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MIXED AUTOMORPHIC FORMS ON SEMISIMPLE LIE GROUPS

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1. Introduction

Let Γ be a discrete subgroup of $PSL(2,\mathbb{R})$ and let $\chi\colon \Gamma \to SL(2,\mathbb{R})$ be a homomorphism of groups. Then both Γ and $\chi(\Gamma)$ operate on the Poincaré upper half plane $\mathcal{H}$ by linear fractional transformations. We assume that there is a holomorphic map $\omega\colon \mathcal{H} \to \mathcal{H}$ satisfying $\omega(gz) = \chi(g)\omega(z)$ for all $g \in \Gamma$ and $z \in \mathcal{H}$. Given a pair of nonnegative integers k and l with k even, a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ satisfying the condition

$$f(gz) = (cz+d)^k (c_\chi \omega(z) + d_\chi)^l f(z)$$

for all $z \in \mathcal{H}$ and

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \qquad \chi(g) = \begin{pmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{pmatrix} \in SL(2,\mathbb{R})$$

is a holomorphic mixed automorphic form of one variable of type (k, l) associated to Γ, ω and χ if f satisfies an additional condition of regularity at the cusps of Γ (see [16]). Certain types of such mixed automorphic forms occur naturally as holomorphic forms of the highest degree on elliptic varieties which are fiber varieties over an arithmetic variety with generic fiber a product of elliptic curves (cf. [10], [15]). Holomorphic mixed automorphic forms of several variables were also introduced in [17] and [18], and it was proved that a certain class of such automorphic forms can be interpreted as holomorphic forms on some families of abelian varieties over an arithmetic variety.

The purpose of this paper is to describe mixed automorphic forms in the setting of representations of semisimple Lie groups following such descriptions for the usual automorphic forms initiated by Selberg and Langlands (e.g., see [3], [4], [9], [21]). More specifically, we define mixed automorphic forms on semisimple Lie groups which generalize holomorphic mixed automorphic forms, and construct Poincaré series and Eisenstein series for such automorphic forms.

2. Mixed automorphic forms on a semisimple Lie group

First, we shall review the definition of the usual automorphic forms on semisimple Lie groups (e.g., see [3], [4], [5], [6]). Let G be a semisimple Lie group, and let $\mathfrak{g}$ be

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its Lie algebra. If V is a finite-dimensional complex vector space, then $\mathfrak{g}$ operates on smooth functions $f\colon G \to V$ by

$$(Y \cdot f)(g) = \frac{d}{dt} f\big((\exp tY)g\big)\Big|_{t=0}$$

for $Y \in \mathfrak{g}$ and $g \in G$. Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Then a vector-valued function $f\colon G \to V$ is said to be $Z(\mathfrak{g})$-*finite* if $Z(\mathfrak{g}) \cdot f$ is a finite-dimensional vector space. If $\alpha\colon G \to GL(W)$ is a finite-dimensional complex representation whose kernel is finite and whose image is closed in $\mathrm{End}(W)$, then we can define a norm $\| \cdot \|_\alpha$ on G by

$$\|g\|_\alpha = \Big(\mathrm{Tr}\big(\alpha(g)^* \cdot \alpha(g)\big)\Big)^{1/2},$$

where $*$ denotes the adjoint with respect to a Hilbert space structure on W invariant under a maximal compact subgroup K of G. If β is another such representation, then there is a constant $M > 0$ and a positive integer m such that

$$\|x\|_\alpha \leq M\|x\|_\beta^m$$

for all $x \in G$. A vector valued function $f\colon G \to V$ is said to be *slowly increasing* if there is a norm $\| \cdot \|$ on G, a constant $C > 0$, and a positive integer m such that

$$|f(g)| \leq C\|g\|^m,$$

where $| \cdot |$ is a norm on V.

Definition 2.1. Let K be a maximal compact subgroup of G, Γ a discrete subgroup of G, and $\sigma\colon K \to GL(V)$ a representation of K in a finite-dimensional complex vector space V. A smooth vector-valued function $f\colon G \to V$ is an *automorphic form for* Γ *and* σ if the following conditions are satisfied:

(i) $f(kg\gamma) = \sigma(k)f(g)$ for all $k \in K$ and $\gamma \in \Gamma$.
(ii) f is $Z(\mathfrak{g})$-finite.
(iii) f is slowly increasing.

Let G, K, Γ, and the representation $\sigma\colon K \to GL(V)$ be as in Definition 2.1, and consider another semisimple Lie group G'. Let K' be a maximal compact subgroup and Γ' a discrete subgroup of G'. Let $\rho\colon G \to G'$ be a homomorphism such that $\rho(K) \subset K'$ and $\rho(\Gamma) \subset \Gamma'$, and let $\sigma'\colon K' \to GL(V')$ be a representation of K' in a finite-dimensional complex vector space V'. Then we obtain the representation $\sigma \otimes (\sigma' \circ \rho|_K)\colon K \to GL(V \otimes V')$ of K in $V \otimes V'$, where $\rho|_K$ denotes the restriction of ρ to K.

Definition 2.2. A *mixed automorphic form for* Γ *of type* (ρ, σ, σ') is an automorphic form for Γ and the representation $\sigma \otimes (\sigma' \circ \rho|_K)$.

Let $f_\sigma\colon G \to V$ (resp. $f'_{\sigma'}\colon G' \to V'$) be an automorphic form for Γ (resp. Γ') and $\sigma\colon K \to GL(V)$ (resp. $\sigma'\colon K' \to GL(V')$), where V (resp. V') is a finite-dimensional complex vector space. We denote by $f_{\rho,\sigma,\sigma'}$ the function from G to $V \otimes V'$ given by

$$f_{\rho,\sigma,\sigma'}(g) = (f_\sigma \otimes (f'_{\sigma'} \circ \rho))(g) = f_\sigma(g) \otimes f'_{\sigma'}(\rho(g))$$

for all $g \in G$.

PROPOSITION 2.3. *The function* $f_{\rho,\sigma,\sigma'}\colon G \to V \otimes V'$ *described above is a mixed automorphic form for* Γ *of type* (ρ, σ, σ').

Proof. We must show that $f_{\rho,\sigma,\sigma'}$ satisfies the conditions (i), (ii) and (iii) in Definition 2.1 for the discrete group Γ and the representation $\sigma \otimes (\sigma' \circ \rho|_K)$. For $g \in G$, $k \in K$ and $\gamma \in \Gamma$, we have

$$\begin{aligned} f_{\rho,\sigma,\sigma'}(kg\gamma) &= f(kg\gamma) \otimes f'(\rho(k)\rho(g)\rho(\gamma)) \\ &= \sigma(k) f(g) \otimes \sigma'(\rho(k)) f'(\rho(g)) \\ &= (\sigma \otimes (\sigma' \circ \rho|_K))(k) \cdot (f \otimes f' \circ \rho)(g), \end{aligned}$$

which shows the condition (i). Now, given $Y \in Z(\mathfrak{g})$, we have

$$\begin{aligned} Y \cdot f_{\rho,\sigma,\sigma'}(g) &= \frac{d}{dt} f_{\rho,\sigma,\sigma'}\big((\exp tY)g\big)\Big|_{t=0} \\ &= \left[\frac{d}{dt} f_\sigma\big((\exp tY)g\big)\right]_{t=0} \otimes f'_{\sigma'}(\rho(g)) \\ &\quad + f_\sigma(g) \otimes \left[\frac{d}{dt} f'_{\sigma'}\big(\rho((\exp tY)g)\big)\right]_{t=0} \\ &= \left[\frac{d}{dt} f_\sigma\big((\exp tY)g\big)\right]_{t=0} \otimes f'_{\sigma'}(\rho(g)) \\ &\quad + f_\sigma(g) \otimes \left[\frac{d}{dt} f'_{\sigma'}\big((\exp td\rho Y)\rho(g)\big)\right]_{t=0}. \end{aligned}$$

Thus the condition (ii) follows from the fact that f_σ is $Z(\mathfrak{g})$-finite and $f'_{\sigma'}$ is $Z(\mathfrak{g}')$-finite. As for the condition (iii), since f_σ and $f'_{\sigma'}$ are slowly increasing, we have

$$|f_{\rho,\sigma,\sigma'}(g)| = |f_\sigma(g) \otimes f'_{\sigma'}(\rho(g))| \leq C_1 \|g\|_\alpha^{m_1} \cdot C_2 \|\rho(g)\|_\beta^{m_2}$$

for some constants C_1, C_2, positive integers m_1, m_2, and representations $\alpha\colon G \to GL(W)$, $\beta\colon G \to GL(W')$. However, we have

$$\|\rho(g)\|_\beta = \|g\|_{\beta\circ\rho} \leq C_3\|g\|_\alpha^{m_3}$$

for some constant C_3 and positive integer m_3. Thus we have

$$|f_{\rho,\sigma,\sigma'}(g)| \leq C_1C_2C_3\|g\|_\alpha^{m_1+m_3},$$

and $f_{\rho,\sigma,\sigma'}$ is slowly increasing. □

Example 2.4. Let $\rho\colon G \to G'$, K, K', V, V' be as above. Assume that the symmetric spaces $D = G/K$ and $D' = G'/K'$ have G-invariant and G'-invariant complex structures, respectively. This assumption implies that D and D' are equivalent to bounded symmetric domains (e.g., see [8]). Let $J\colon G \times D \to GL(V)$ and $J'\colon G' \times D' \to GL(V')$ be automorphy factors, and let $\tau\colon D \to D'$ be a holomorphic map satisfying

$$\tau(gz) = \rho(g)\tau(z)$$

for all $g \in G$ and $z \in D$. Then we can define a mixed automorphic vector bundle $\mathcal{M}$ (see [18]) on the Shimura variety $X = \Gamma\backslash D$ whose sections can be considered as holomorphic mappings $f\colon D \to V \otimes V'$ satisfying

$$f(gz) = \big(J(g,z) \otimes J'(\rho(g), \tau(z))\big)f(z)$$

for all $g \in G$ and $z \in D$. Given such a bundle $\mathcal{M}$ associated to J, J', ρ and τ, we define mappings $\sigma\colon G \to GL(V)$ and $\sigma'\colon G' \to GL(V')$ by

$$\sigma(k) = J(k,0), \quad \sigma'(k') = J'(k',0'),$$

where $0 \in D$ and $0' \in D'$ are the fixed points of K and K', respectively. Then σ and σ' are representations of G and G', respectively, and the sections of $\mathcal{M}$ are mixed automorphic forms for Γ of type (ρ, σ, σ').

Remark 2.5. In Example 2.4, if G' is a symplectic group and if Γ is cocompact, then it was shown in [18] that for some specific automorphy factors J and J' the sections of the mixed automorphic vector bundle $\mathcal{M}$ can be identified with holomorphic forms on certain families of abelian varieties parametrized by a Shimura variety. In the elliptic modular case, that is, when $G = SL(2, \mathbb{R})$, various results have been obtained concerning the corresponding mixed automorphic forms (e.g., see [10], [15], [16]). Similar geometric aspects for the Siegel modular case were treated in [17].

3. Poincaré series

In this section, as our first examples of mixed automorphic forms, we shall construct Poincaré series. Let Γ (resp. Γ') be a discrete subgroup of a semisimple Lie group G (resp. G'), and let K (resp. K') be a maximal compact subgroup of G (resp. G'). Let $\rho\colon G \to G'$ be a homomorphism such that $\rho(K) \subset K'$. If $f\colon G \to V$ is a vector-valued function and if h is an element of G, we denote by $l(h)$ and $r(h)$ the translation operators given by

$$l(h)f(g) = f(h^{-1}g), \quad r(h)f(g) = f(gh)$$

for all $g \in G$.

Definition 3.1. A vector-valued function $f\colon G \to V$ on G is said to be *left* (resp. *right*) *K-finite* if the set of left (resp. right) translations

$$\{l(k)f \mid k \in K\} \quad (\text{resp. } \{r(k)f \mid k \in K\})$$

of f by elements of K spans a finite-dimensional vector space.

PROPOSITION 3.2. *Let V and V' be finite-dimensional complex vector spaces, and assume that the following conditions are satisfied*:

(i) $f \otimes (f' \circ \rho) \in L^1(G) \otimes (V \otimes V')$, *where $L^1(G)$ denotes the set of integrable functions on G.*
(ii) *f is $Z(\mathfrak{g})$-finite and f' is $Z(\mathfrak{g}')$-finite.*
(iii) *f is right K-finite and f' is right K'-finite.*

Then the series $P_{\rho,f,f'}(g)$ defined by

$$P_{\rho,f,f'}(g) = \sum_{\gamma\in\Gamma}(f \otimes (f' \circ \rho))(g \cdot \gamma)$$

converges absolutely and uniformly on compact sets. Furthermore, the series

$$\sum_{\gamma\in\Gamma}|(f \otimes (f' \circ \rho))(g \cdot \gamma)|$$

is bounded on G where $|\cdot|$ denotes the norm on $V \otimes V'$.

Proof. As in the proof of Proposition 2.3 we can show that the function $f \otimes (f' \circ \rho)\colon G \to V \otimes V'$ is $Z(\mathfrak{g})$-finite using condition (ii). For $k \in K$ and $g \in G$, we have

$$\begin{aligned} r(k)(f \otimes (f' \circ \rho))(g) &= (f \otimes (f' \circ \rho)) \\ (gk) &= f(gk) \otimes f'(\rho(g)\rho(k)) \\ &= r(k)f(g) \otimes r(\rho(k))f'(\rho(k)). \end{aligned}$$

Hence condition (iii) implies that $f \otimes (f' \circ \rho)$ is right K-finite. Therefore the proposition follows from [1, Theorem 23], [2, Theorem 5.4], or [3, Theorem 9.1]. □

Definition 3.3. The series

$$P_{\rho,f,f'}(g) = \sum_{\gamma\in\Gamma}(f \otimes (f' \circ \rho))(g \cdot \gamma)$$

is called a *Poincaré series associated to* ρ, f *and* f'.

COROLLARY 3.4. *Let* $\sigma\colon K \to GL(V)$ *and* $\sigma'\colon K' \to GL(V')$ *be finite-dimensional representations of* K *and* K', *respectively. Assume the* f *and* f'*satisfy the conditions* (*i*), (*ii*) *and* (*iii*) *of Proposition* 3.2 *together with the condition that*

$$f(kg) = \sigma(k)f(g), \quad f'(k'g') = \sigma'(k')f'(g')$$

for $k \in K$, $g \in G$, $k' \in K'$ *and* $g' \in G'$. *Then the Poincaré series* $P_{\rho,f,f'}$ *is a mixed automorphic form for* Γ *of type* (ρ, σ, σ').

Proof. Since $f \otimes (f' \circ \rho)$ is $Z(\mathfrak{g})$-finite on the left, so is the Poincaré series $P_{\rho,f,f'}$. From the definition of $P_{\rho,f,f'}(g)$ the right Γ-invariance of $P_{\rho,f,f'}$ follows immediately, and $P_{\rho,f,f'}$ is slowly increasing by Proposition 3.2. As in the proof of Proposition 2.3, we have

$$(f \otimes (f' \circ \rho))(kg) = (\sigma \otimes (\sigma' \circ \rho|_K))(k)(f \otimes (f' \circ \rho))(g);$$

hence it follows that

$$P_{\rho,f,f'}(kg) = (\sigma \otimes (\sigma' \circ \rho|_K))(k)P_{\rho,f,f'}(g),$$

and the proof of the corollary is complete. □

Let G, G', K, K', ρ, and τ be as in Example 2.4. In particular, $D = G/K$ and $D' = G'/K'$ are symmetric domains, and $\tau\colon D \to D'$ is a holomorphic mapping satisfying

$$\tau(gz) = \rho(g)\tau(z)$$

for all $g \in G$ and $z \in D$. Let $J\colon G \times D \to GL(V)$ (resp. $J'\colon G' \times D' \to GL(V')$) be the canonical automorphy factor (see [2, §1.8], [20, §II.5]), and let $j\colon G \times D \to \mathbb{C}$ (resp. $j'\colon G' \times D' \to \mathbb{C}$) be the corresponding Jacobian determinant function.

LEMMA 3.5. *If* l *and* m *are nonnegative integers with* $l \geq 2$, *then the mapping*

$$g \mapsto j(g, 0)^l \otimes j'(\rho(g), 0')^m\colon G \to \mathbb{C}$$

is in $L^1(G)$.

Proof. The mapping $g \mapsto |j(g,0)|^l$ (resp. $g' \mapsto |j'(g',0')|^m$) is left and right K-invariant (resp. K'-invariant) (see [2, §5.8]); hence it can be considered as a function on D (resp. D'), and we have

$$\int_G |j(g,0)^l j'(\rho(g),0')^m| \, dg = \int_D |j(z,0)|^l |j'(\tau(z),0')|^m d\mu(z),$$

where $d\mu(z)$ denotes the invariant Bergmann measure on D (cf. [1, §4.3]). If dz is the usual Euclidean measure on D, then we have

$$d\mu(z) = |j(z,0)|^{-2} dz$$

up to a positive factor. Thus we have

$$\int_G |j(g,0)^l j'(\rho(g),0')^m| \, dg = \int_D |j(z,0)|^{l-2} |j'(\tau(z),0')|^m dz.$$

However, both $|j(z,0)|$ and $|j(\tau(z),0')|$ are bounded by [2, Proposition 1.12]; hence the lemma follows. □

Let $F\colon D \to V$ and $F'\colon D' \to V'$ be mappings such that $F \otimes (F' \circ \tau)\colon D \to V \otimes V'$ is a polynomial mapping. Given nonnegative integers l, m with $l \geq 2$, we define the functions $f\colon G \to V$ and $f'\colon G' \to V'$ by

$$f(g) = j(g,0)^l F(\pi g), \quad f'(g') = j'(g',0')^m F'(\pi' g')$$

for $g \in G$ and $g' \in G'$, where $0 \in D$ and $0' \in D'$ are the fixed points of K and K', respectively, and $\pi\colon G \to D$ and $\pi'\colon G' \to D'$ are canonical projection mappings. We set

$$\sigma(k) = j(k,0)^l, \quad \sigma'(k') = j(k',0')^m$$

for $k \in K$ and $k' \in K'$. Then σ and σ' are representations of G and G' in V and V', respectively. We set

$$P^{l,m}_{\rho,j,j'}(g) = \sum_{\gamma \in \Gamma} (f \otimes (f' \circ \rho))(g\gamma)$$

for $g \in G$, which is the Poincaré series $P_{\rho,f,f'}$ in the sense of Definition 3.3.

THEOREM 3.6. *The Poincaré series $P^{l,m}_{\rho,j,j'}(g)$ is a mixed automorphic form for Γ of type (ρ, σ, σ').*

Proof. For each $g \in G$ we have

$$\begin{aligned}(f \otimes (f' \circ \rho))(g) &= \big(j(g,0)^l F(\pi g)\big) \otimes \big(j'(\rho(g),0')^m F'(\pi'(\rho(g)))\big) \\ &= \big(j(g,0)^l \cdot j'(\rho(g),0')^m\big)\big(F(\pi g) \otimes F'(\omega(\pi g))\big).\end{aligned}$$

Thus from Lemma 3.5 and the fact that $F \otimes (F' \circ \omega)$ is a polynomial mapping on the bounded symmetric domain D it follows that

$$\int_G |(f \otimes (f' \circ \rho))(g)|\, dg < \infty;$$

hence $f \otimes (f' \circ \rho)$ is in $L^1(G) \otimes (V \otimes V')$. Since $f(kg) = \sigma(k) f(g)$ and $f'(k'g') = \sigma'(k') f'(g')$ for $k \in K$, $k' \in K'$, $g \in G$ and $g' \in G'$, it follows that f is $Z_{\mathfrak{g}}$-finite and f' is $Z_{\mathfrak{g}'}$-finite. For $k \in K$ and $g \in G$ we have $j(gk, 0) = j(g, 0) j(k, 0)$, and the set of right translates $r(k)F(\pi(gk))$ are polynomials of the same degree as F; hence f is right K- finite. Similarly, f' is right K'-finite. Thus it follows from Proposition 3.2 that the series $\sum_{\gamma \in \Gamma} |(f \otimes (f' \circ \rho))(g\gamma)|$ is bounded on G. In particular, $P^{l,m}_{\rho,j,j'}$ is slowly increasing. As in the proof of Proposition 2.3, $f \otimes (f' \circ \rho)$ is $Z_{\mathfrak{g}}$-finite, and for $k \in K$ and $g \in G$ we have

$$(f \otimes (f' \circ \rho))(kg) = (\sigma \otimes (\sigma' \circ \rho|_K))(k)(f \otimes (f' \circ \rho))(g);$$

hence it follows that

$$P^{l,m}_{\rho,j,j'}(kg) = (\sigma \otimes (\sigma' \circ \rho|_K))(k) P^{l,m}_{\rho,j,j'}(g).$$

Therefore $P^{l,m}_{\rho,j,j'}(g)$ is an automorphic form for Γ of type (ρ, σ, σ') in the sense of Definition 2.1. □

4. Eisenstein series

In order to discuss Eisenstein series, instead of the usual semisimple Lie group G we shall first consider an algebraic group whose set of real points will coincide with G. Let $\mathbb{G}$ be a connected, semisimple, linear algebraic group defined over a subfield k of $\mathbb{R}$, and let $\mathbb{P}$ be a k-parabolic subgroup of $\mathbb{G}$ containing a maximal k-split torus $\mathbb{S}$ of $\mathbb{G}$. Let $\mathbb{P}_0$ be a minimal k-parabolic subgroup of $\mathbb{G}$ such that $\mathbb{S} \subset \mathbb{P}_0 \subset \mathbb{P}$. We define an ordering on the set Σ_k of k-roots of $\mathbb{G}$ with respect to $\mathbb{S}$ as follows: A k-root $\alpha \in \Sigma_k$ is positive if and only if the subgroup of $\mathbb{G}$ generated by the α-eigenspace of the adjoint representation of $\mathbb{S}$ is contained in $\mathbb{P}_0$. We denote by Δ_k the set of simple positive k-roots in Σ_k.

If Ξ is a subset of Δ_k, we set

$$\mathbb{S}_\Xi = \left(\bigcap_{\alpha \in \Xi} \operatorname{Ker} \alpha \right)^0,$$

where $(\,)^0$ denotes the connected component of the identity. Let Θ be the subset of Δ_k such that $\mathbb{P}$ is generated by the unipotent radical $\mathbb{U}_0$ of $\mathbb{P}_0$ and by the centralizer

$Z(\mathbb{S}_\Theta)$ of $\mathbb{S}_\Theta$. Let $\mathbb{U}_\Theta$ be the unipotent radical of $\mathbb{P}$, and let $\mathbb{M}_\Theta$ be a subgroup of $\mathbb{G}$ such that $Z_{\mathbb{G}}(\mathbb{S}_\Theta) = \mathbb{S}_\Theta \cdot \mathbb{M}_\Theta$ with $\mathbb{S}_\Theta \cap \mathbb{M}_\Theta$ finite. We set

$$P = \mathbb{P}(\mathbb{R}), \quad A = \mathbb{S}_\Theta(\mathbb{R})^0, \quad M = \mathbb{M}_\Theta(\mathbb{R}), \quad U = \mathbb{U}_\Theta(\mathbb{R}).$$

Then we obtain the Langlands decomposition

$$P = MAU$$

of P and the corresponding decomposition

$$G = KP = KMAU$$

of $G = G(\mathbb{R})$. If $g = kmau \in G$ with $k \in K$, $m \in M$, $a \in A$ and $u \in U$, then $k \cdot m$, a and u are uniquely determined. We write $a = a(g)$.

Let $\{\Lambda_\alpha\}_{\alpha \in \Delta_k}$ be the set of fundamental dominant k-weights of $\mathbb{G}$ that satisfy

$$\langle \Lambda_\alpha, \beta \rangle = d_\alpha \delta_{\alpha\beta}$$

for all $\alpha, \beta \in \Delta_k$, where $\delta_{\alpha\beta}$ is the Kronecker delta and d_α is a positive real number (see [3, §11]). Let $\mathfrak{u}$ be the Lie algebra of U and let $\chi = \det \mathrm{Ad}_{\mathfrak{u}}$ be the character of P with $p^\chi = \det \mathrm{Ad}_{\mathfrak{u}} p$ for $p \in P$. We set $\tilde{\Theta} = \Delta_k - \Theta$. Then χ is a positive linear combination of the Λ_α for $\alpha \in \tilde{\Theta}$, i.e.,

$$\chi = \sum_{\alpha \in \tilde{\Theta}} e_\alpha \Lambda_\alpha \qquad \text{with } e_\alpha > 0.$$

If $\{s_\alpha\}_{\alpha \in \tilde{\Theta}}$ is a set of complex numbers $s_\alpha \in \mathbb{C}$ associated to each $\alpha \in \tilde{\Theta}$, and if $p \in P$, then we set

$$p^{\Lambda_s} = \prod_{\alpha \in \tilde{\Theta}} |p^{\Lambda_\alpha}|^{s_\alpha}.$$

LEMMA 4.1. *Let Γ be a discrete subgroup of G and let Γ_∞ be a subgroup of $\Gamma \cap MU$. Suppose that there is a set $\{s_\alpha\}_{\alpha \in \tilde{\Theta}}$ of complex numbers satisfying the following conditions*:

(i) $a(\gamma)^{\Lambda_\alpha} \geq \epsilon > 0$ *for all* $\gamma \in \Gamma$ *and* $\alpha \in \tilde{\Theta}$.
(ii) MU/Γ_∞ *has finite measure.*
(iii) $\mathrm{Re}\, s_\alpha > e_\alpha$ *for all* $\alpha \in \tilde{\Theta}$.

Then the series

$$E(g, s) = \sum_{\gamma \in \Gamma/\Gamma_\infty} a(g\gamma)^{-\Lambda_s}$$

converges uniformly on any compact subset of G.

Proof. See Lemma 4 in [1] or Lemma 11.1 in [3] (see also [7]). □

Let $G' = \mathbb{G}'(\mathbb{R})$ be another semisimple Lie group associated to a connected algebraic group $\mathbb{G}'$ defined over a subfield k of $\mathbb{R}$. We consider the corresponding subgroups K', P', M', A', U', etc. defined in a way similar to the case of G above. Thus we have decompositions

$$P' = K'A'U', \quad G' = K'P' = K'M'A'U'.$$

Let $\rho\colon G \to G'$ be a Lie group homomorphism such that

$$\rho(K) \subset K', \quad \rho(P) \subset P', \quad \rho(A) \subset A'.$$

THEOREM 4.2. *Let Γ and Γ_∞ be as in Lemma* 4.1, *and let $f\colon G \to V$ and $f'\colon G' \to V'$ be smooth vector-valued functions, where V and V' are finite-dimensional vector spaces. Suppose that there is a set $\{s_\alpha\}_{\alpha\in\tilde{\Theta}}$ of complex numbers satisfying the following conditions*:

(i) $a(\gamma)^{\Lambda_\alpha} \geq \epsilon > 0$ *for all* $\gamma \in \Gamma$ *and* $\alpha \in \tilde{\Theta}$.
(ii) MU/Γ_∞ *has finite measure.*
(iii) $\operatorname{Re} s_\alpha > e_\alpha$ *for all* $\alpha \in \tilde{\Theta}$.
(iv) $(f \otimes (f' \circ \rho))(g\gamma) = (f \otimes (f' \circ \rho))(g)$ *for all* $\gamma \in \Gamma_\infty$.
(v) $|f(gp)||f'(\rho(gp))|p^{\Lambda_s}$ *is bounded for* $p \in P$ *and* g *in a fixed compact set.*

Then the series

$$E_{\rho,f,f'}(g) = \sum_{\gamma\in\Gamma/\Gamma_\infty} (f \otimes (f' \circ \rho))(g\gamma)$$

converges absolutely and uniformly on any compact set of G.

Proof. Since $G = KP$, we have

$$\begin{aligned}(f \otimes (f' \circ \rho))(g) \cdot a(g)^{\Lambda_s} &= (f \otimes (f' \circ \rho))(kp) \cdot a(kp)^{\Lambda_s} \\ &= (f \otimes (f' \circ \rho))(kp) \cdot a(p)^{\Lambda_s} \\ &= (f \otimes (f' \circ \rho))(kp) \cdot p^{\Lambda_s}\end{aligned}$$

for $g = kp$ with $k \in K$ and $p \in P$. Hence by (v) $|(f \otimes (f' \circ \rho))(g) \cdot a(g)^{\Lambda_s}|$ is bounded for $g \in G$. Therefore, the series defining $E_{f,f',\rho}$ is majorized by a constant times the series

$$\sum_{\gamma\in\Gamma/\Gamma_\infty} a(g\gamma)^{-\Lambda_s},$$

which converges uniformly on any compact set by Lemma 4.1. Hence the theorem follows. □

Definition 4.3. The series

$$E_{\rho,f,f'}(g) = \sum_{\gamma \in \Gamma/\Gamma_\infty} (f \otimes (f' \circ \rho))(g\gamma)$$

is called an *Eisenstein series for* Γ *associated to* ρ, f *and* f'.

5. Eisenstein series as mixed automorphic forms

In this section we discuss a special class of Eisenstein series and show that they are mixed automorphic forms. Let G, P, M, A, and U be as in §4. Thus we have decompositions

$$P = MAU, \quad G = KP = KMAU.$$

Let $\sigma\colon K \to GL(V)$ be a representation of K in a finite-dimensional complex vector space V. Let $\pi\colon MU \to M$ be the natural projection, and let $K_M = \pi(K \cap MU)$. Then K_M is a maximal compact subgroup of M, and σ induces the representation σ_M of K_M given by

$$\sigma_M(\pi(k)) = \sigma(k)$$

for all $k \in K \cap MU$. Let Γ be an arithmetic subgroup of G, and let $\Gamma_M = \pi(\Gamma \cap MU)$ be the corresponding arithmetic subgroup of M. We denote by $L^2(M/\Gamma_M, \sigma_M)$ the space of square-integrable functions $\varphi\colon M \to V$ satisfying

$$\varphi(km\gamma) = \sigma_M(k)\varphi(m)$$

for all $k \in K_M$, $m \in M$ and $\gamma \in \Gamma_M$. Any function $\varphi\colon M \to V$ satisfying $\varphi(km) = \sigma_M(k)\varphi(m)$ for $k \in K_M$ and $m \in M$ can be extended to a function $\varphi_G\colon G \to V$ on G by the formula

$$\varphi_G(kmau) = \sigma(k)\varphi(m)$$

for all $k \in K$, $m \in M$, $a \in A$ and $u \in U$. Then φ_G is σ-equivariant; i.e.,

$$\varphi_G(kg) = \sigma(k)\varphi_G(g)$$

for $k \in K$ and $g \in G$. Although a decomposition $g = kmau$ is not unique, the extension φ_G is uniquely determined. We shall identify a σ_M-equivariant function φ on M and the corresponding σ-equivariant function φ_G on G. Thus each element of $L^2(M/\Gamma_M, \sigma_M)$ will be regarded as a function of G into V.

Let $\mathfrak{a}_{\mathbb{C}}^*$ be the dual space of the complexification $\mathfrak{a}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{a}$ of A, and let

$$(\mathfrak{a}_{\mathbb{C}}^*)^- = \{\Lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \langle \operatorname{Re}\Lambda + \rho, \alpha\rangle < 0 \text{ for all } \alpha \in \Sigma^0\},$$

where Σ^0 is the set of simple roots of the Lie algebra $\mathfrak{g}$ of G. For $\varphi \in L^2(M/\Gamma_M, \sigma_M)$ and $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we set

$$\varphi_\Lambda(x) = \varphi(x)e^{(\Lambda-\rho)(H(x))}$$

for $x \in G$, where $H(x)$ denotes $\log a(x)$. Then we have

$$\varphi_\Lambda(kg\gamma) = \sigma(k)\varphi_\Lambda(g)$$

for all $g \in G, k \in K$ and $\gamma \in U(\Gamma \cap P)$. Given $\varphi \in L^2(M/\Gamma_M, \sigma_M)$ and $\Lambda \in (\mathfrak{a}_{\mathbb{C}}^*)^-$, the series

$$E(\Lambda, \varphi, x) = \sum_{\Gamma/\Gamma\cap P} \varphi_\Lambda(x\gamma)$$

is called an *Eisenstein series* (see [9]; also see [11], [12], [13], [14], [19]).

Let χ be a representation of $Z(\mathfrak{g})$ in V. We denote by $L^2(M/\Gamma_M, \sigma_M, \chi)$ the subgroup of $L^2(M/\Gamma_M, \sigma_M)$ consisting of functions $f\colon G \to V$ satisfying the condition

$$(Y \cdot f)(g) = f(g)\chi(Y)$$

for all $Y \in Z(\mathfrak{g})$.

LEMMA 5.1. *If $\varphi \in L^2(M/\Gamma_M, \sigma_M, \chi)$ and $\gamma \in U(\Gamma \cap P)$, then there are a positive real number C and a positive integer N such that*

$$\sum_{\Gamma/\Gamma\cap P} |\varphi_\Lambda(x\gamma)| \le C\|x\|^N$$

for all $x \in G$.

Proof. This follows from [9, Lemma 24]. □

Let G' be another semisimple Lie group, and consider the corresponding objects K', P', M', A', U', $\mathfrak{a}'$, $\mathfrak{a}_{\mathbb{C}}'^*$, $(\mathfrak{a}_{\mathbb{C}}'^*)^-$ and the representation $\sigma'\colon K' \to GL(V')$ of K' in a finite-dimensional complex vector space V'. Let $\rho\colon G \to G'$ be a Lie group homomorphism such that

$$\rho(K) \subset K', \quad \rho(P) \subset P', \quad \rho(A) \subset A'.$$

As in the case of G, for $\varphi' \in L^2(M'/\Gamma'_{M'}, \sigma'_{M'})$ and $\Lambda' \in (\mathfrak{a}_{\mathbb{C}}'^*)^-$, we set

$$\varphi'_{\Lambda'}(y) = \varphi'(y)e^{(\Lambda'-\rho')(H'(y))}$$

for $y \in G'$, where $H'(y)$ denotes $\log a'(y)$.

PROPOSITION 5.2. *Let χ and χ' be representations of $Z(\mathfrak{g})$ and $Z(\mathfrak{g}')$ in V and V' respectively, and let $\varphi_\Lambda \in L^2(M/\Gamma_M, \sigma_M, \chi)$ and $\varphi'_{\Lambda'} \in L^2(M'/\Gamma'_{M'}, \sigma'_{M'}, \chi')$ with $\Lambda \in (\mathfrak{a}_{\mathbb{C}}^*)^-$ and $\Lambda' \in (\mathfrak{a}_{\mathbb{C}}'^*)^-$. Then there are a positive real number C_0 and a positive integer N_0 such that*

$$\sum_{\gamma \in \Gamma/\Gamma\cap P} |(\varphi_\Lambda \otimes (\varphi'_{\Lambda'} \circ \rho))(x\gamma)| \leq C_0 \|x\|^{N_0}$$

for all $x \in G$.

Proof. By Lemma 5.1 there are C, C', N, and N' such that

$$\sum_{\gamma \in \Gamma/\Gamma\cap P} |\varphi_\Lambda(x\gamma)| \leq C\|x\|^N, \qquad \sum_{\gamma' \in \Gamma'/\Gamma'\cap P'} |\varphi'_{\Lambda'}(x'\gamma')| \leq C'\|x'\|^{N'}$$

for all $x \in G$ and $x' \in G'$. In particular, we have

$$\begin{aligned} \sum_{\gamma \in \Gamma/\Gamma\cap P} |\varphi'_{\Lambda'}(\rho(x\gamma))| &= \sum_{\gamma \in \Gamma/\Gamma\cap P} |\varphi'_{\Lambda'}(\rho(x)\rho(\gamma))| \\ &\leq C'\|\rho(x)\|^{N'} \leq C'\|\rho\|^{N'}\|x\|^{N'} \end{aligned}$$

for all $x \in G$, since $\rho(\gamma) \in \Gamma'/\Gamma' \cap P'$ whenever $\gamma \in \Gamma/\Gamma \cap P$. Thus we obtain

$$\begin{aligned} \sum_{\gamma \in \Gamma/\Gamma\cap P} |(\varphi_\Lambda \otimes (\varphi'_{\Lambda'} \circ \rho))(x\gamma)| &= \sum_{\gamma \in \Gamma/\Gamma\cap P} |\varphi_\Lambda(x\gamma)| \cdot |\varphi'_{\Lambda'}(\rho(x\gamma))| \\ &\leq \sum_{\gamma \in \Gamma/\Gamma\cap P} |\varphi_\Lambda(x\gamma)| \cdot \sum_{\gamma \in \Gamma/\Gamma\cap P} |\varphi'_{\Lambda'}(\rho(x\gamma))| \\ &\leq C \cdot C' \cdot \|\rho\|^{N'} \cdot \|x\|^{N+N'}; \end{aligned}$$

hence the proposition follows. □

We set

$$E_{\rho,\varphi,\varphi'}(\Lambda, \Lambda', x) = \sum_{\gamma \in \Gamma/\Gamma\cap P} (\varphi_\Lambda \otimes (\varphi'_{\Lambda'} \circ \rho))(x\gamma)$$

for $x \in G$, which is an Eisenstein series for Γ associated to ρ, φ_Λ, and $\varphi'_{\Lambda'}$ in the sense of Definition 4.3 with $\Gamma_\infty = \Gamma \cap MU = \Gamma \cap P$ (see [9, p. 6]).

THEOREM 5.3. *The Eisenstein series $E_{\rho,\varphi,\varphi'}(\Lambda, \Lambda', x)$ is a mixed automorphic form for Γ of type (ρ, σ, σ').*

Proof. Recall that φ_Λ can be regarded as a function $\varphi_\Lambda\colon G \to V$ on G satisfying $\varphi_\Lambda(kmau) = \sigma(k)\varphi_\Lambda(m)$. Thus we have

$$\varphi_\Lambda(kg) = \sigma(k)\varphi_\Lambda(g)$$

for all $k \in K$ and $g \in G$. Similarly, we have

$$\varphi'_{\Lambda'}(k'g') = \sigma'(k')\varphi'_{\Lambda'}(g')$$

for all $k' \in K'$ and $g' \in G'$. Hence it follows that

$$(\varphi_\Lambda \otimes (\varphi'_{\Lambda'} \circ \rho))(kg) = (\sigma \otimes (\sigma' \circ \rho|_K))(k)(\varphi_\Lambda \otimes (\varphi'_{\Lambda'} \circ \rho))(g)$$

and

$$E_{\rho,\varphi,\varphi'}(kg) = (\sigma \otimes (\sigma' \circ \rho|_K))(k)E_{\rho,\varphi,\varphi'}(g)$$

for all $k \in K$ and $g \in G$. By Theorem 5.2 we have

$$\left| \sum_{\gamma \in \Gamma/\Gamma\cap P} (\varphi_\Lambda \otimes (\varphi'_{\Lambda'} \circ \rho))(x\gamma) \right| \leq \sum_{\gamma \in \Gamma/\Gamma\cap P} |(\varphi_\Lambda \otimes (\varphi'_{\Lambda'} \circ \rho))(x\gamma)| \leq C_0 \|x\|^{N_0},$$

and consequently the mapping $g \mapsto E_{\rho,\varphi,\varphi'}(\Lambda, \Lambda', x)$ is slowly increasing. Now from the conditions

$$(Y \cdot \varphi_\Lambda)(g) = \varphi_\Lambda(g)\chi(Y), \qquad (Y' \cdot \varphi'_{\Lambda'})(g') = \varphi'_{\Lambda'}(g')\chi'(Y')$$

for $Y \in Z(\mathfrak{g})$, $Y' \in Z(\mathfrak{g}')$, $g \in G$ and $g' \in G'$, it follows that φ_Λ is $Z(\mathfrak{g})$-finite and $\varphi'_{\Lambda'}$ is $Z(\mathfrak{g}')$-finite; hence, as in the proof of Proposition 2.3, $\varphi_\Lambda \otimes (\varphi'_{\Lambda'} \circ \rho)$ is $Z(\mathfrak{g})$-finite. Therefore $E_{\rho,\varphi,\varphi'}$ is also $Z(\mathfrak{g})$-finite, and the theorem follows. □

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