "differentially algebraic", (DA). They both mean that the object in question satisfies a non-trivial algebraic differential equation.) A series with Hadamard gaps ($\lim n_{k+1}/n_k > 1$) certainly satisfies (1.1) so that a DA power series cannot have Hadamard gaps (unless it is a polynomial). For handling smaller gaps, there is a convenient condition which implies 1.1. Suppose that $|n_k - \varphi(k)| < \mathscr{C}$, where $\mathscr{C}$ is some constant, and suppose that $\varphi(x) = \exp \rho(\log x)$, where $\rho'(x) \uparrow \infty$ and $\rho'(x)/x \to \infty$. Then condition (1.1) holds. A special case of this is

$$\varphi(x) = e^{(\log x)^{2+\varepsilon}}, \quad \varepsilon > 0.$$

Going now to Problem 7, there is an example in [GOL], of which I was unaware until recently, of a DA function that is analytic exactly on the complement of a certain Cantor set, so that the answer to Problem 7 is "yes", contrary to what I initially expected. This throws new light on Problems 5 and 6.

We recall:

PROBLEM 8. *If $w(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies an* ADE, *$P(z, \overrightarrow{w(z)}) = 0$, must there exist another* ADE, *$Q(z, \overrightarrow{s(z)}) = 0$, that is satisfied by every partial sum $s(z) = s_n(z) = \sum_{k=0}^{n} a_k z^k$?*

Some definite progress was made in [BRU], in which examples were given where the answer is "no". In the special case, though, where P is *linear* (i.e., of degree 1), then it was shown there that the answer is "yes". It was shown there also that if the partial sums *do* all satisfy one and the same ADE, then the sequence (a_n) of coefficients, even when the index set is restricted to an arbitrary arithmetic progression, must satisfy an algebraic *difference* equation.

Problem 9 asked whether there exists an algorithm to decide whether a given ADE has a global solution on a given open interval I. A partial answer, depending on the context, was given in [DEL-I, II]. (Much earlier, [JAS] outlined a proof of a related result.) First of all, Denef and Lipshitz provided an algorithm to decide when a given system of ADE's, say with rational coefficients, has a formal power series solution. Singer earlier proved in [SIN] that there exists no algorithm to decide whether it has a non-zero power series solution. [DEL-I] shows that there is no algorithm to decide whether it has a power series solution that converges in some neighborhood of the origin, but it does provide an algorithm to decide whether the system has a solution that is C^∞ in some neighborhood of the origin. They show [DEL-II] that there is no algorithm for deciding whether the radius of convergence of a solution is < 1 or ≥ 1. Moreover, they have related results about systems of algebraic *partial* differential equations that we won't describe here. Finally, we mention an earlier undecidability result of Adler in [ADL].

An affirmative answer to Problem 12 was given in [RUS], in a much more general form. This requires the notion of a DA function of *several* variables, say $F(x_1, \ldots, x_n)$. This means that it is DA as a function of each variable when the others are held fixed.

THEOREM [RUS]. *Let $G(t_{-1}, t_0, t_1, \ldots, t_n)$ be a non-trivial analytic differentially algebraic function of its $n + 2$ variables, and let $y(x)$ be an analytic solution, on an interval I, of the differential equation*

$$G(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0.$$

Then $y(x)$ solves some algebraic differential equation on I.

An extension of this result to the C^∞ case is given in Corollary 3.1., p. 50, of [HOR].

Problem 16 has been partially answered in [LAO], the Ph.D. thesis of Vichian Laohakosol, written under my direction. Problem 17 is partially answered in [LIR-I]. Problem 19 has not been solved yet, but an ADE with "almost" the required properties has been described by Boshernitzan in [BOS]. Namely, for every compact interval I in $\mathbf{R}$, and for every real continuous function φ on I, and for every $\varepsilon > 0$, there exists a *polynomial* solution p of the ADE such that $|p(x) - \varphi(x)| < \varepsilon$ for all $x \in I$. Problem 23 was partially answered in [LIR-I], but most of it remains untouched.

2. New problems

A theme that runs through many of these problems is—what would classical complex analysis look like in the context of *differentially algebraic* analytic functions only? Thus for example, Problem 33 asks for a differentially-algebraic form of the Riemann mapping theorem.

The first three questions are closely related. The second and third appeared in Part I as Problems 21 and 20, respectively, but I renumber them here because of their importance and apparent difficulty—I have a lot to say about Problem 28 especially, although I am no closer to a solution. I rephrase Problem 21 in a more emphatic form.

Let $\mathscr{E}$ denote the space of all entire functions in the topology of uniform convergence on compact sets in the complex plane. Let $\mathscr{E} \cap \mathrm{DA}$ denote the space of those entire functions that are differentially algebraic. It is easy to see that a subset S of $\mathscr{E}$ is σ-bounded (a countable union of bounded sets) if and only if there exists a countable number $\varphi_1, \varphi_2, \varphi_3, \ldots$ of positive continuous functions on $\mathbf{C}$ such that for each $f \in S$, there exists an $n \in \mathbf{N}$ such that $|f(z)| \leq \varphi_n(z)$ for all $z \in \mathbf{C}$.

PROBLEM 26. *Is $\mathscr{E} \cap \mathrm{DA}$ σ-bounded in $\mathscr{E}$?*

We rephrase this more colloquially.

PROBLEM 26′. *Is there any a priori restriction on the growth of a differentially algebraic entire function?*

We now go to specific conjectured bounds. Let $e_1(x) = \exp x$, $e_{n+1}(x) = \exp(e_n(x))$.

PROBLEM 26″. *If $f \in \mathscr{E} \cap \mathrm{DA}$ satisfies an n-th order* ADE, *must there exist constants n, A, and α such that*

$$|f(z)| \leq A e_n(|z|^{\alpha})? \tag{2.1}$$

On page 47 of [BOR], a weaker statement than this is apparently asserted. But it is not clear in [BOR] just what is proved, what is asserted to be proved, or what is just asserted. Suffice it to say that counterexamples have since been found to one of the major assertions there—see [BAN], [BBV], [RUB-II], and [VIJ].

A function that violates all the bounds (2.1) is called "transexponential". So our question is whether there exists a transexponential DA entire function. Given an entire function f, we define the iterates of f by $f^{[1]}(z) = f(z)$, $f^{[n+1]}(z) = f(f^{[n]}(z))$. One approach to a "no" answer to Problem 26 would be to produce a reasonably large f whose iterates all satisfy one and the same ADE. In the terminology of [BRU], the iterates of f form a coherent (or uniformly differentially algebraic) family. The "Bourbaki polynomials" z^m have this property, as well as the Tchebycheff polynomials $T_m(z) = \cos(m \arccos z)$, and A.E. Eremenko has communicated a proof to me that these are the only (suitably normalized) polynomials whose iterates are coherent. (See [BRU] for the quadratic case.)

PROBLEM 27. *Does there exist a transcendental entire function whose iterates are uniformly differentially algebraic?*

Some work related to Problem 27 can be found in [KNE]. Let us return to Problem 26. For reasons that are probably improper, I entertain from time to time the possibility that it is independent of the ZFC axioms of set theory. One reason why such independence is not too implausible is the close connection (see [SHA], [POE], [LIR-II]) between ADE's and analog computers. Be that as it may, a positive answer to Problem 26 would settle Problems 28 and 29 which we now state.

PROBLEM 28 (Old Problem 21). *Given a sequence (z_m) of distinct complex numbers such that $z_n \to \infty$, and any sequence (w_n) of complex numbers, does there exist a differentially algebraic entire function f such that $f(z_n) = w_n$ for all $n = 1, 2, 3, \ldots$?*

This problem is associated with corresponding problems for functions holomorphic on other domains than the full complex plane. It is a classical result that the answer is "yes" if the "differentially algebraic" requirement on f be dropped.

Remark. The problem in Problem 28 is one of *simple* interpolation. The corresponding problem with the multiplicites counted has a negative solution, as we now show. The (z_n) are as before, but the (w_n) now become $(w_{n,j})$, $n = 1, 2, 3, \ldots,\ j = 0, 1, \ldots, j_n$, and the condition on f is that $f^{(j)}(z_n) = w_{n,j}$ for all n and j as above. Now choose $(w_{n,j})$ so that the set of $(p+1)$-tuples $(w_{n,0}, w_{n,1}, \ldots, w_{n,p})$, with $j_n \geq p$ of course, is dense in $\mathbf{C}^{p+1}$, for $p = 0, 1, 2, \ldots$. Suppose now that there was an interpolating function f that was DA. Then f would satisfy not only an ADE but an *autonomous* one:

$$P(f(z), f'(z), \ldots, f^{(p)}(z)) = 0.$$

Consequently

$$P(w_{n,0}, w_{n,1}, \ldots, w_{n,p}) = 0$$

for all large n. Thus P vanishes on a dense subset of $\mathbf{C}^{p+1}$ and thus $P = 0$, and the result is proved by contradiction.

PROBLEM 29 (Old Problem 20). *Given a sequence z_n (counting multiplicities) that approaches ∞ in the complex plane, must there exist a* DA *entire function f whose zero-set is exactly (z_n)?*

We remark again that even the case $z_n = n$ for $n = 1, 2, 3, \dots$ is still undecided. For the present, we will discuss Problem 28 in extenso. First, we rephrase the problem. Let $D = \mathscr{E} \cap \mathrm{DA}$ and let $T = \{f \in \mathscr{E}\colon f(z_n) = 0$ for $n = 1, 2, 3, \dots\}$. Then we have, equivalently to Problem 28:

PROBLEM 28. *Is $\mathscr{E} = D + T$ (as a sum of subspaces)?*

This may seem unlikely since T is just an "average" closed subspace of $\mathscr{E}$, while D is a very small (non-closed) subspace of $\mathscr{E}$. It was shown in [RUB-III] that D is of Baire category 1 in $\mathscr{E}$. Here we prove a stronger result.

DEFINITION. Let V be a topological vector space and D a linear subspace of V. We say that D is of Baire category zero in V to mean that the intersection of D with any infinite-dimensional subspace E of V is of Baire category one.

THEOREM. DA *is of Baire category* 0 *in* $\mathscr{E}$.

Proof. Let M be a subspace of $\mathscr{E}$. We must prove that either $\mathrm{DA} \cap M$ is Baire-1 or is finite-dimensional. Suppose that, $\dim(M \cap \mathrm{DA}) = \infty$. By the Ritt-Gourin Theorem, every function in DA satisfies an algebraic differential equation $P = 0$ where P is a differential polynomial with integer coefficients. We can enumerate such P. Let $\mathscr{N}$ be a neighborhood in $\mathrm{DA} \cap M$. Then N has the form

$$\mathscr{N} = \{f \in \mathrm{DA} \cap M\colon |f(z) - f_0(z)| < \varepsilon,\ z \in K\},$$

where $\varepsilon > 0$ and K is a compact set in $\mathbf{C}$. Let $P^{\perp}$ be the set of entire functions annihilated by P. Clearly $P^{\perp}$ is closed. It is enough to prove that for any differential polynomial P, $P^{\perp}$ contains no neighborhood $\mathscr{N}$. Suppose it did. Then $f \in M \cap \mathrm{DA}$, $|f - f_0| < \varepsilon$ on K, implies $P(\vec{f}) = 0$. Take $g \in M \cap \mathrm{DA}$ and $|\varepsilon|$ really small. Then $P(f_0 + \varepsilon g) = 0$. But this is a polynomial in ε that vanishes on a disc in $\mathbf{C}$, and hence vanishes identically. So $P(f_0 + \varepsilon g) = 0$ for all $\varepsilon \in \mathbf{C}$, and $g \in M \cap \mathrm{DA}$. Hence $P(f)$ vanishes for all $f \in M \cap \mathrm{DA}$. The desired result follows from the next lemma. (We call a differential polynomial P autonomous if $P(f) = P(f(z), f'(z), \dots, f^{(N)}(z))$ (i.e., P does not explicitly mention the independent variable z) and we call N the *order* of P. It is a well-known fact (involving resultants), that we may take the polynomials P in the conclusion of the Ritt-Gourin theorem to be also *autonomous*.)

LEMMA. *If P is an autonomous differential polynomial such that $P^{\perp}$ contains an infinite-dimensional vector space in $\mathscr{E}$, then $P \equiv 0$.*

Proof. Call the vector space V, and take $f_0, f_1, \dots, f_N$ linearly independent in V, where N is the order of P. (So we will actually prove that $\dim V \leq N$). Let

$$W(z) = \det\left[f_i^{(j)}(z)\right]_{\substack{i=0,1,\dots,N \\ j=0,1,\dots,N}}.$$

Since we are dealing with *analytic* functions f_i, we know (see [KOL]) that $W(z)$ is *not* $\equiv 0$. Let $\mathscr{Z} = \{z : W(z) = 0\}$. For $z \in \mathbf{C} \setminus \mathscr{Z}$ and any complex numbers $w_0, w_1, \dots, w_N$, solve

$$\begin{aligned}
\varepsilon_0 f_0(z) \quad &+ \cdots + \quad \varepsilon_N f_N(z) = w_0 \\
\varepsilon_0 f_0'(z) \quad &+ \cdots + \quad \varepsilon_N f_N'(z) = w_1 \\
&\vdots \\
\varepsilon_0 f_0^{(N)}(z) \quad &+ \cdots + \quad \varepsilon_N f_N^{(N)}(z) = w_N
\end{aligned}$$

where $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in \mathbf{C}$. We then have $P(w_0, w_1, \dots, w_N) = 0$. Let

$$f(z) = \varepsilon_0 f_0(z) + \cdots + \varepsilon_N f_N(z),$$

supposedly a solution of $P = 0$. Since $w_0, w_1, \dots, w_N$ are arbitrary, we have $P \equiv 0$.

Remark. Jean Saint-Raymond has written to me that in case V is a separable metrizable vector space and D is an analytic linear subspace of V, then D is of Baire category zero if and only if D contains no complete infinite-dimensional subspace of V. At one time it seemed to me that one could not have $\mathscr{E} = D + T$ where D is an algebra of Baire category zero and T is a closed subspace of infinite codimension in $\mathscr{E}$, the space of all entire functions discussed above. However, Saint-Raymond has sent me an example of this, where $T = \{f \in \mathscr{E}: f(n) = 0,\ n = 0, \pm 1, \pm 2, \dots\}$. So the approach by category seems to be a false lead.

Here is what seems to be a promising approach, but, so far, nobody has been able to make it work. The idea is to consider a DA function as a function of the initial conditions of an ADE satisfied by the function. To make matters simpler (and a Baire category argument reduces it almost to this), suppose (z_n) given and suppose that there is *one* differential polynomial P such that for any sequence (w_n) there is an entire solution of $P(\vec{f}) = 0$ such that $f(z_n) = w_n$ for $n = 1, 2, 3, \dots$. For simplicity, we will take $z_n = n$ for all $n = 1, 2, 3, \ldots$. Let p be the order of P and let N be much larger than p. Along with the equation $P(\vec{g}) = 0$ consider the initial conditions $g(0) = c_0$, $g'(0) = c_1, \ldots, g^{(p)}(0) = c_p$. Putting aside certain difficulties for the moment, consider g as a function of $c_0, c_1, \ldots, c_p$, say $g = g(z : c_0, c_1, \ldots, c_p)$. Look at the mapping F: $\mathbf{C}^{p+1} \to \mathbf{C}^N$ given by

$$F(c_0, c_1, \ldots, c_p) = (g(1), g(2), \ldots, g(N)).$$

By the interpolation hypothesis, F maps *onto* $\mathbf{C}^N$—in other words, F is a *dimension-raising* map. This seems extremely implausible. True, there do exist dimension-raising maps (think of a space-filling curve), but they cannot have any of the analyticity properties we would expect F to have.

Let us be a little critical of the "argument" just sketched. First of all, F need not be defined everywhere. Here is an extreme example of this, due to Leonard Lipshitz. Consider the functions

$$f_\lambda(z) = \frac{\lambda e^{z-1} - 1}{z - 1},$$

where λ is a parameter. It is easy to write down a first-order ADE whose solutions are exactly the functions f_λ, and $f_\lambda(0) = 1 - \lambda e^{-1} = \mu$, say. But f_λ is *entire* exactly for *one* value of λ ($\lambda = 1$ or $\mu = 1 - e^{-1}$), and for no others. Moreover, even if we suppose that there is a unique solution to our initial-value problem for every $(p + 1)$-tuple $c = (c_0, c_1, \ldots, c_p)$, it is not clear what kind of a function it is of c. So we state:

PROBLEM 30. *Let $G(z_1, \ldots, z_n)$ be an analytic function of $(z_1, \ldots, z_n)$ in a neighborhood of $(0, 0, \ldots, 0)$ that is differentially algebraic as a function of z_1. Define $\rho = \rho(z_2, z_3, \ldots, z_n)$ to be the radius of convergence in the z_1-plane of the Taylor series for G as a function of z_1. Can you say something stronger about ρ in this case than if you drop the hypothesis about G being differentially algebraic?*

We note what is known in the unrestricted case (see [LEL]). If we let

$$R(z_2, \ldots, z_n) = \log \rho(z_2, \ldots, z_n),$$

then the lower-semicontinuous regularization of R is a plurisuperharmonic function of $(z_2, \ldots, z_n)$. In particular, its set of discontinuities is a pluripolar set. In the case where G is DA in z_1 (perhaps supposing G to be DA in $z_2, \ldots, z_n$ as well), must the set of discontinuities of ρ be contained in a countable union of varieties of dimension less than p? Here is an illuminating example from [DEL-I]. Let

$$f(x, \alpha) = \sum_{n=0}^{\infty} \alpha(\alpha - 1) \cdots (\alpha - n) x^n.$$

It is easy to see that, as a formal power series, $f(x, \alpha)$ is DA in x for each α. But for $\alpha \notin \mathbf{N}$ it has zero radius of convergence, while for $\alpha \in \mathbf{N}$ it has infinite radius of convergence. Another example would be $F(z, w) = f(z)g(w)$, where, say, $f(z) = (1 - z)^{-1}$ and g is a DA entire function. If w_0 is a zero of g, then $F(z, w_0)$ has infinite radius of convergence, otherwise radius of convergence one.

PROBLEM 31. *Given a "nice" initial-value problem for a system of algebraic differential equations in the dependent variables $y_1, \ldots, y_n$, must $y_1(x_0)$ be differentially algebraic as a function of the* **initial conditions**, *for each x_0?*

We won't say more about what "nice" means except that the problem should have a unique solution for each initial condition in a suitable open set. Even a very simple version of the problem, due to Lou van den Dries, seems very elusive.

PROBLEM 31. *Consider the initial-value problem*

$$\begin{cases} y' = P(x, y) \\ y(0) = t, \end{cases}$$

where P is a polynomial in (x, y). Suppose that it is "nice" so that there is a unique and well-behaved solution on the interval $[0, 1]$. *Write*

$$\varphi(t) = y(1).$$

Must φ be differentially algebraic?

About Problem 28, I have proved in [RUB-IV] that if simple interpolation *is* possible in DA then DA is not elementarily equivalent (as a ring) to $\mathscr{E}$.

Consider the following Bezout problem about DA.

PROBLEM 32. *If f and g are differentially algebraic entire functions with no common zeros, must there exist differentially algebraic entire functions a and b such that $af + bg = 1$?*

Note that if we choose $f = \cos \pi z$ and $g = \cos \pi\lambda z$ where λ is extremely Liouvillean—i.e., has very close rational approximations, then b would have to be extremely large on some of the zeros of f. This reflects on Problem 26. Problem 32 is closely related to Problem 28 since $af + bg = 1$ will hold if b interpolates $1/g$ (counting multiplicities) on the zeros of f. (Choose $a = (1 - bg)/f$).

We now turn our attention to some unrelated problems.

PROBLEM 33. *Let G be a "nice" region, say $G = \{(x + iy) \in \mathbf{C}\colon y < p(x)\}$, where $p(x)$ is a given polynomial, and let f map G conformally onto the lower half-plane $\{(x + iy) \in \mathbf{C}\colon y < 0\}$. Must f be differentially algebraic?*

Here, the meaning of "nice" is that the boundary of G is given by algebraic conditions. We mention in this connection the oval of Cassini

$$G = \{(x, y) \in \mathbf{C}\colon x^4 + y^4 < 1\}$$

for whose mapping function f Lummer in his 1920 Leipzig thesis found an explicit formula which implies that f is DA. Even the case

$$G = \{(x + iy) \in \mathbf{C}\colon y < x^3\}$$

has not been settled.

Hilbert's 13th problem, (see [LOR]) about superpositions of functions, turned out to be stated too broadly. After all, Hilbert was principally interested in nomograms for certain seventh-degree algebraic functions so that the positive solution of Kolmogorov-Arnold in the context of merely continuous functions may not have too much bearing on Hilbert's actual intent. The following problem is due to Leonard Lipshitz.

PROBLEM 34. *If $F(x, y)$ is a differentially algebraic (and real-analytic?) function of x and y, can we write F as a superposition of differentially algebraic functions of one variable, with $+$ being the only allowed function of two variables?*

The restriction to just two variables x and y is only for simplicity.

PROBLEM 35. *If $f(z_1, z_2, \ldots, z_n)$ is an entire function on $\mathbf{C}^n$ that is irreducible as a differentially algebraic function, must it be irreducible as an entire function?*

Lou van den Dries has shown me a proof that if $f(z_1, \ldots, z_n)$ is an irreducible *polynomial*, then it is irreducible as an entire function.

The next cluster of problems has to do with different kinds of convolutions of DA functions. The first kind, Hadamard convolution, is defined by

$$\sum a_n x^n * \sum b_n x^n = \sum a_n b_n x^n.$$

In [LIR], two DA power series $\Sigma a_n x^n$ and $\Sigma b_n x^n$ were given whose Hadamard convolution was not DA.

PROBLEM 36. *Describe the class of Hadamard convolutions of differentially algebraic power series.*

This is closely related to the next problem, about *diagonals*—the diagonal of $\Sigma a_{mn} x^m y^n$ is $\Sigma a_{mm} x^m$. In [LIP], it was proved that the diagonal of a *linearly* differentially algebraic power series in two variables is linearly differentially algebraic, but in [LIR-I] this assertion was shown to be false if the word "linearly" is omitted in both places.

PROBLEM 37. *Describe the class of diagonals of differentially algebraic power series in several variables.*

PROBLEM 38. *The same as Problem* 36, *but with* f, g *real analytic and differentially algebraic functions on* $[0, \infty)$ *and*

$$(f * g)(x) = \int_0^x f(x - t) g(t)\, dt.$$

The next problem concerns the general-purpose analog computer (GPAC) (see [SHA] and [POE]) which may be presented as an initial-value problem,

$$\begin{cases} \displaystyle\sum_{i, j, k = 1, 2, \ldots, n} c_{ijk} y_j \frac{dy_k}{dx} = 0 \\ y_i(0) = c_i, \qquad i = 2, 3, \ldots, n \end{cases}$$

with $y_1(x) \equiv x$.

PROBLEM 39. *If all the* $c_{i,j,k}$ *and the* c_i *are rational, is there an algorithm to decide whether the machine runs for a positive length of time? (That is, whether there is an interval* $[0, \varepsilon]$, $\varepsilon > 0$, *with a solution existing on it.)*

It was shown in [DEL-II] that the answer is "no" for *general* systems of ADE's but the above system is special—it is quasilinear and first-order.

We now turn to a version of Church's Thesis that all reasonable definitions of computability boil down to the same notion. It was originally intended in the context of discrete (i.e. *digital*) computers, but has recently been broadened to include continuous (i.e. *analog*) computers. In [RUB-V], it was shown that digital simulation of a large class of GPAC's is always possible.

PROBLEM 40. *Is analog simulation of digital computation always possible?*

This problem can be interpreted in various ways. One of them is, given a recursive sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ of zeros and ones, to construct a GPAC with rational data one of whose outputs $u(x)$, defined for all $x > 0$, satisfies

$$|u(n) - \varepsilon_n| < 1/10 \quad \text{for } n = 1, 2, 3, \ldots .$$

Leonard Lipshitz has shown me a proof that one can, given $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$, always choose a $\lambda \in \mathbf{R}$ so that

$$u(x) = \sin \lambda e^{e^x}$$

works. The trouble with this is that λ may be some crazy irrational number.

We now turn our attention to the (non-separable) Banach space $H^\infty(\mathbf{D})$, consisting of all bounded analytic functions in $\mathbf{D} = \{z \in \mathbf{C}: |z| < 1\}$, with the norm

$$\|f\| = \sup\{|f(z)|: z \in \mathbf{D}\}.$$

PROBLEM 41. *Is $H^\infty \cap$ DA dense in H^∞?*

Here "$H^\infty \cap$ DA" is shorthand for "the differentially algebraic functions that are bounded and holomorphic in $\mathbf{D}$". The non-separability presents us, for example, from simply taking the polynomials (which are certainly DA) with rational coefficients as a countable dense subset of $H^\infty(\mathbf{D})$. Problem 41 is due to Alan L. Horwitz and me, jointly.

Fatou's Theorem (see [HOF]) says that for each $f \in H^\infty(\mathbf{D})$

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

(the "radial boundary value of f") exists for almost all $\theta \in [-\pi, \pi]$.

PROBLEM 42. *If $f \in H^\infty(\mathbf{D}) \cap$ DA, must $f^*(e^{i\theta})$ exist with at most countably many exceptional θ?*

If the answer to Problem 42 is "yes", then the answer to Problem 41 is "no", for the class of all f in $H^\infty(\mathbf{D})$ with at most countably many exceptional θ is closed in the H^∞-norm. With a little work, I have produced an $f \in H^\infty(\mathbf{D}) \cap$ DA where the exceptional set is countably infinite. This problem may be related to the result in [GOL] mentioned earlier in connection with Problem 7. It may also be related to the following problem, which also came up in connection with my work on Problem 30.

PROBLEM 43. *Let μ be a singular but continuous measure in the plane. Can the logarithmic potential $\hat{\mu}$ of μ be differentially algebraic?*

Here

$$\hat{\mu}(x, y) = \frac{1}{2} \int\int_{\xi, n} \log \frac{1}{(x-\xi)^2 + (y-n)^2} \, d\xi \, dn.$$

Similarly:

PROBLEM 44. *Let μ be a singular but continuous measure on the unit circle. Can the Poisson integral P_μ of μ be differentially algebraic?*

Here,

$$P_\mu(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \, d\mu(\varphi).$$

Special cases of this, where μ is a Cantor measure, lead to:

PROBLEM 45. *Is $f(x) = \prod_{n=0}^{\infty}(1 + x^{3^n})$ differentially algebraic?*

Note that this f satisfies the functional equation $f(x^3) = f(x)/(1+x)$.

Note also that $f(x) = \sum \varepsilon_n x^n$ where $\varepsilon_n = 1$ if the triadic expansion of n involves only 0's and 1's: otherwise $\varepsilon_n = 0$.

Since this paper was written, Bernard Randé of Bordeaux has solved Problem 45, as well as some closely related problems, in the *negative*.

PROBLEM 46. *Can a Riesz product*

$$f(x) = \prod_{\nu=0}^{\infty} (1 + a_\nu \cos n_\nu x)$$

$$-1 \le a_\nu \le 1, \; a_\nu \ne 0 \textit{ for all } \nu, \; n_{\nu+1}/n_\nu \ge 3$$

satisfy an algebraic differential equation?

Consideration of the ε_n in connection with Problem 45 leads to the following problem.

PROBLEM 47. *Let $f(x) = \sum_{n=0}^{\infty} \varepsilon_n x^n$, $\varepsilon_n = 0, 1$, and suppose that every initial block of ε_n's occurs infinitely often in the sequence of ε_n's. If $f(x)$ is differentially algebraic, must it be a rational function?*

Notice that for the $f(x)$ of Problem 45, we have $\varepsilon_n = 0$ for $2 \cdot 3^k \le n < 3^{k+1}$. Except for an irksome side condition, a theorem in [OSG] would show that $f(x)$ could therefore not be DA. This side condition could be handled easily if the discussion of Problem 8 in this Part II could be favorably resolved.

PROBLEM 48. *If S is the spectrum of a differentially algebraic power series, must S then be a finite union of arithmetic progressions, give or take a set of arithmetic density zero?*

(The spectrum of $\Sigma a_n x^n$ is $S = \{n\colon\ a_n \neq 0\}$.) This is related to the Skolem-Mahler-Lech Theorem (see [LAO] and our remarks about Problem 16).

PROBLEM 49. *What can you say about the spectrum of a differentially algebraic Laurent series?*

(You must put some convergence restrictions on the series in order to be able to meaningfully plug it into an ADE.)

PROBLEM 50. *Suppose Σ is a system of algebraic partial differential equations for which every C^∞ solution on $\mathbf{R}^n$ is real-analytic. Then must every solution which is differentiable enough to plug in be real-analytic on a dense open set in $\mathbf{R}^n$?*

Example. Let the system Σ be

$$(u_x - 2x)(u_x + 2x) = 0, \quad u_y = 0.$$

The C^∞ solutions are $u = x^2 + c$, $u = -x^2 + c$, which are certainly real-analytic. But Σ has the C^1-solution

$$u = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

which is real-analytic off $\{x = 0\}$.

For the next cluster of problems, I suppose some familiarity with (Nevanlinna) value-distribution theory (see [HAY].)

PROBLEM 51. *What does Nevanlinna theory look like if we restrict it to differentially algebraic entire or meromorphic functions?*

To be more specific:

PROBLEM 52. *Does there exist a differentially algebraic entire function of non-rational order, say π or $\sqrt{2}$?*

PROBLEM 53. *How about the defects of a differentially algebraic entire or meromorphic function? Can* $\delta(a, f) = 1/\sqrt{2}$ *say, for such a function?*

Since this paper was written, I discovered that it is possible to have $\delta(0, f)$ *transcendental* for a differentially algebraic entire function. Consider Dugué's function

$$w(z) = \frac{e^{2\pi i e^{z}} - 1}{e^{2\pi i e^{-z}} - 1}$$

(see R. Nevanlinna, Eindentige Analytische Funktionen, 2nd edition, Springer-Verlag, Berlin, 1953, pp. 274–275, §230). Let $f(z) = w(z - 1)$. It is shown that

$$\frac{1 - \delta(0, f)}{1 - \delta(\infty, f)} = e^{2}$$

so that at least one of $\delta(0, f)$ and $\delta(\infty, f)$ must be transcendental. (If it is $\delta(\infty, f)$, replace f by $F = 1/f$ to get $\delta(0, F)$ transcendental.)

PROBLEM 54. *Suppose f is a differentially algebraic meromorphic function on* **C**. *Can we write $f = g/h$ where g and h are entire and differentially algebraic?*

This is related to Problem 20. The answer is classically affirmative if the adjective "differentially algebraic" is dropped entirely.

We now turn to a question about orthogonal polynomials. Let $w(x)$ be a positive weight on $[0, 1]$, say, and let $\{p_n\}$ be the orthogonal polynomials with respect to w, i.e.

$$\int_0^1 p_m(x) p_n(x) w(x)\, dx = \delta_{m,n}.$$

PROBLEM 55. *For what weights $w(x)$ will there be a single algebraic differential equation satisfied by every p_n, $n = 0, 1, 2, \ldots$?*

A famous theorem of Bers (see [LUR]) says that if G and G' are two regions in **C**, then G is conformally (or anticonformally) equivalent to G' if and only if $H(G)$ is isomorphic to $H(G')$, where $H(G)$ is the ring of holomorphic functions on G. This has been extended to $M(G)$, the field of meromorphic functions on G, in [ISS] by Hironaka.

PROBLEM 56. *Suppose* DA(G) *is isomorphic to* DA(G'). *Must G be conformally or anticonformally equivalent to G'? In the conformal case, must the mapping be differentially algebraic?*

Here, DA(G) is the ring of all differentially algebraic holomorphic functions on G. There is a companion problem for meromorphic functions.

The following result (Lemma 3.2 on page 142 of [BHR]) has proved useful (see [NAN]) in dealing with ring homomorphisms of algebras of holomorphic functions. It is a minor variant of a result in [NIT].

THE "DUTCH" RESULT. *If $\beta \in \mathbf{R}$ and $\alpha \in \mathbf{C} \setminus \mathbf{R}$, then there exists an entire function f such that $f(\beta) = \beta$, $f(\alpha) \neq \alpha$, and f maps $\mathbf{Q}$ onto $\mathbf{Q}$, and the restriction of f to $\mathbf{R}$ is strictly increasing.*
($\mathbf{Q}$ is not essential here—any countable dense subset of $\mathbf{R}$ will do.)

PROBLEM 57. *Can we moreover make f differentially algebraic?*

PROBLEM 58. *Suppose f and g are entire and differentially algebraic functions and suppose that $h = f/g$ is entire. Can we find a non-trivial entire and differentially algebraic function k so that gk is "simpler" than fk?*

To say that G is "simpler" than F is to say that G satisfies a simpler ADE than any satisfied by F. An illuminating example, due to Ozawa, is $f = \sin \pi z^2$, $g = \sin \pi z$.

PROBLEM 59. *Does there exist an entire function f such that $f(z^2)/f(z)$ is differentially algebraic, but $f(z)$ is not differentially algebraic?*

PROBLEM 60. *Does there exist a differentially algebraic entire function f such that the sequence $f, f', f'', \ldots$ is dense in the space of all entire functions, in the topology of uniform convergence on compact subsets of $\mathbf{C}$?*

This was done, *without* the DA requirement, in [MAC] (see also [BLR-I] and [BLR-II].) Notice that if $f(z) = e^{-z^2}$ then the n-th derivative of f is just e^{-z^2} multiplied by the n-th Hermite polynomial, so that the *span* of $f, f', f'', \ldots$ is dense, while e^{-z^2} is certainly DA.

For the next problem, we require the notion from [RUB-VI] of an N-solution of an ADE. We say that $u(x)$ is locally an N solution of $P = 0$ at x_0 if there is an open interval I containing x_0 such that, for some differential polynomial $Q = Q_I$, u is on I an actual solution of $Q = 0$ and $P \leq Q$, which means that every analytic solution of $Q = 0$ is a solution of $P = 0$. By the differential Nullstellensatz, (see [RIT-I]) $P \leq Q$ iff P belongs to the radical differential ideal generated by Q, which means that

$$P^r = \sum A_\alpha Q^{(\alpha)}$$

where the sum is over a finite index set,

$$Q^{(\alpha)} = \frac{d^\alpha}{dx^\alpha} Q,$$

and r is a positive integer. (In reality, one should consider, in place of Q, a *system* Σ of differential polynomials.) We say that u is an N-solution of $P = 0$ on an open interval J if it is locally an N-solution at each $x_0 \in J$.

PROBLEM 61. *Describe all N-solutions of* $y^{(n)} = 0$ *or, more generally, of* $L(y) = 0$ *where L is a linear differential operator with constant coefficients.*

It was shown in [RUB-IV] that every N-solution of $P = 0$ is a spline of actual solutions, and that in the case of $y^{(9)} = 0$, the knots of the spline can form an arbitrary nowhere dense closed set. In [RUB-VII], the N-solutions of $y''' = 0$ were studied, and it was shown that the knots can have a finite limit point.

PROBLEM 62. *In the case of* $y''' = 0$, *can there exist limit points of limit points of the knots?*

Also in [RUB-VII] an example was constructed of two N-solutions u_1 and u_2 of $y''' = 0$, such that $u_1 + u_2$ is *not* an N-solution of this equation. However, $u_1 + u_2$ is not DA in this case.

PROBLEM 63. *If* u_1 *and* u_2 *are N-solutions of* $y^{(n)} = 0$ *and if* $u_1 + u_2$ *satisfies some algebraic differential equation, must* $u_1 + u_2$ *be an N-solution of* $y^{(n)} = 0$?

Even the case $n = 3$ has not been resolved.

We now turn to an investigation of DA forms of Runge's theorem. We recall (see [BRU]) that a sequence (p_n) of functions is called "uniformly differentially algebraic", or "coherent", if there is *one and the same* ADE satisfied by *all* the p_n.

PROBLEM 64. *Given a region G in* $\mathbf{C}$ *without holes, and a differentially-algebraic function f that is holomorphic on G, must there exist a sequence* (p_n) *of polynomials, that is uniformly differentially algebraic, and such that* $p_n \to f$ *uniformly on compact subsets of G?*

PROBLEM 65. *The same problem, but you let G have holes, and you take the* p_n *to be rational functions with poles in the holes.*

Problem 64 is challenging even in case $G = \mathbf{D}$, the open unit disc. In general (see our remarks on Problem 8), we may *not*, in general, take p_n to be the n-th partial sum of the Taylor series of f, as the example $f(z) = z/(e^z - 1)$ shows. However, in this particular case, suitable p_n *can* be constructed.

The following special case of Problem 64 seems almost doable.

PROBLEM 64*. *Given two disjoint compact sets K_1 and K_2 in* $\mathbf{C}$, *with no holes, construct a coherent sequence (p_n) of polynomials such that p_n converges uniformly to 0 on K_1 and uniformly to 1 on K_2.*

I thought I had such a construction but there was a gap in my argument.

PROBLEM 66. *Given a curve $f(x, y) = 0$, where f is differentially algebraic (or even a polynomial), when is there a global parametrization $x = \varphi(t)$, $y = \psi(t)$, where φ and ψ are differentially algebraic functions?*

PROBLEM 67. *Study* DA *manifolds.*

A DA manifold is one for which the coordinate transformations between coordinate patches are (uniformly?) differentially algebraic. For example:

PROBLEM 68. *Are there* DA *spheres that are exotic?*

Ritt showed in [RIT-II] and [RIT-III] that if

$$f(\lambda z) = R(f(z))$$

where R is a rational function, and if f is differentially algebraic, then f is essentially $\cos z$, e^z or $\wp(z)$, where $\wp$ is the Weierstrass p-function.

PROBLEM 69. *How about*

$$f(\lambda z) = R(z, f(z)),$$

where R is a rational function of two variables?

It turns out that both A. Erememko and I have graduate students working on this problem.

My final topic is coercive ADE's. We say that an ADE is coercive on an interval I, if the limit of solutions is again a solution. There are dozens of precise notions of "coercive", depending on the kind of convergence, and on what properties the limit function is supposed to have. One could even interpret "solution" in different ways (see [RUB-VI].) The simplest notion of coercivity, say of $P = 0$, is that the pointwise limit of solutions is again a solution, provided only it is differentiable enough to plug into P. The equation $y' - y = 0$ is clearly coercive in this sense. The universal ADE's of [RUB-II] or [BOS] are very far from being coercive, since "every" function is a uniform limit of solutions.

PROBLEM 70. *Develop the theory of coercive algebraic differential equations.*

For example, give reasonable conditions on an ADE that it be coercive, or see whether one kind of coercivity implies another. A step in the first direction is the following.

THEOREM. *Let*

$$(*) \qquad p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_0(x)y(x) = 0$$

be a linear algebraic differential equation, and let $f_1(x), f_2(x), \ldots$ be solutions of $(*)$ on an interval I, with $f_k(x)$ converging pointwise on I to a function $f(x)$. Then $f(x)$ is a solution of $(*)$.

Proof. Let $I = (-\infty, \infty)$ for convenience of exposition, and let V denote the vector space of all solutions of $(*)$. Put two topologies on V. The first, T_1, is the topology of pointwise convergence. The second, T_2, is the topology of uniform convergence of the function and its first n derivatives on compact subsets of $\mathbf{R}$.

LEMMA. *V is finite-dimensional.*

Given the lemma, which we shall prove at the end, we observe that T_1 and T_2 must coincide, because (see [RUD]) any two linear topologies on V must be the same. Now suppose that $f_1, f_2, f_3, \ldots$ is a sequence of solutions of $(*)$ that converges pointwise to f. Then (f_n) is a T_1-Cauchy sequence and therefore a T_2-Cauchy sequence. Since the f_j are analytic off the zeros of $p_n(x)$, we see that $f', f'', \ldots, f^{(n)}$ exist on $\mathbf{R}$ and that, uniformly on compact subsets of $\mathbf{R}$, $f_j' \to f', f_j'' \to f'', \ldots, f_j^{(n)} \to f^{(n)}$. We may therefore take limits in $(*)$ to see that f is indeed a solution.

Proof of the lemma. Suppose that there are k distinct real zeroes of $p_n(x)$, and let $N = (k + 1)(n + 1)$. Suppose there were $N + 1$ linearly independent solutions $g_0, \ldots, g_N$ of $(*)$. Let $I_1, I_2, \ldots, I_{k+1}$ be the open intervals of $\mathbf{R}$ left after the zeros of p_n are removed from $\mathbf{R}$. Choose $x_j \in I_j$ for each j. Then there is a non-trivial linear combination $g = a_0 g_0 + \cdots + a_N g_N$

of the g_i so that

$$g(x_j) = g'(x_j) = \cdots = g^{(n)}(x_j) = 0 \quad \text{for } j = 1, \dots, k+1.$$

This is because we are placing N linear constraints on a vector space of dimension $N + 1$ (the linear combinations of $g_0, g_1, \dots, g_N$). By a fundamental theorem, g must $= 0$ on each of the I_j, and hence by continuity $g \equiv 0$ on $\mathbf{R}$, a contradiction.

PROBLEM 71. *If f and g are differentially algebraic functions of k variables, and if there exists a non-trivial entire function F of two variables such that $F(f, g) \equiv 0$, then can we choose F to be moreover differentially algebraic?*

One can prove, via the Jacobian matrix that if f and g are polynomials, then we can choose F to be moreover a polynomial.

I will stop here and begin saving problems for a possible third installment, who knows how many years from now.

REFERENCES

[ADL] A. ADLER, *Some recursively unsolvable problems in analysis*, Proc. Amer. Math. Soc., vol. 122 (1969), pp. 523–526.

[BAN] STEVEN B. BANK, *Some results on analytic and meromorphic solutions of algebraic differential equations*, Adv. in Math., vol. 15 (1975), pp. 41–62.

[BBV] N. BASU, S. BOSE and T. VIJAYARAGHAVAN, *A simple example for a theorem of Vijayaraghavan*, J. London Math. Soc., vol. 12 (1937), pp. 250–252.

[BHR] JOSEPH BECKER, C. WARD HENSON and LEE A. RUBEL, *First-order conformal invariants*, Ann. of Math., vol. 112 (1980), pp. 123–178.

[BLR-I] CHARLES BLAIR and LEE A. RUBEL, *A universal entire function*, Amer. Math. Monthly, vol. 90 (1983), pp. 331–332.

[BLR-II] ______, *A triply universal entire function*, Enseign. Math., vol. 30 (1984), pp. 269–274.

[BOR] ÉMILE BOREL, *Mémoire sur les séries divergentes*, Ann. Sci. École Norm. Sup., vol. 16 (1899), pp. 9–136.

[BOS] MICHAEL BOSHERNITZAN, *Universal formulae and universal differential equations*, Ann. of Math., vol. 124 (1986), pp. 273–291.

[BRU] MICHAEL BOSHERNITZAN and LEE A. RUBEL, *Coherent families of polynomials*, Analysis, vol. 6 (1986), pp. 339–389.

[DEL-I] J. DENEF and L. LIPSHITZ, *Power series solutions of algebraic differential equations*, Math. Ann., vol. 267 (1984), pp. 213–238.

[DEL-II] ______, *Decision problems for differential equations*, J. Symbolic Logic, vol. 54 (1989), pp. 941–950.

[GOL] VLADIMIR V. GOLUBEW, *Vorlesungen über Differentialgleichungen im Komplexen*, VEB Deutschen Verlag der Wissenschaften, Berlin, 1958.

[HAY] WALTER K. HAYMAN, *Meromorphic functions*, Oxford Univ. Press, Oxford, 1964.

[HOF] KENNETH HOFFMAN, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[HOR] ALAN L. HORWITZ, *Differentially analytic functions*, Complex Variables, vol. 14 (1990), pp. 45–52.

[ISS] Hej Iss'sa, *On the meromorphic function field of a Stein variety*, Ann. of Math., vol. 83 (1966), pp. 34–46.

[JAS] S. Jaskowski, *Example of a class of systems of ordinary differential equations having no decision method for existence problems*, Bull. Acad. Polonaise Sci., Classes III, vol. 2 (1954), pp. 155–157.

[KNE] Helmut Kneser, *Reelle analytische Lösungen der Gleichung*, $\varphi(\varphi(x)) = e^x$ *und verwandter Funktionalgleichungen*, J. Reine Angew. Math., vol. 187 (1950), pp. 56–67.

[KOL] E.R. Kolchin, *Differential algebra and algebraic groups*, Academic Press, New York, 1973.

[LAO] Vichian L. Laohakosol, *Some extensions of the Skolem-Mahler-Lech theorem*, Exposition. Math., vol. 7 (1989), pp. 137–189.

[LEL] P. Lelong, *Plurisubharmonic functions and positive differential forms*, Gordon and Breach, New York, 1969.

[LIP] Leonard Lipshitz, *The diagonal of a D-finite power series is D-finite*, J. Algebra, vol. 113 (1988), pp. 373–378.

[LIR-I] Leonard Lipshitz and Lee A. Rubel, *A gap theorem for power series that satisfy an algebraic differential equation*, Amer. J. Math., vol. 108 (1986), pp. 1193–1214.

[LIR-II] ______, *A differentially algebraic replacement theorem and analog computability*, Proc. Amer. Math. Soc. 99 (1987), 367–372.

[LOR] G.G. Lorentz, "The 13-th problem of Hilbert" in *Mathematical developments arising from Hilbert problems, I*, Felix E. Browder, ed., Proceedings of Symposia in Pure Mathematics, vol. XXVIII, Amer. Math. Soc., Providence, R.I., 1976, pp. 419–430.

[LUM] Fritz Lummer, *Über der konforme Abbildung bizirkulärer Kurven vierter Ordnung und die analytische Spiegelung an ihnen*, Thesis, Leipzig, 1920.

[LUR] Daniel Luecking and Lee A. Rubel, *Complex analysis: A functional analysis approach*, Springer-Verlag, New York 1984.

[MAC] Gerald MacLane, *Sequences of derivatives and normal families*, J. d'Analyse Math., vol. 2 (1952), pp. 72–87.

[NAN] N.R. Nandakumar, *Ring homomorphisms on algebras of analytic functions*, preprint 1990.

[NIT] J.W. Nienhuys and J.G.F. Thiemann, *On the existence of entire functions mapping countable dense sets on each other*, Proc. Dutch Academy of Science, Ser. A., vol. 79 (1976), pp. 331–334.

[OSG] Charles F. Osgood, *Differential algebra and the simultaneous approximation by rational functions of power series solutions to algebraic differential equations*, Amer. J. Math., vol. 103 (1981), pp. 469–497.

[POE] Marian B. Pour-El, *Abstract computability and its relation to the general purpose analog computer* (*some connections between logic, differential equations, and analog computers*), Trans. Amer. Math. Soc., vol. 199 (1974), pp. 1–28.

[RIT-I] Joseph Fels Ritt, *Differential algebra*, Amer. Math. Soc. Colloq. Publ., vol. 33, Amer. Math. Soc., Providence, R.I. 1950.

[RIT-II] ______, *Transcendental transcendency of certain functions of Poincaré*, Math. Ann., vol. 95 (1926), pp. 671–682.

[RIT-III] ______, *Meromorphic functions with addition or multiplication theorems*, Trans. Amer. Math. Soc., vol. 29 (1927), pp. 341–360.

[RUB-I] Lee A. Rubel, *Some research problems about algebraic differential equations*, Trans. Amer. Math. Soc. 280 (1983), 43–52.

[RUB-II] ______, *A universal differential equation*, Bull. Amer. Math. Soc. (N.S.), vol. 4 (1981), pp. 345–349.

[RUB-III] ______, *A survey of transcendentally transcendental functions*, Amer. Math. Monthly, vol. 96 (1989), pp. 777–788.

______, *On the ring of differentially-algebraic entire functions*, J. Symbolic Logic, vol. 57 (1992), pp. 449–451.

[RUB-V] ______, *Digital simulation of analog computation*, J. Symbolic Logic, vol. 54 (1989), pp. 1011–1017.

[RUB-VI] ______, *Generalized solutions of algebraic differential equations*, J. Differential Equations, vol. 62 (1986), pp. 242–251.

[RUB-VII] ______, *The differential equation $y''' = 0$ revisited*, J. Differential Equations, vol. 91 (1991), pp. 78–87.

[RUD] Walter Rudin, *Functional analysis*, McGraw Hill, New York, 1973.

[RUS] Lee A. Rubel and Michael F. Singer, *A differentially algebraic elimination theorem with application to analog computability in the calculus of variations*, Proc. Amer. Math. Soc., vol. 94 (1985), pp. 653–658.

[SHA] Claude E. Shannon, *Mathematical theory of the differential analyzer*, J. Math, Phys. (M.I.T.), vol. 20 (1941), pp. 347–354.

[SIN] Michael F. Singer, *The model theory of ordered differential fields*, J. Symbolic Logic, vol. 43 (1978), pp. 82–91.

[VIJ] T. Vijayaraghavan, *Sur la croissance des fonctions définies par les équations différentielles*, C. R. Acad. Sci. Paris, vol. 194 (1932), pp. 827–829.

University of Illinois
Urbana, Illinois