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BOUNDEDNESS OF SOME SUBLINEAR OPERATORS ON HERZ SPACES

XINWEI LI AND DACHUN YANG

1. Introduction

It is well known that Beurling [2] and Herz [11] introduced some new spaces that characterize certain properties of functions. These new spaces are called the Herz spaces. Many studies involving these spaces can be found in the literature. One of the main reasons is that Hardy space theory associated with Herz spaces is very rich. Actually, these new Hardy spaces are a sort of local version of the ordinary Hardy spaces; the former, sometimes, are good substitutes of the latter when considering, for example, the boundedness of non-translation invariant singular integral operators. This paper is motivated by previous work of Lu, Hernández and the second author (see [14] and [10]), and also by more applications, such as the boundedness of bilinear operators and the regularity of solutions of the Laplacian and the wave equations on Herz-type spaces. See [12] and [16]. Our main interest is to study the boundedness of some sublinear operators on these spaces under certain weak size conditions (see (2.1) and (2.2) below). These conditions are similar to those introduced by Soria and Weiss in [18], and are satisfied by most of the operators in harmonic analysis (see [18]). Let us first introduce some notations.

Let $B_k = \{x \in \mathbb{R}^n\colon |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Let $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, where χ_E is the characteristic function of the set E.

Definition 1.1. Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$.

(a) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})\colon \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\right\}^{1/p} < \infty,$$

with the usual modifications made when $p = \infty$ and/or $q = \infty$.

(b) The non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)\colon \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

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where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \|f\chi_{B_0}\|_{L^q(\mathbb{R}^n)}^p + \sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} < \infty.$$

with the usual modifications made when $p = \infty$ and/or $q = \infty$.

Obviously, $\dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) = K_p^{0,p}(\mathbb{R}^n)$ for all $0 < p \leq \infty$.

The spaces $K_q^{n(1-1/q),1}(\mathbb{R}^n) \equiv A^q$ are particular cases of the spaces introduced by Beurling [2] with a different, but equivalent, norm. The spaces A^q are the so-called Beurling algebras. The norm of the spaces A^q used in Definition 1.1 was first introduced by Feichtinger [5]. In addition, the spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ are first introduced by Herz [11] with different norms and notations. Flett [6] gave a characterization of the Herz spaces which is easily seen to be equivalent to Definition 1.1. More interesting accounts and applications of these spaces can also be found in [1].

In §2 of this paper, we will prove the boundedness of some sublinear operators on the Herz spaces. These results are the complement of the corresponding results in [14], [16] and [10], and are the best possible under the conditions of the theorems. It is worth pointing out that our method is somewhat different from the one used in [14], [16] and [10]. Some of our techniques are similar to those used by Soria and Weiss in [18].

Let $g(f)$ be the standard g-function in [19]. It is well known that

$$c_1\|f\|_{L^p(\mathbb{R}^n)} \leq \|g(f)\|_{L^p(\mathbb{R}^n)} \leq c_2\|f\|_{L^p(\mathbb{R}^n)} \tag{1.1}$$

for all $1 < p < \infty$ (see [19] or [20]). Notice that when $p < q$, $\dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n) \cup K_q^{n(1/p-1/q),p}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and, when $p = q$, they are just the space $L^p(\mathbb{R}^n)$. It is natural to ask if we can generalize (1.1) to Herz spaces; that is, is (1.1) still true if we replace the $L^p(\mathbb{R}^n)$ norm by the Herz space norms if $1 < p,\ q < \infty$ and $\alpha = n(1/p - 1/q)$? In §3, we will give an affirmative answer to this question using the results in §2. Moreover, we will use those theorems in §2 to establish the generalized Littlewood-Paley function characterizations of Herz spaces. Our results are, in some sense, best possible. Also the theorems in §2 allow us to determine the relations between Herz spaces and Herz-type Hardy spaces that have been studied in recent years; see [3], [7], [8], [13], [15], [17] and [22].

2. Main theorems and their proofs

Theorem 2.1 (below) in the case $0 < \alpha < n(1 - 1/q)$ can be found in [14]. In [10], Hernández and the second author generalize the theorem in [14] to the case $-n/q < \alpha < n(1 - 1/q)$ with $\alpha \neq 0$. Here, we use a different method to obtain the case with $\alpha = 0$. Actually, the same procedure works for all α in the range $-n/q < \alpha < n(1 - 1/q)$.

THEOREM 2.1. *Let* $0 < p \le \infty$, $1 < q < \infty$ *and* $-n/q < \alpha < n(1 - 1/q)$. *Suppose a sublinear operator* T *satisfies the size conditions*

$$|Tf(x)| \le c\|f\|_{L^1(\mathbb{R}^n)}/|x|^n, \tag{2.1}$$

when supp $f \subseteq A_k$ *and* $|x| \ge 2^{k+1}$ *with* $k \in \mathbb{Z}$, *and*

$$|Tf(x)| \le c2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}, \tag{2.2}$$

when supp $f \subseteq A_k$ *and* $|x| \le 2^{k-2}$ *with* $k \in \mathbb{Z}$. *Then, if* T *is bounded on* $L^q(\mathbb{R}^n)$, T *is also bounded on* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

Proof. Because of the above remarks and the hypotheses of the theorem, we only need to show the theorem in the case $\alpha = 0$ and $p \neq q$. We also assume $0 < p < \infty$; the proof of the case $p = \infty$ is simpler. We write

$$\begin{aligned}
\|Tf\|_{\dot{K}_q^{0,p}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} \|\chi_k Tf\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\le \left\{\sum_{k=-\infty}^{\infty} \left\| \left(\sum_{\ell=-\infty}^{\infty} |T(f\chi_\ell)(x)|\right) \chi_k(x) \right\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\le c_p \left\{\sum_{k=-\infty}^{\infty} \left\| \left(\sum_{\ell=-\infty}^{k-2} |T(f\chi_\ell)(x)|\right) \chi_k(x) \right\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\quad + c_p \left\{\sum_{k=-\infty}^{\infty} \left\| \left(\sum_{\ell=k-1}^{k+1} |T(f\chi_\ell)(x)|\right) \chi_k(x) \right\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\quad + c_p \left\{\sum_{k=-\infty}^{\infty} \left\| \left(\sum_{\ell=k+2}^{\infty} |T(f\chi_\ell)(x)|\right) \chi_k(x) \right\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\equiv c_p(I_1 + I_2 + I_3).
\end{aligned}$$

For I_2, using the $L^q(\mathbb{R}^n)$–boundedness of T, we obtain

$$\begin{aligned}
I_2 &\le c\left\{\sum_{k=-\infty}^{\infty} \left(\sum_{\ell=k-1}^{k+1} \|T(f\chi_\ell)\|_{L^q(\mathbb{R}^n)}^p\right)\right\}^{1/p} \\
&\le c\left\{\sum_{k=-\infty}^{\infty} \left(\sum_{\ell=k-1}^{k+1} \|f\chi_\ell\|_{L^q(\mathbb{R}^n)}^p\right)\right\}^{1/p} \\
&\le c\left\{\sum_{k=-\infty}^{\infty} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p} = c\|f\|_{\dot{K}_q^{0,p}(\mathbb{R}^n)}.
\end{aligned}$$

Here, as in other cases, the value of c can vary.

For I_1, we use the facts that $\ell \le k-2$ and $x \in A_k$; by (2.1), we have

$$|T(f\chi_\ell)(x)| \le c2^{-kn}\|f\chi_\ell\|_{L^1(\mathbb{R}^n)}. \tag{2.3}$$

In what follows, if $1 < p \le \infty$, we let $1/p + 1/p' = 1$. From (2.3) and Hölder's inequality, we deduce

$$\begin{aligned}
I_1 &\le c\left\{\sum_{k=-\infty}^{\infty}\left\|\left(\sum_{\ell=-\infty}^{k-2}\|f\chi_\ell\|_{L^1(\mathbb{R}^n)}\right)2^{-kn}\chi_k(x)\right\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p}\\
&= c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{\ell=-\infty}^{k-2}\|f\chi_\ell\|_{L^1(\mathbb{R}^n)}\right)^p 2^{kn(1/q-1)p}\right\}^{1/p}\\
&\le c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{\ell=-\infty}^{k-2}\|f\chi_\ell\|_{L^q(\mathbb{R}^n)}2^{(k-\ell)n(1/q-1)}\right)^p\right\}^{1/p}\\
&\le c\begin{cases}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{\ell=-\infty}^{k-2}\|f\chi_\ell\|_{L^q(\mathbb{R}^n)}^p 2^{(k-\ell)n(1/q-1)p}\right)\right\}^{1/p} & \text{if } 0<p\le 1\\
\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{\ell=-\infty}^{k-2}\|f\chi_\ell\|_{L^q(\mathbb{R}^n)}^p 2^{(k-\ell)n/2(1/q-1)p}\right)\right. & \\
\qquad\left.\times\left(\sum_{\ell=-\infty}^{k-2}2^{(k-\ell)n/2(1/q-1)p'}\right)^{p/p'}\right\}^{1/p} & \text{if } 1<p<\infty\end{cases}\\
&\le c\begin{cases}\left\{\sum_{\ell=-\infty}^{\infty}\|f\chi_\ell\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p} & \text{if } 0<p\le 1\\
\left\{\sum_{\ell=-\infty}^{\infty}\|f\chi_\ell\|_{L^q(\mathbb{R}^n)}^p\left(\sum_{k=\ell+2}^{\infty}2^{(k-\ell)n/2(1/q-1)p}\right)\right\}^{1/p} & \text{if } 1<p<\infty\end{cases}\\
&\le c\|f\|_{\dot K_q^{0,p}(\mathbb{R}^n)}.
\end{aligned}$$

For I_3, we use $\ell \ge k+2$ and (2.2); we have

$$|T(f\chi_\ell)(x)|\chi_k(x) \le c2^{-\ell n}\|f\chi_\ell\|_{L^1(\mathbb{R}^n)} \le c2^{-\ell n/q}\|f\chi_\ell\|_{L^q(\mathbb{R}^n)}.$$

From this and Hölder's inequality, it follows that

$$\begin{aligned}
I_3 &\le c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{\ell=k+2}^{\infty}2^{(k-\ell)n/q}\|f\chi_\ell\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&\le c\begin{cases}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{\ell=k+2}^{\infty}2^{(k-\ell)pn/q}\|f\chi_\ell\|_{L^q(\mathbb{R}^n)}^p\right)\right\}^{1/p} & \text{if } 0<p\le 1\\
\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{\ell=k+2}^{\infty}2^{(k-\ell)pn/(2q)}\|f\chi_\ell\|_{L^q(\mathbb{R}^n)}^p\right)\right. & \\
\qquad\left.\times\left(\sum_{\ell=k+2}^{\infty}2^{(k-\ell)np'/(2q)}\right)^{p/p'}\right\}^{1/p} & \text{if } 1<p<\infty\end{cases}
\end{aligned}$$

$$\leq c \begin{cases} \left\{ \sum_{\ell=-\infty}^{\infty} \|f\chi_\ell\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} & \text{if } 0 < p \leq 1 \\ \left\{ \sum_{\ell=-\infty}^{\infty} \|f\chi_\ell\|_{L^q(\mathbb{R}^n)}^p \left(\sum_{k=-\infty}^{\ell-2} 2^{(k-\ell)pn/(2q)} \right) \right\}^{1/p} & \text{if } 1 < p < \infty \end{cases}$$

$$\leq c\|f\|_{\dot{K}_q^{0,p}(\mathbb{R}^n)}.$$

Combining the estimates on I_1, I_2 and I_3, we obtain

$$\|Tf\|_{\dot{K}_q^{0,p}(\mathbb{R}^n)} \leq c\|f\|_{\dot{K}_q^{0,p}(\mathbb{R}^n)},$$

the desired result. This finishes the proof of Theorem 2.1.

We have a similar theorem for the non-homogeneous spaces whose proof is similar to that of Theorem 2.1.

THEOREM 2.2. *Let p, q and α be as in Theorem 2.1. Suppose a sublinear operator T satisfies the size conditions*

$$|Tf(x)| \leq c\|f\|_{L^1(\mathbb{R}^n)}/|x|^n, \tag{2.4}$$

when supp $f \subseteq B_0$ *and* $|x| > 2$ *or* supp $f \subseteq A_k$ *and* $|x| \geq 2^{k+1}$ *with* $k \in \mathbb{N}$, *and*

$$|Tf(x)| \leq c2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}, \tag{2.5}$$

when supp $f \subseteq A_k$ *and* $|x| \leq 2^{k-2}$ *with* $k \geq 2$. *Then, if T is bounded on $L^q(\mathbb{R}^n)$, T is also bounded on $K_q^{\alpha,p}(\mathbb{R}^n)$.*

COROLLARY 2.1. *Let p, q and α be as in Theorem 2.1. If a sublinear operator T satisfies the condition*

$$|Tf(x)| \leq c\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy, \quad x \notin \text{supp } f \tag{2.6}$$

for any integrable function f with compact support and T is bounded on $L^q(\mathbb{R}^n)$, then T is bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$.

We remark that (2.6) is satisfied by many operators in harmonic analysis, such as Calderón-Zygmund operators, the Carleson maximal operator, C. Fefferman's singular multiplier operator, R. Fefferman's singular integral operator and the Bochner-Riesz means at the critical index and so on; see [18]. In particular, the Hardy-Littlewood maximal function $M(f)$ also satisfies the hypotheses of Theorems 2.1 and 2.2. It is worth pointing out that Theorems 2.1 and 2.2 and Corollary 2.1 are best possible. In another words, when $\alpha \geq n(1-1/q)$ or $\alpha \leq -n/q$, these theorems are false; see [14] for a counterexample in the case $\alpha \geq n(1-1/q)$ and [10] for another example when $\alpha \leq -n/q$.

Using the method in the proof of Theorem 2.1, we can prove the following extensions of these results:

THEOREM 2.3. *Let $0 < \ell < n$. Suppose a sublinear operator $I_\ell(f)$ satisfies*

$$|I_\ell(f)(x)| \leq c|x|^{-(n-\ell)}\|f\|_{L^1(\mathbb{R}^n)}, \tag{2.7}$$

when supp $f \subseteq A_k$ *and* $|x| \geq 2^{k+1}$ *with* $k \in \mathbb{Z}$, *and*

$$|I_\ell(f)(x)| \leq c2^{-k(n-\ell)}\|f\|_{L^1(\mathbb{R}^n)}, \tag{2.8}$$

when supp $f \subseteq A_k$ *and* $|x| \leq 2^{k-2}$ *with* $k \in \mathbb{Z}$. *Also assume* $1 < q_1 < n/\ell$, $1/q_2 = 1/q_1 - \ell/n$, $-n/q_1 + \ell < \alpha < n(1 - 1/q_1)$, $0 < p_1 \leq p_2 \leq \infty$ *and that* $I_\ell(f)$ *maps* $L^{q_1}(\mathbb{R}^n)$ *into* $L^{q_2}(\mathbb{R}^n)$. *Then* $I_\ell(f)$ *maps* $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ *into* $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$.

THEOREM 2.4. *Let* $0 < \ell < n$, α, q_1, q_2, p_1 *and* p_2 *as in Theorem* 2.3. *Suppose a sublinear operator* $I_\ell(f)$ *satisfies*

$$|I_\ell(f)(x)| \leq c|x|^{-(n-\ell)}\|f\|_{L^1(\mathbb{R}^n)}, \tag{2.9}$$

when supp $f \subseteq B_0$ *and* $|x| \geq 2$ *or* supp $f \subseteq A_k$ *and* $|x| \geq 2^{k+1}$ *with* $k \in \mathbb{N}$, *and*

$$|I_\ell(f)(x)| \leq c2^{-k(n-\ell)}\|f\|_{L^1(\mathbb{R}^n)}, \tag{2.10}$$

when supp $f \subseteq A_k$ *and* $|x| \leq 2^{k-2}$ *with* $k \geq 2$. *Also assume that* $I_\ell(f)$ *maps* $L^{q_1}(\mathbb{R}^n)$ *into* $L^{q_2}(\mathbb{R}^n)$. *Then* $I_\ell(f)$ *maps* $K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ *into* $K_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$.

Remark 2.1. If $I_\ell(f)$ satisfies

$$|I_\ell(f)(x)| \leq c\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\ell}}\,dy, \quad x \notin \text{supp } f \tag{2.11}$$

for any integrable function f with compact support, then $I_\ell(f)$ obviously satisfies (2.7) and (2.10). In particular, if $I_\ell(f)$ is a (standard) fractional integral, then $I_\ell(f)$ obviously satisfies (2.11) and, therefore, all the conditions of Theorems 2.3 and 2.4. The fractional maximal function $M_\ell(f)$, defined by

$$M_\ell(f)(x) = \sup_{r>0} r^{-(n-\ell)}\int_{B(x,r)} |f(x)|\,dx,$$

also satisfies the conditions of Theorems 2.3 and 2.4, where $B(x,r) = \{y \in \mathbb{R}^n\colon |y - x| \leq r\}$.

Remark 2.2. If $\ell = 0$, Theorems 2.3 and 2.4 are just Theorems 2.1 and 2.2. Moreover, Theorems 2.3 and 2.4 are also best possible; that is, if $\alpha \geq n(1 - 1/q_1)$ or $\alpha \leq -n/q_1 + \ell$, Theorems 2.3 and 2.4 are false.

Notice that if $p_2 \geq p_1$, then

$$\dot{K}_{q_2}^{\alpha,p_1}(\mathbb{R}^n) \subset \dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n) \text{ and } K_{q_2}^{\alpha,p_1}(\mathbb{R}^n) \subset K_{q_2}^{\alpha,p_2}(\mathbb{R}^n). \tag{2.12}$$

Thus, we only need to show Theorems 2.3 and 2.4 in the case $p_2 = p_1$. Since Theorem 2.4 can be proved in a similar way to Theorem 2.3, we only show Theorem 2.3.

Proof of Theorem 2.3. Just as in the proof of Theorem 2.1, we write

$$\begin{aligned}
\|I_\ell(f)\|_{\dot K_{q_2}^{\alpha,p_1}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\||I_\ell(f)|\chi_k\|_{L^{q_2}(\mathbb{R}^n)}^{p_1}\right\}^{1/p_1}\\
&\le c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left\|\left(\sum_{j=-\infty}^{k-2}|I_\ell(f\chi_j)|\right)\chi_k\right\|_{L^{q_2}(\mathbb{R}^n)}^{p_1}\right\}^{1/p_1}\\
&\quad + c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left\|\left(\sum_{j=k-1}^{k+1}|I_\ell(f\chi_j)|\right)\chi_k\right\|_{L^{q_2}(\mathbb{R}^n)}^{p_1}\right\}^{1/p_1}\\
&\quad + c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left\|\left(\sum_{j=k+2}^{\infty}|I_\ell(f\chi_j)|\right)\chi_k\right\|_{L^{q_2}(\mathbb{R}^n)}^{p_1}\right\}^{1/p_1}\\
&\equiv cII_1 + cII_2 + cII_3.
\end{aligned}$$

For II_2, using the fact that $I_\ell(f)$ maps $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
II_2 &\le c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=k-1}^{k+1}\|I_\ell(f\chi_j)\|_{L^{q_2}(\mathbb{R}^n)}\right)^{p_1}\right\}^{1/p_1}\\
&\le c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=k-1}^{k+1}\|f\chi_j\|_{L^{q_1}(\mathbb{R}^n)}\right)^{p_1}\right\}^{1/p_1}\\
&\le c\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\|f\chi_k\|_{L^{q_1}(\mathbb{R}^n)}^{p_1}\right\}^{1/p_1} = c\|f\|_{\dot K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}.
\end{aligned}$$

For II_1, notice that $j \le k-2$; by (2.7), we have

$$|I_\ell(f\chi_j)(x)|\chi_k(x) \le c2^{-k(n-\ell)}\|f\chi_j\|_{L^1(\mathbb{R}^n)}.$$

From this, it follows that

$$\begin{aligned}
II_1 &\le c\left\{\sum_{k=-\infty}^{\infty} 2^{k(\alpha-(n-\ell)+n/q_2)p_1}\left(\sum_{j=-\infty}^{k-2}\|f\chi_j\|_{L^1(\mathbb{R}^n)}\right)^{p_1}\right\}^{1/p_1}\\
&\le c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha}\|f\chi_j\|_{L^{q_1}(\mathbb{R}^n)}2^{(j-k)(n(1-1/q_1)-\alpha)}\right)^{p_1}\right\}^{1/p_1}\\
&\le c\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p_1}\|f\chi_j\|_{L^{q_1}(\mathbb{R}^n)}^{p_1}\right\}^{1/p_1} = c\|f\|_{\dot K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)},
\end{aligned}$$

where in the last inequality, we estimated as we did for I_1 in the proof of Theorem 2.1 since $\alpha < n(1 - 1/q_1)$.

For II_3, notice that $j \geq k + 2$; by (2.8), we have

$$|I_\ell(f\chi_j)(x)|\chi_k(x) \leq c2^{-j(n-\ell)}\|f\chi_j\|_{L^1(\mathbb{R}^n)} \leq c2^{-jn/q_2}\|f\chi_j\|_{L^{q_1}(\mathbb{R}^n)}.$$

From this, we deduce that

$$II_3 \leq c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty} 2^{j\alpha}\|f\chi_j\|_{L^{q_1}(\mathbb{R}^n)}2^{(k-j)(n/q_2+\alpha)}\right)^{p_1}\right\}^{1/p_1}.$$

Using the same argument as for I_3 in the proof of Theorem 2.1, we obtain

$$II_3 \leq c\left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p_1}\|f\chi_j\|^{p_1}_{L^{q_1}(\mathbb{R}^n)}\right\}^{1/p_1} = c\|f\|_{\dot{K}^{\alpha,p_1}_{q_1}(\mathbb{R}^n)}$$

since $n/q_2 + \alpha > 0$.

Combining II_1, II_2 and II_3, we obtain

$$\|Tf\|_{\dot{K}^{\alpha,p_1}_{q_2}(\mathbb{R}^n)} \leq c\|f\|_{\dot{K}^{\alpha,p_1}_{q_1}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 2.3.

We can vary the index α in Theorems 2.3 and 2.4 and still obtain similar results. More precisely, we consider the case where $p_1 \leq q_1$, $\alpha_2 = \alpha_1 + \ell(p_1/q_1 - 1)$ and $1/q_2 = 1/q_1 - p_1\ell/(q_1 n)$. Then $1/q_2 \geq 1/q_1 - \ell/n \equiv 1/q_0$, and

$$\dot{K}^{\alpha_1,p_2}_{q_0}(\mathbb{R}^n) \subset \dot{K}^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n) \text{ and } K^{\alpha_1,p_2}_{q_0}(\mathbb{R}^n) \subset K^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n).$$

These inclusions, together with Theorems 2.3 and 2.4, easily imply the following result:

COROLLARY 2.2. *Let* $0 < \ell < n$. *Suppose that a sublinear* $I_\ell(f)$ *satisfies* (2.7) *and* (2.8) *or* (2.9) *and* (2.10). *If* $1 < q_1 < \infty$, $0 < p_1 \leq \min\{q_1, p_2\}$, $-n/q_1 + \ell < \alpha_1 < n(1 - 1/q_1)$, $1/q_2 = 1/q_1(1 - \ell p_1/n)$, $\alpha_2 = \alpha_1 + \ell(p_1/q_1 - 1)$, *and* $I_\ell(f)$ *maps* $L^{q_1}(\mathbb{R}^n)$ *into* $L^{q_0}(\mathbb{R}^n)$, *where* $1/q_0 = 1/q_1 - \ell/n$, *then* $I_\ell(f)$ *maps* $\dot{K}^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)$ *into* $\dot{K}^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n)$ *or* $K^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)$ *into* $K^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n)$.

Remark 2.3. If $I_\ell(f)$ is a (standard) fractional integral, then Theorems 2.3 and 2.4 and Corollary 2.2 with $0 < \alpha_1 < n(1 - 1/q_1)$ have been obtained by Lu and Yang in [16].

3. Some applications

Using Theorem 2.1 and 2.2, we are able to characterize those Herz spaces of Banach type in several ways, especially, by means of the (generalized) Littlewood-Paley g-function, the Lusin area function and the Littlewood-Paley g_λ^*-function.

Recall the following definitions (see [19] and [20]). Suppose ψ is integrable on $\mathbb{R}^n$ and

(i) $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$,
(ii) $|\psi(x)| \le c(1+|x|)^{-(n+\alpha)}$, for some $\alpha > 0$,
(iii) $\int_{\mathbb{R}^n} |\psi(x+y) - \psi(x)|\,dx \le c|y|^{\mathbb{R}^n}$, all $y \in \mathbb{R}^n$, for some $\gamma > 0$.

Let $\psi_t(x) = t^{-n}\psi(x/t)$ with $t > 0$ and $x \in \mathbb{R}^n$. For f in $L^2(\mathbb{R}^n)$ with compact support (these functions are dense in Herz spaces), the Littlewood-Paley g-function of f is defined by

$$g(f)(x) = \left\{\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t}\right\}^{1/2}; \tag{3.1}$$

the Lusin area function of f is defined by

$$S_{\psi,a}(f)(x) = \left(\frac{1}{a^n|B_0|}\int_{\Gamma_a(x)} |f * \psi_t(x)|^2 t^{-n}\,dy\frac{dt}{t}\right)^{1/2}, \tag{3.2}$$

where $|B_0|$ is the Lebesgue measure of the unit ball B_0 of $\mathbb{R}^n$, and $\Gamma_a(x) = \{(y,t) \in \mathbb{R}_+^{n+1}\colon |x-y| < at\}$; and the Littlewood-Paley g_λ^*-function of f is defined by

$$g_{\psi,\lambda}^*(f)(x) = \left\{\int_0^\infty \int_{\mathbb{R}^n} \frac{|f * \psi_t(y)|^2}{\left(1+\frac{|x-y|}{t}\right)^{2\lambda}} t^{-n}\,dy\frac{dt}{t}\right)^{1/2}. \tag{3.3}$$

It is well known (see [19]) that, in the classical situation where ψ is related to the gradient of the Poisson kernel, all those functions have $L^p(\mathbb{R}^n)$ norms equivalent to $\|f\|_{L^p(\mathbb{R}^n)}$ when f in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This equivalence actually holds for any general Littlewood-Paley function defined in terms of such ψ. See Torchinsky [20] for details.

We will show, in the next theorem, that this equivalence holds also for the spaces $\dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n)$ and $K_q^{n(1/p-1/q),p}(\mathbb{R}^n)$ with $1 < p, q < \infty$ and, therefore, we answer the question posed in §1. Actually, we will show the equivalence for more general Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ with $1 \le p \le \infty$, $1 < q < \infty$ and $-n/q < \alpha < n(1-1/q)$; within these ranges of p and q, the corresponding Herz spaces are Banach spaces (see [10]).

It is known, by [20], for example, that for any general Littlewood-Paley function associated with ψ, the inequality

$$S_{\psi,a}(f)(x) \le c g_{\psi,\lambda}^*(f)(x), \quad x \in \mathbb{R}^n \tag{3.4}$$

holds for all $a,\ \lambda > 0$. Wilson in [21] proved that, for any compactly supported ψ satisfying (i), (ii) and (iii), there is another compactly supported radial function ρ of this type, such that

$$g_\psi(f)(x) \leq cS_{\rho,2}(f)(x), \quad x \in \mathbb{R}^n. \tag{3.5}$$

This inequality is also true if $g_\psi(f)$ and $S_{\rho,2}(f)$ are replaced by the standard g-function and Lusin-function, respectively. See [19]. However, it is not known whether (3.5) is true or not for general (not necessarily compactly supported) functions ψ and ρ satisfying (i), (ii) and (iii).

THEOREM 3.1. *Let* $1 \leq p \leq \infty,\ 1 < q < \infty,\ -n/q < \alpha < n(1-1/q)$ *and let* ψ *satisfy (i), (ii) and (iii). Then there exist absolute constants* $c_1,\ c_2,\ c_3$ *and* c_4 *such that*

$$\begin{aligned}\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} &\leq c_1\|g_\psi(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leq c_2\|S_{\psi,a}(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}\\ &\leq c_3\|g^*_{\psi,\lambda}(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leq c_4\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}\end{aligned} \tag{3.6}$$

for all $a > 0$ *and* $\lambda > 3n/2$. *The same is true for the spaces* $K_q^{\alpha,p}(\mathbb{R}^n)$.

Remark 3.1. From the proof below, we can deduce that the boundedness of $g_\psi(f)$, $S_{\psi,a}(f)$ and $g^*_{\psi,\lambda}(f)$ on $\dot K_q^{\alpha,p}(\mathbb{R}^n)$ holds for all p such that $0 < p \leq \infty$. When $\alpha = 0,\ p = q > 1$, we recover the classical results.

Proof. We only prove the case for the homogeneous Herz spaces. The other case is similar. The main effort in this proof is devoted to showing

$$\|g_\psi(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leq c\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \tag{3.7}$$

and

$$\|g^*_{\psi,\lambda}(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leq c\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}. \tag{3.8}$$

Once this is done, we obtain $\|S_{\psi,a}(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leq c\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}$ from (3.4) and (3.8). Standard arguments give $\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leq c_1\|g_\psi(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}$ and $\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leq c_2\|S_{\psi,a}(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}$. Using (3.4) again, we obtain $\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \leq c_3\|g^*_{\psi,\lambda}(f)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}$. The reader can see that the chain of inequalities (3.6) follows easily from these.

(1) *The case* $g_\psi(f)$.

Assume $\operatorname{supp} f \subseteq A_k$ and $|x| \geq 2^{k+1}$. We then have

$$\begin{aligned}|f * \psi_t(x)| &\leq \frac{1}{t^n}\int_{\mathbb{R}^n}|f(x-y)|\left|\psi\left(\frac{y}{t}\right)\right|dy\\ &\leq ct^\alpha\int_{\mathbb{R}^n}|f(x-y)|\frac{dy}{(t+|y|)^{n+\alpha}}\end{aligned} \tag{3.9}$$

by (ii) in the definition of ψ. Notice that $|x-y| \le 2^k$ implies $|y| \ge |x|/2$. We write

$$\begin{aligned} g_\psi(f)(x) &\le \left(\int_0^{|x|} |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2} + \left(\int_{|x|}^\infty |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2} \\ &\equiv I_1(x) + I_2(x). \end{aligned}$$

For $I_1(x)$, by (3.9),

$$|f * \psi_t(x)| \le c \frac{t^\alpha}{|x|^{n+\alpha}} \|f\|_{L^1(\mathbb{R}^n)}.$$

Thus,

$$I_1(x) \le \frac{c}{|x|^{n+\alpha}} \|f\|_{L^1(\mathbb{R}^n)} \left(\int_0^{|x|} t^{2\alpha} \frac{dt}{t}\right)^{1/2} \le \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{|x|^n}.$$

For $I_2(x)$, from (3.9), we obtain

$$|f * \psi_t(x)| \le \frac{c}{|x|^{n-1}} \frac{\|f\|_{L^1(\mathbb{R}^n)}}{t}.$$

Thus,

$$I_2(x) \le \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{|x|^{n-1}} \left(\int_{|x|}^\infty \frac{dt}{t^3}\right)^{1/2} \le \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{|x|^n}.$$

Hence we have verified (2.1) for $g_\psi(f)$.

Next we consider the case when supp $f \subseteq A_k$ and $|x| \le 2^{k-2}$. In this case, $|y| \ge 2^{k-2}$. We write

$$\begin{aligned} g_\psi(f)(x) &\le \left(\int_0^{2^{k-2}} |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2} + \left(\int_{2^{k-2}}^\infty |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2} \\ &\equiv II_1(x) + II_2(x). \end{aligned}$$

In $II_1(x)$, we obtain

$$|f * \psi_t(x)| \le \frac{ct^\alpha}{2^{k(n+\alpha)}} \|f\|_{L^1(\mathbb{R}^n)}.$$

Thus,

$$II_1(x) \le c \frac{\|f\|_{L^1(\mathbb{R}^n)}}{2^{k(n+\alpha)}} \left(\int_0^{2^{k-2}} t^{2\alpha} \frac{dt}{t}\right)^{1/2} \le c \frac{\|f\|_{L^1(\mathbb{R}^n)}}{2^{kn}}.$$

In $II_2(x)$, we have

$$|f * \psi_t(x)| \le c \frac{1}{2^{k(n-1)}} \frac{\|f\|_{L^1(\mathbb{R}^n)}}{t}.$$

Thus, similarly, we obtain

$$II_2(x) \le c\frac{\|f\|_{L^1(\mathbb{R}^n)}}{2^{kn}}.$$

Therefore (2.2) is verified, and, by Theorem 2.1 for $T = g_\psi$, we obtain (3.7).

(2) *The case* $g^*_{\psi,\lambda}(f)$.

Let supp $f \subseteq A_k$ and $|x| \ge 2^{k+1}$. We write

$$\begin{aligned} g^*_{\psi,\lambda}(f)(x) &= \left(\int_0^\infty \int_{\mathbb{R}^n} \frac{|f*\psi_t(y)|^2}{(t+|x-y|)^{2\lambda}}\, dy \frac{dt}{t^{n+1-2\lambda}}\right)^{1/2} \\ &\le \left(\int_0^{|x|} \int_{\mathbb{R}^n} \frac{|f*\psi_t(y)|^2}{(t+|x-y|)^{2\lambda}}\, dy \frac{dt}{t^{n+1-2\lambda}}\right)^{1/2} \\ &\quad + \left(\int_{|x|}^\infty \int_{\mathbb{R}^n} \frac{|f*\psi_t(y)|^2}{(t+|x-y|)^{2\lambda}}\, dy \frac{dt}{t^{n+1-2\lambda}}\right)^{1/2} \\ &\equiv E_1(x) + E_2(x)\,. \end{aligned}$$

Recalling (3.9) we now have

$$|f*\psi_t(y)| \le ct^\alpha \int_{\mathbb{R}^n} |f(y-z)| \frac{dz}{(t+|z|)^{n+\alpha}}. \tag{3.10}$$

Furthermore, we write

$$\begin{aligned} E_1(x) &= \left(\int_0^{|x|} \int_{|y|\le \frac{3|x|}{4}} \frac{|f*\psi_t(y)|^2}{(t+|x-y|)^{2\lambda}}\, dy \frac{dt}{t^{n+1-2\lambda}}\right)^{1/2} \\ &\quad + \left(\int_0^{|x|} \int_{|y|> \frac{3|x|}{4}} \frac{|f*\psi_t(y)|^2}{(t+|x-y|)^{2\lambda}}\, dy \frac{dt}{t^{n+1-2\lambda}}\right)^{1/2} \\ &\equiv F_1(x) + F_2(x). \end{aligned}$$

For $F_1(x)$, $|x-y| \ge |x| - |y| > \frac{|x|}{4}$, and, we obtain $|f*\psi_t(y)| \le ct^{-n}\|f\|_{L^1(\mathbb{R}^n)}$ from (3.10). Consequently,

$$F_1(x) \le c\frac{\|f\|_{L^1(\mathbb{R}^n)}}{|x|^\lambda}\left(\int_0^{|x|} \int_{|y|\le\frac{3|x|}{4}} dy \frac{dt}{t^{3n+1-2\lambda}}\right)^{1/2} \le c\frac{\|f\|_{L^1(\mathbb{R}^n)}}{|x|^n},$$

whenever $\lambda > 3n/2$.

For the estimate of $F_2(x)$, we use (3.10) with $|z| \ge |y|/3$ which gives us

$$|f*\psi_t(y)| \le \frac{ct^\alpha}{|y|^{n+\alpha}}\|f\|_{L^1(\mathbb{R}^n)}.$$

This relation $|z| \geq |y|/3$ follows from $|y-z| \leq 2^k$ and $|x| \geq 2^{k+1}$, which give us $|y-z| \leq |x|/2 \leq 2|y|/3$.

Thus,

$$\begin{aligned} F_2(x) &\leq c\|f\|_{L^1(\mathbb{R}^n)} \left(\int_0^{|x|} \int_{|y|>\frac{3|x|}{4}} \frac{1}{(t+|x-y|)^{2\lambda}} \frac{1}{|y|^{2n+2\alpha}}\, dy \frac{dt}{t^{n+1-2\lambda-2\alpha}} \right)^{1/2} \\ &\leq c\|f\|_{L^1(\mathbb{R}^n)} \left(\int_0^{|x|} \int_{\frac{3|x|}{4}<|y|\leq 2|x|} \frac{1}{(t+|x-y|)^{2\lambda}} \frac{1}{|y|^{2n+2\alpha}}\, dy \frac{dt}{t^{n+1-2\lambda-2\alpha}} \right)^{1/2} \\ &\quad + c\|f\|_{L^1(\mathbb{R}^n)} \left(\int_0^{|x|} \int_{|y|>2|x|} \frac{1}{(t+|x-y|)^{2\lambda}} \frac{1}{|y|^{2n+2\alpha}}\, dy \frac{dt}{t^{n+1-2\lambda-2\alpha}} \right)^{1/2} \\ &\equiv J_1(x) + J_2(x). \end{aligned}$$

For $J_1(x)$, we choose ℓ such that $0 < \ell < 2\lambda$ and $n - 2\alpha < \ell < n$, and, since $|x-y| \leq |x| + |y| \leq 3|x|$, we obtain

$$\begin{aligned} J_1(x) &\leq \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{|x|^{n+\alpha}} \left(\int_0^{|x|} \int_{\frac{3|x|}{4}<|y|\leq 2|x|} \frac{1}{|x-y|^{\ell}}\, dy \frac{dt}{t^{n+1-2\alpha-\ell}} \right)^{1/2} \\ &\leq \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{|x|^n}. \end{aligned}$$

For $J_2(x)$, noting that $|x-y| \geq |y|/2$ we have

$$\begin{aligned} J_2(x) &\leq c\|f\|_{L^1(\mathbb{R}^n)} \left(\int_0^{|x|} \int_{|y|>2|x|} \frac{dy}{|y|^{2n+2\alpha+2\lambda}} \frac{dt}{t^{n+1-2\lambda-2\alpha}} \right)^{1/2} \\ &\leq \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{|x|^n}, \end{aligned}$$

since $\lambda + \alpha > n/2$.

Now for $E_2(x)$, we use $|f * \psi_t(y)| \leq ct^{-n}\|f\|_{L^1(\mathbb{R}^n)}$ and obtain

$$\begin{aligned} E_2(x) &\leq c\|f\|_{L^1(\mathbb{R}^n)} \left(\int_{|x|}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(t+|x-y|)^{2\lambda}}\, dy \frac{dt}{t^{3n+1-2\lambda}} \right)^{1/2} \\ &\leq c\|f\|_{L^1(\mathbb{R}^n)} \left(\int_{|x|}^{\infty} \int_{|y|\leq 2|x|} \frac{1}{(t+|x-y|)^{2\lambda}}\, dy \frac{dt}{t^{3n+1-2\lambda}} \right)^{1/2} \\ &\quad + c\|f\|_{L^1(\mathbb{R}^n)} \left(\int_{|x|}^{\infty} \int_{|y|>2|x|} \frac{1}{(t+|x-y|)^{2\lambda}}\, dy \frac{dt}{t^{3n+1-2\lambda}} \right)^{1/2} \\ &\equiv K_1(x) + K_2(x), \end{aligned}$$

where

$$K_1(x) \leq c\|f\|_{L^1(\mathbb{R}^n)} |x|^{n/2} \left(\int_{|x|}^{\infty} t^{-3n-1}\, dt \right)^{1/2} \leq \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{|x|^n},$$

and

$$K_2(x) \le c\|f\|_{L^1(\mathbb{R}^n)} \left(\int_{|x|}^{\infty} \int_{|y|>2|x|} \frac{dy}{|y|^{2\ell}} \frac{dt}{t^{3n+1-\ell}}\right)^{1/2} \le \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{|x|^n}$$

since $|x-y| \ge |y| - |x| \ge |y|/2$ and we can choose ℓ such that $n/2 < \ell < 3n$ with $0 < \ell < 2\lambda$. Therefore, (2.1) holds for $g^*_{\psi,\lambda}(f)$.

Now we verify (2.2) for $g^*_{\psi,\lambda}(f)$. Suppose $|x| \le 2^{k-2}$. We write

$$\begin{aligned} g^*_{\psi,\lambda}(f)(x) &\le \left(\int_0^{2^{k-2}} \int_{\mathbb{R}^n} \frac{|f * \psi_t(y)|^2}{(t+|x-y|)^{2\lambda}} dy \frac{dt}{t^{n+1-2\lambda}}\right)^{1/2} \\ &+ \left(\int_{2^{k-2}}^{\infty} \int_{\mathbb{R}^n} \frac{|f * \psi_t(y)|^2}{(t+|x-y|)^{2\lambda}} dy \frac{dt}{t^{n+1-2\lambda}}\right)^{1/2} \\ &\equiv L_1(x) + L_2(x). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} L_1(x) &\le \left(\int_0^{2^{k-2}} \int_{|y|\le\frac{3}{2}\cdot 2^{k-2}} \frac{|f * \psi_t(y)|^2}{(t+|x-y|)^{2\lambda}} dy \frac{dt}{t^{n+1-2\lambda}}\right)^{1/2} \\ &+ \left(\int_0^{2^{k-2}} \int_{|y|>\frac{3}{2}\cdot 2^{k-2}} \frac{|f * \psi_t(y)|^2}{(t+|x-y|)^{2\lambda}} dy \frac{dt}{t^{n+1-2\lambda}}\right)^{1/2} \\ &\equiv M_1(x) + M_2(x). \end{aligned}$$

For $M_1(x)$, since $|z-y| > 2^{k-1}$ we have $|z| \ge |z-y| - |y| > \frac{1}{2}\cdot 2^{k-1} \ge c2^k$. Thus,

$$\begin{aligned} M_1(x) &\le c\frac{\|f\|_{L^1(\mathbb{R}^n)}}{2^{k(n+\alpha)}} \left(\int_0^{2^{k-2}} \int_{|y|\le\frac{3}{2}\cdot 2^{k-2}} \frac{dy}{|x-y|^{\ell}} \frac{dt}{t^{n+1-2\alpha-\ell}}\right)^{1/2} \\ &\le \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{2^{kn}} \end{aligned}$$

whenever $n - 2\alpha < \ell < n$. For $M_2(x)$, $|x-y| \ge |y| - |x| \ge \frac{1}{3}|y|$. Consequently,

$$\begin{aligned} M_2(x) &\le c\|f\|_{L^1(\mathbb{R}^n)} \left(\int_0^{2^{k-2}} \int_{|y|>\frac{3}{2}\cdot 2^{k-2}} \frac{dy}{|y|^{2\lambda}} \frac{dt}{t^{3n+1-2\lambda}}\right)^{1/2} \\ &\le \frac{c\|f\|_{L^1(\mathbb{R}^n)}}{2^{kn}} \end{aligned}$$

whenever $\lambda > 3n/2$.

Finally, for $L_2(x)$, we have

$$\begin{aligned} L_2(x) &\leq c\|f\|_{L^1(\mathbb{R}^n)}\left(\int_{2^{k-2}}^{\infty}\int_{\mathbb{R}^n}\frac{1}{(t+|x-y|)^{2\lambda}}\,dy\frac{dt}{t^{3n+1-2\lambda}}\right)^{1/2} \\ &\leq c\|f\|_{L^1(\mathbb{R}^n)}\left(\int_{2^{k-2}}^{\infty}\int_{|y|\leq 2^{k-1}}dy\frac{dt}{t^{3n+1}}\right)^{1/2} \\ &\quad + c\|f\|_{L^1(\mathbb{R}^n)}\left(\int_{2^{k-2}}^{\infty}\int_{|y|>2^{k-1}}\frac{1}{|y|^{\ell}}\,dy\frac{dt}{t^{3n+1-\ell}}\right)^{1/2} \end{aligned}$$

where we use $|x-y| > |y|/2$ to get the second summand. Both summands above are dominted by $\frac{c\|f\|_{L^1(\mathbb{R}^n)}}{2^{kn}}$ whenever $n < \ell < 3n$. Therefore we have verified (2.2) for $g^*_{\psi,\lambda}(f)$ also.

Theorem 3.1 is proved.

In recent years, several authors have considered the theory of Hardy spaces associated with the Herz spaces (see [3], [7], [8], [13], [15], [17] and [22]). First, we introduce the following definitions.

Let $\phi \in C_0^\infty(\mathbb{R}^n)$, supp $\phi \subseteq B_1$ and $\int_{\mathbb{R}^n}\phi(x)\,dx = 1$. And we set that $\phi_t(x) = t^{-n}\phi(x/t)$ with $t > 0$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the vertical maximal function $\phi^*(f)(x)$ and the local vertical maximal function $\widetilde{\phi}^*(f)(x)$ by

$$\phi^*(f)(x) = \sup_{t>0}|(f*\phi_t)(x)|$$

and

$$\widetilde{\phi}^*(f)(x) = \sup_{0<t\leq 1}|(f*\phi_t)(x)|.$$

The Herz-type Hardy spaces are defined as follows.

Definition 3.1. Let $0 < p \leq \infty$, $0 < q < \infty$, $\alpha \in \mathbb{R}$ and ϕ be as above.

(a) The homogeneous Herz-type Hardy spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $h\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ associated with $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ are defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)\colon\ \phi^*(f) \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}$$

and

$$h\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)\colon\ \widetilde{\phi}^*(f) \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|\phi^*(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$ and $\|f\|_{h\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|\widetilde{\phi}^*(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

(b) The non-homogeneous Herz-type Hardy spaces $HK_q^{\alpha,p}(\mathbb{R}^n)$ and $hK_q^{\alpha,p}(\mathbb{R}^n)$ associated with $K_q^{\alpha,p}(\mathbb{R}^n)$ are defined by

$$HK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)\colon\ \phi^*(f) \in K_q^{\alpha,p}(\mathbb{R}^n)\}$$

and

$$hK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)\colon \widetilde{\phi}^*(f) \in K_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} = \|\phi^*(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)}$ and $\|f\|_{hK_q^{\alpha,p}(\mathbb{R}^n)} = \|\widetilde{\phi}^*(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)}$.

Obviously, $H\dot{K}_p^{0,p}(\mathbb{R}^n) = H^p(\mathbb{R}^n) = HK_p^{0,p}(\mathbb{R}^n)$ and $h\dot{K}_p^{0,p}(\mathbb{R}^n) = h^p(\mathbb{R}^n) = hK_p^{0,p}(\mathbb{R}^n)$ with $0 < p < \infty$. Here the spaces $H^p(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$ are the standard Hardy spaces and local Hardy spaces respectively studied by Fefferman and Stein [4] and Goldberg [9]. Moreover, by the characterizations established in [7], [8], [10], [13], [15], [17] and [22], we know that Definition 3.1 is independent of the choice of the function ϕ.

THEOREM 3.2. *Let $0 < p \le \infty$, $1 < q < \infty$ and $-n/q < \alpha < n(1 - 1/q)$. Then $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n) = h\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n) = K_q^{\alpha,p}(\mathbb{R}^n) = hK_q^{\alpha,p}(\mathbb{R}^n)$.*

Proof. Notice that $K_q^{\alpha,p}(\mathbb{R}^n) \subset L^q_{\mathrm{loc}}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. From the simple inequality

$$|f(x)| \le \min\{\phi^*(f)(x),\ \widetilde{\phi}^*(f)(x)\} \text{ a.e. in } \mathbb{R}^n,$$

we obtain

$$K_q^{\alpha,p}(\mathbb{R}^n) \supseteq \{HK_q^{\alpha,p}(\mathbb{R}^n) \cup hK_q^{\alpha,p}(\mathbb{R}^n)\}. \tag{3.11}$$

On the other hand, by the definitions of $\phi^*(f)$ and $\widetilde{\phi}^*(f)$, we easily verify that $\phi^*(f)$ and $\widetilde{\phi}^*(f)$ satisfy the conditions of Corollary 2.1. Thus, we have

$$\|\phi^*(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)} + \|\widetilde{\phi}^*(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)} \le c\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)},$$

where C is independent of f. From this, we deduce that

$$K_q^{\alpha,p}(\mathbb{R}^n) \subseteq \{HK_q^{\alpha,p}(\mathbb{R}^n) \cap hK_q^{\alpha,p}(\mathbb{R}^n)\}. \tag{3.12}$$

By (3.11) and (3.12), we obtain

$$HK_q^{\alpha,p}(\mathbb{R}^n) = K_q^{\alpha,p}(\mathbb{R}^n) = hK_q^{\alpha,p}(\mathbb{R}^n).$$

Similarly, by Corollary 2.1, we easily prove

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \subseteq \{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \cap h\dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}.$$

To see the converse, we only need to show that if $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, then $f \in L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n)$ for $0 < q_1 \le q$ and $q_1 < n/(\alpha + n/q)$. Therefore, if $-n/q < \alpha < n(1 - 1/q)$, then

$f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In fact, we only need to verify that $f \in L^{q_1}(B_1)$. By Definition 1.1, we have

$$\begin{aligned}\|f\|^{q_1}_{L^{q_1}(B_1)} &= \sum_{k=-\infty}^{0} \|f\chi_k\|^{q_1}_{L^{q_1}(\mathbb{R}^n)} \le \sum_{k=-\infty}^{0} \|f\chi_k\|^{q_1}_{L^{q}(\mathbb{R}^n)} 2^{kn(1-q_1/q)} \\ &\le c\|f\|^{q_1}_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} \sum_{k=-\infty}^{0} 2^{k(n(1-q_1/q)-\alpha q_1)} < \infty\end{aligned}$$

since $n(1-q_1/q)-\alpha q_1 > 0$.

This finishes the proof of Theorem 3.2.

Remark 3.2. For g_ψ, the conclusions of Theorem 3.1 with $\alpha \ne 0$ and $\psi \in C^\infty_0(\mathbb{R}^n)$ can be found in [10]. For the spaces $H\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ and $HK^{\alpha,p}_q(\mathbb{R}^n)$, if $\alpha = n(1/p-1/q)$ with $1 < p < \infty$, Theorem 3.2 can be found in [14]; if $0 < \alpha < n(1-1/q)$, Theorem 3.2 can be found in [15]; if $-n/q < \alpha < n(1-1/q)$ with $\alpha \ne 0$, Theorem 3.2 has been obtained by Hernández and Yang in [10].

We also remark that if $1 < q < \infty$, $\alpha \ge n(1-1/q)$ and $0 < p < \infty$, then $hK^{\alpha,p}_q(\mathbb{R}^n)$ also equals $K^{\alpha,p}_q(\mathbb{R}^n)$; but, $h\dot{K}^{\alpha,p}_q(\mathbb{R}^n) \ne \dot{K}^{\alpha,p}_q(\mathbb{R}^n) \ne H\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ and $HK^{\alpha,p}_q(\mathbb{R}^n) \ne K^{\alpha,p}_q(\mathbb{R}^n)$. We also remark that $H\dot{K}^{\alpha,p}_q(\mathbb{R}^n) \subsetneqq h\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$; see [7], [8], [13], [15] and [22] for the details.

Remark 3.3. As we pointed out before, inequality (3.6) holds, in particular, for $\alpha = n(1/p-1/q)$ and $1 < p,\ q < \infty$. We remark that Theorem 3.1 is best possible under this restriction on α. In fact, if $\alpha = n(1/p-1/q)$ and $0 < p \le 1 < q \le 2$, Theorem 3.1 fails. Otherwise, $H\dot{K}^{\alpha,p}_q(\mathbb{R}^n) = \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ and $HK^{\alpha,p}_q(\mathbb{R}^n) = K^{\alpha,p}_q(\mathbb{R}^n)$ by the characterizations established in [8] (see also [3], [7], [13] and [15]). But this is not possible by the above remark.

Remark 3.4. By the results of Fefferman and Stein [4], we know that (1.1) is still true if we replace $L^p(\mathbb{R}^n)$ by the standard Hardy space $H^p(\mathbb{R}^n)$ when $0 < p \le 1$. In another words, the standard Hardy space $H^p(\mathbb{R}^n)$ can be characterized by the Littlewood-Paley g-functions. It is natural to ask if this is still true for Herz-type Hardy spaces ? Lu and Yang [13] obtained such a characterization on $H\dot{K}^{n/2,1}_2(\mathbb{R}^n)$ and $HK^{n/2,1}_2(\mathbb{R}^n)$. Later on, García-Cuerva and Herrero [8] obtained the characterization for $H\dot{K}^{n(1/p-1/q),p}_q(\mathbb{R}^n)$ and $HK^{n(1/p-1/q),p}_q(\mathbb{R}^n)$ for $0 < p \le 1 < q \le 2$. The methods used in the above papers can be trivially adapted to the spaces $H\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ and $HK^{\alpha,p}_q(\mathbb{R}^n)$ with $\alpha \ge n(1-1/q)$ $1 < q \le 2$ and $0 < p \le \infty$. However, it is still not known if there is such a characterization on Herz-type Hardy spaces with $q > 2$.

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WASHINGTON UNIVERSITY
ST. LOUIS, MISSOURI

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BEIJING, THE PEOPLE'S REPUBLIC OF CHINA