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POINTWISE MULTIPLIERS FROM THE HARDY SPACE TO THE BERGMAN SPACE

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ABSTRACT. For which regions G is the Hardy space $H^2(G)$ contained in the Bergman space $L_a^2(G)$? This paper relates the above problem to that of finding the multipliers of $H^2(\mathbb{D})$ into $L_a^2(\mathbb{D})$. When G is a simply connected region this leads to a solution of the above problem in terms of Lipschitz conditions on the Riemann map of $\mathbb{D}$ onto G. For arbitrary regions G, it is shown that if G is the range of a function whose derivative is a multiplier from $H^2(\mathbb{D})$ to $L_a^2(\mathbb{D})$, then $H^2(G)$ is contained in $L_a^2(G)$. Also, if G has a piecewise smooth boundary, then it is shown that $H^2(G)$ is contained in $L_a^2(G)$ if and only if the angles at all the "corner" points are at least $\pi/2$. Examples of multipliers from $H^2(\mathbb{D})$ to $L_a^2(\mathbb{D})$ are given; and in particular, every Bergman inner function is such a multiplier.

Preliminaries

If G is a region in the complex plane $\mathbf{C}$ and $1 \leq p < \infty$, then the Bergman space $L_a^p(G)$ is the space of all analytic functions f on G so that $|f|^p$ is integrable with respect to area measure on G. Endowed with the usual L^p norm, $L_a^p(G)$ becomes a Banach space. The Hardy space $H^p(G)$ is the space of all analytic functions f on G so that $|f|^p$ has a harmonic majorant. Among all the harmonic majorants of $|f|^p$ there is a smallest one, u_f, called the least harmonic majorant. In order to put a norm on $H^p(G)$, first choose a point $a \in G$, then define $\|f\|_{H^p}^p = u_f(a)$. With this norm $H^p(G)$ becomes a Banach space. For $p = 2$, both the Bergman space and Hardy space are separable Hilbert spaces. See [5] and [6] for more information on Hardy spaces and Bergman spaces, both on the unit disk and in more general regions.

This paper addresses the problem of characterizing those regions G with the property that $H^p(G)$ is contained in $L_a^p(G)$ when $1 \leq p < \infty$. For example, if G is a bounded by a finite number of smooth curves then $H^p(G) \subseteq L_a^p(G)$; see [3], p. 21. More generally, if G is a region so that every positive harmonic function on G is integrable with respect to area measure, then since functions in $H^p(G)$ have harmonic majorants, $H^p(G) \subseteq L_a^p(G)$. However, there are simply connected regions G where $H^p(G) \subseteq L_a^p(G)$, yet not every positive harmonic function is integrable.

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The integrability of positive harmonic functions has been studied by several people; see [14] and the references there. Also, Axler and Shields in [2] construct a region G bounded by a rectifiable Jordan curve so that the Dirichlet space is not contained in the Bergman space on G, thus necessarily $H^2(G)$ is not contained in $L^2_a(G)$.

This work relates the containment problem mentioned above to multipliers of $H^p(\mathbb{D})$ into $L^p_a(\mathbb{D})$. These multipliers have been characterized by Stegenga [13]. We, however, are interested in a different type of characterization of the multipliers and it will not depend on his work.

If μ is a positive regular Borel measure on G and X is a Banach space of analytic functions on G, then we say that μ is a p-Carleson measure for X, if $X \subseteq L^p(\mu)$. The usual definition of a Carleson measure, in our terminology, is simply a 2-Carleson measure for $H^2(\mathbb{D})$. It is well known that if μ is a measure on the open unit disk $\mathbb{D}$, then μ is a p-Carleson measure for $H^p(\mathbb{D})$ if and only if there exists a constant C such that $\mu(S_h(\theta)) \leq Ch$ for every Carleson square $S_h(\theta) = \{z \in \mathbb{D} : 1 - |z| \leq h$ and $\theta - h \leq \arg(z) \leq \theta + h\}$. See [7], p. 156 or [9], p. 33 for a proof. In particular since the condition above on Carleson squares is independent of p, we see that a measure μ is a p-Carleson measure for $H^p(\mathbb{D})$ if and only if μ is a 2-Carleson measure for $H^2(\mathbb{D})$.

1. Basic properties of multipliers

In this section a relation is given between the containment of the Hardy space inside the Bergman space and multiplication operators that map $H^p(\mathbb{D})$ into $L^p_a(\mathbb{D})$. Also certain growth conditions are given for functions that multiply $H^p(\mathbb{D})$ into $L^p_a(\mathbb{D})$.

PROPOSITION 1.1. *Suppose G is a simply connected region and $\tau : \mathbb{D} \to G$ is a Riemann map. Then for $1 \leq p < \infty$ the following are equivalent:*

(*a*) $H^p(G) \subseteq L^p_a(G)$;
(*b*) $(\tau')^{2/p} H^p(\mathbb{D}) \subseteq L^p_a(\mathbb{D})$; *that is* $(\tau')^{2/p} f \in L^p_a(\mathbb{D})$ *for each* $f \in H^p(\mathbb{D})$;
(*c*) $\mu = |\tau'|^2 dA$ *is a p-Carleson measure for* $H^p(\mathbb{D})$.

Proof. Let $h \in H^p(G)$. Since τ is univalent, by the usual change of variables we have

$$(*) \qquad \int_G |h|^p \, dA = \int_D |h \circ \tau|^p \left|\tau'\right|^2 dA.$$

Now as h varies over all of $H^p(G)$, $h \circ \tau$ varies over all of $H^p(\mathbb{D})$, since the Hardy spaces are conformally invariant. So, if (a) holds then each f in $H^p(\mathbb{D})$ has the form $f = h \circ \tau$ for some h in $H^p(G)$, thus $(\tau')^{2/p} f = (h \circ \tau)(\tau')^{2/p}$ and the $L^p_a(\mathbb{D})$ norm of this last expression is exactly the right hand side of $(*)$. But by (a), the left hand side of $(*)$ is finite, so (a) implies (b).

Now if (b) holds, then for each f in $H^p(\mathbb{D})$, $(\tau')^{2/p} f \in L^p_a(\mathbb{D})$. Thus $\int_D |f|^p |\tau'|^2 dA < \infty$ and hence $H^p(\mathbb{D}) \subseteq L^p(\mu)$, where $\mu = |\tau'|^2 dA$. So μ is a p-Carleson measure for $H^p(\mathbb{D})$ and (c) follows. Finally, if (c) holds, then the right hand side of $(*)$ is finite for each h in $H^p(G)$. Hence the left hand side is also and so (a) follows. □

COROLLARY 1.2. *If $1 \leq p < \infty$ and G is a simply connected region, then $H^p(G) \subseteq L^p_a(G)$ if and only if $H^2(G) \subseteq L^2_a(G)$.*

This can be seen by checking condition (b) above or by observing as mentioned before that a measure μ is a p-Carleson measure for $H^p(\mathbb{D})$ if and only if μ is a Carleson measure for $H^2(\mathbb{D})$.

In view of Corollary 1.2 we will mainly restrict our attention to $p = 2$, occasionally considering or commenting on other values of $p \geq 1$.

Proposition 1.1 makes precise the relation between the containment of $H^p(G)$ in $L^p_a(G)$ and multipliers on the unit disk. Notice that when $p = 2$ and $\tau : \mathbb{D} \to G$ is a Riemann map then $H^2(G) \subseteq L^2_a(G)$ if and only if τ' is a multiplier of $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$. Thus we shall try to understand which functions multiply $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$. In particular, motivated by the univalent case, we want to understand which functions ϕ have the property that ϕ' is a multiplier of $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$. Unless otherwise stated, whenever we say multiplier we mean an analytic function on the unit disk that multiplies $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$. Let $M(H^2, L^2_a)$ denote the set of multipliers. Each multiplier f induces a multiplication operator, denoted by M_f.

THEOREM 1.3. *(a) If f is a multiplier of $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$, then $f \in L^2_a(\mathbb{D})$ and there is a constant C such that $(1 - |z|^2)|f(z)|^2 \leq C$ for all z in $\mathbb{D}$.*

(b) If f is any analytic function on $\mathbb{D}$, then f is a multiplier of $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$ if and only if there is a constant K so that $\int_{S_h} |f|^2 dA \leq Kh$ for each Carleson square S_h of size h.

Proof. (a) If $f \in M(H^2, L^2_a)$, then, since the constants are in $H^2(\mathbb{D})$, f must be in $L^2_a(\mathbb{D})$. Let $K_w = \frac{(1-|w|^2)^{1/2}}{(1-\bar{w}z)}$ and $B_w = \frac{(1-|w|^2)}{(1-\bar{w}z)^2}$ be the normalized reproducing kernels in $H^2(\mathbb{D})$ and $L^2_a(\mathbb{D})$, respectively. Since fK_w is in $L^2_a(\mathbb{D})$ and $\frac{1}{(1-\bar{w}z)^2}$ is the reproducing kernel in $L^2_a(\mathbb{D})$, for each w in $\mathbb{D}$ we have

$$|\langle fK_w, B_w\rangle| = (1 - |w|^2)\,|(fK_w)(w)| = (1 - |w|^2)^{1/2}\,|f(w)|.$$

It follows that for all w in $\mathbb{D}$,

$$(**) \qquad (1 - |w|^2)^{1/2}\,|f(w)| = |\langle fK_w, B_w\rangle| = \left|\langle M_f K_w, B_w\rangle\right| \leq \left\|M_f\right\|$$

Hence (a) holds.

Now suppose f is analytic on $\mathbb{D}$, then $f \in M(H^2, L^2_a)$ if and only if $\int_D |h|^2 |f|^2 dA < \infty$ for each $h \in H^2(\mathbb{D})$. But this is equivalent to $\mu = |f|^2 dA$ being a Carleson measure and condition (b) is exactly the condition mentioned in the previous section that characterizes Carleson measures. □

Recall that the Dirichlet space $D(G)$ consists of all analytic functions on G whose derivative is in $L^2_a(G)$. Also a function f defined on a region G is Lipschitz of order α if there is a constant C so that $|f(z) - f(w)| \le C|z-w|^\alpha$ for all $z, w \in G$.

COROLLARY 1.4. *If φ is an analytic function on $\mathbb{D}$ and $\varphi' \in M(H^2, L^2_a)$, then φ is in the Dirichlet space and is Lipschitz of order 1/2.*

Proof. Clearly $\varphi' \in L^2_a(\mathbb{D})$ as φ' is a multiplier, so φ is in the Dirichlet space. The fact that φ is Lipschitz of order 1/2 follows from [7], p. 74 since Theorem 1.3 (a) implies that there is a constant C such that $|\varphi'(z)| \le \frac{C}{(1-|z|^2)^{1/2}}$. □

COROLLARY 1.5. *Suppose G is a simply connected region and $\tau : \mathbb{D} \to G$ is a Riemann map. If $H^2(G) \subseteq L^2_a(G)$, then τ is Lipschitz of order 1/2. In particular τ is continuous on the closed unit disk and G is bounded.*

The previous corollary gives many examples of simply connected regions where $H^2(G)$ is not contained in $L^2_a(G)$. For example, such a containment fails whenever ∂G is not locally connected, because then the Riemann map will not extend continuously to $\partial\mathbb{D}$.

Corollary 1.5 also raises an interesting question about the boundedness of G, when G is not simply connected. Namely, if G is any region such that $H^2(G) \subseteq L^2_a(G)$, does G have to be bounded? Clearly since $H^2(G)$ contains the constants, G must have finite area. As noted above, if G is simply connected, then G must be bounded, but for arbitrary regions of finite area it is not clear that boundedness holds.

By a compact multiplier we mean a function f whose multiplication operator M_f is compact.

THEOREM 1.6. *(a) If f induces a compact multiplier of $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$, then $(1-|z|^2)|f(z)|^2 \to 0$ as $|z| \to 1$.*

(b) If f is an analytic function on $\mathbb{D}$, then f induces a compact multiplier of $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$ if and only if $\int_{S_h} |f|^2 dA = o(h)$ as $h \to 0$.

Proof. (a) Consider the normalized reproducing kernels, K_w and B_w as in Theorem 1.3. Since M_f is compact and K_w converges weakly to zero as $|w| \to 1$, we have that fK_w converges to zero in norm in $L^2_a(\mathbb{D})$. Thus since B_w has norm one, $|\langle M_f K_w, B_w\rangle| \to 0$ as $|w| \to 1$, thus equation (**) above gives the desired conclusion. To see that (b) holds, notice that M_f is compact if and only if the inclusion of $H^2(\mathbb{D})$ into $L^2(|f|^2 dA)$ is compact, and this means, by definition, that $|f|^2 dA$ is a "vanishing Carleson measure". Further, these measures are characterized exactly by the condition stated in (b); see [15], p. 172. □

Notice that the estimate in (a) of Theorems 1.3 and 1.6 can also be proved by using the Carleson measure estimate, part (b) from Theorems 1.3 and 1.6, together with the subharmonicity of $|f|^2$. This technique works for other values of p where reproducing kernel arguments are not as readily available.

We are interested in finding to what extent the necessary conditions of Corollary 1.4 are sufficient to guarantee that φ' is a multiplier. We shall see that a stronger condition holds on the valence of φ. Although, if φ is univalent, the conditions of Corollary 1.4 are both necessary and sufficient.

2. The valence function

In Corollary 1.4 it was shown that if φ is analytic on $\mathbb{D}$ and φ' is a multiplier, then φ is in the Dirichlet space and is Lipschitz of order 1/2. In this section it is shown that the converse holds for a large class of functions, including the univalent ones.

In order to do this, we need a change of variables formula for non-univalent functions. So suppose $\varphi : G \to \mathbf{C}$ is an analytic function on an open set G. Define its valence function or counting function $n_\varphi(w)$ as the number of points, counting multiplicity, in the pre-image, $\varphi^{-1}(w)$. So $n_\varphi(w)$ is defined on all of $\mathbf{C}$, but is zero off the range of φ. Also, notice that φ is univalent precisely when $n_\varphi(w) \leq 1$ everywhere on $\mathbf{C}$ and that $n_\varphi(w)$ may equal infinity. The following is a change of variables formula for non-univalent functions. It is well known in geometric measure theory and is useful in the study of analytic functions. We include a proof for completeness.

THEOREM 2.1. *Suppose $\varphi : G \to \Omega$ is a non-constant analytic function with valence function $n_\varphi(w)$. If $f : \Omega \to [0, \infty)$ is any Borel function, then*

$$\int_G f(\varphi(z))\,|\varphi'(z)|^2\,dA(z) = \int_\Omega f(w)n_\varphi(w)dA(w).$$

Proof. If $Z = \{z \in G : \varphi'(z) = 0\}$, then Z is a discrete set in G and hence has area zero. So φ is univalent on some small disk about each point of $G - Z$. Using Vitali's covering lemma we can find a sequence of disjoint disks B_n inside G so that $\varphi|B_n$ is univalent for each n and the area of $G - \bigcup_{n=1}^{\infty} B_n$ is zero. If χ_E is the characteristic function of the set E, then $n_\varphi(w) = \sum_{n=1}^{\infty} \chi_{\varphi(B_n)}$ a.e. (area). This is because the B_n's cover almost all of G and since analytic functions always map sets of area zero onto sets of area zero, their images $\varphi(B_n)$ cover almost all of the range of φ. Also, this expression for n_φ shows that it is a measurable function. So for each n, since $\varphi|B_n$ is univalent, the usual change of variables formula gives $\int_{B_n} f(\varphi(z))|\varphi'(z)|^2 dA(z) = \int_{\varphi(B_n)} f(w)dA(w)$. Since the disks are pairwise disjoint, we get

$$\begin{aligned}\int_G f(\varphi(z))\,|\varphi'(z)|^2\,dA(z) &= \sum_{n=1}^{\infty}\int_{\varphi(B_n)} f(w)dA(w) = \sum_{n=1}^{\infty}\int_\Omega f(w)\chi_{\varphi(B_n)}dA(w)\\ &= \int_\Omega f(w)\sum_{n=1}^{\infty}\chi_{\varphi(B_n)}dA(w) = \int_\Omega f(w)n_\varphi(w)dA(w).\end{aligned}$$

Notice that we are allowed to change the integral and the sum because everything is positive. □

COROLLARY 2.2. *If $\varphi : G \to \Omega$ is a non-constant analytic function and n_φ is its valence function, then* $\int_G |\varphi'(z)|^2 dA(z) = \int_\Omega n_\varphi(w) dA(w)$.

Notice this says that φ is in the Dirichlet space if and only if its valence function is an L^1 function. Thus assuming a function is in the Dirichlet space is simply imposing a restriction on the growth of its valence function.

Now, consider an analytic function φ on $\mathbb{D}$ that is Lipschitz of order 1/2 and in the Dirichlet space. Is such a function a multiplier? If we impose a stronger condition on the valence of the function φ, then the Lipschitz condition will guarantee that φ is a multiplier. We will consider the case when n_φ is essentially bounded, that is, there is a constant C so that $n_\varphi(w) \leq C$ for all w except a set of area zero.

THEOREM 2.3. *If φ is analytic on $\mathbb{D}$ and n_φ is essentially bounded, then φ' is a multiplier if and only if φ is Lipschitz of order* 1/2.

Proof. In view of Corollary 1.4 and Theorem 1.3 (b) it suffices to show that if φ is Lipschitz of order 1/2 then $\mu = |\varphi'|^2 dA$ is a Carleson measure. So, if S_h is a Carleson square of size h, then by Corollary 3.2 we have

$$\begin{aligned} \mu(S_h) &= \int_{S_h} |\varphi'|^2 \, dA = \int_{\varphi(S_h)} n_\varphi(w) dA \\ &\leq \|n_\varphi\|_\infty \operatorname{Area}\{\varphi(S_h)\} \leq \pi \, \|n_\varphi\|_\infty \operatorname{diam}\{\varphi(S_h)\}^2 \leq Ch \end{aligned}$$

where the constant C depends only on the norm of n_φ and the Lipschitz constant of φ. Thus μ is a Carleson measure on $\mathbb{D}$. □

Notice that the assumption that n_φ is essentially bounded is stronger than assuming that φ is only in the Dirichlet space, that is $n_\varphi \in L^1$. But there is still an ample supply of such functions. First, all the univalent functions are in this class and also for any bounded region G, there is an analytic function φ on $\mathbb{D}$ so that $\varphi(\mathbb{D}) = G$ and n_φ is essentially bounded. So such functions come in all shapes and sizes.

COROLLARY 2.4. *If φ is a univalent function on $\mathbb{D}$, then φ' is a multiplier of $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$ if and only if φ is Lipschitz of order* 1/2.

COROLLARY 2.5. *If G is a simply connected region and $\tau : \mathbb{D} \to G$ is a Riemann map, then $H^2(G) \subseteq L^2_a(G)$ if and only if τ is Lipschitz of order* 1/2.

Smith and Stegenga in [12] have given a geometric characterization, in terms of the hyperbolic metric, of simply connected regions for which the Riemann map satisfies a Lipschitz condition.

Now suppose that G is bounded by a finite number of piecewise C^1-smooth disjoint Jordan curves, each having one-sided derivatives at the "corner points". Consider the angles θ_j, where $0 \leq \theta_j \leq 2\pi$, formed by the tangent lines at the corner points. So, if $\theta_j = 0$ at a corner point w, then there is an outward cusp at w and if $\theta_j = 2\pi$ there is an inward cusp at w. However, if $\theta_j = \pi$, then the curve is actually smooth at w.

COROLLARY 2.6. *If G is as above, then $H^2(G) \subseteq L_a^2(G)$ if and only if $\theta_j \geq \pi/2$ for all j.*

Proof. First suppose that G is simply connected and bounded by a piecewise smooth Jordan curve having corners at the points $\{w_j\}$ forming angles $\{\theta_j\}$. Let $\tau : \mathbb{D} \to G$ be a Riemann map and $z_j \in \partial\mathbb{D}$ satisfy $\tau(z_j) = w_j$. If the angle $\theta_j = \alpha_j\pi$ where $0 \leq \alpha_j \leq 2$, then $(z - z_j)^{1-\alpha_j}\tau'(z)$ has a non-zero finite limit at z_j; see Pommerenke[11], p. 52. However as previously mentioned, τ is Lipschitz of order 1/2 if and only if $|\tau'(z)| \leq \frac{C}{(1-|z|^2)^{1/2}}$ holds for some constant C; see [7], p. 74. Thus τ is Lipschitz of order 1/2 if and only if $\alpha_j \geq 1/2$ for all j. This, together with Corollary 2.5, gives the desired conclusion in this case. The general case when G is finitely connected may be reduced to the simply connected case because $H^2(G)$ may be decomposed as a direct sum of Hardy spaces over simply connected regions in a canonical way; see Conway [4]. □

Example. If G is a triangle, then $H^2(G) \not\subseteq L_a^2(G)$; however if G is a rectangle, then $H^2(G) \subseteq L_a^2(G)$. If G has an outward cusp ($\theta = 0$), then $H^2(G) \not\subseteq L_a^2(G)$.

It is interesting that it is rather difficult to construct a region G bounded by a rectifiable Jordan curve such that the Dirichlet space on G is not contained in the Bergman space on G, see Axler and Shields [2].

Now it is shown that there is a slightly stronger necessary condition on an analytic function φ in order to have φ' a multiplier. Corollary 1.4 shows that if φ' is a multiplier, then φ is in the Dirichlet space; that is, $n_\varphi \in L^1$. Thus, if we let $\mu = n_\varphi(w)dA(w)$, then Corollary 1.4 says that μ is a finite measure whenever φ' is a multiplier. It is now shown that whenever φ' is a multiplier μ is a Carleson measure for $H^2(G)$, where $G = \varphi(\mathbb{D})$.

THEOREM 2.7. *Suppose φ is analytic on $\mathbb{D}$ and $G = \varphi(\mathbb{D})$. If φ' is a multiplier of $H^2(\mathbb{D})$ into $L_a^2(\mathbb{D})$, then the measure $\mu = n_\varphi\, dA$ is a Carleson measure for $H^2(G)$. That is, $H^2(G) \subseteq L^2(\mu)$.*

Proof. Let $h \in H^2(G)$. Since $\varphi : \mathbb{D} \to G$ is analytic, we have $h(\varphi(z)) \in H^2(\mathbb{D})$

and, because φ' is a multiplier, $h(\varphi(z))\varphi'(z) \in L^2_a(\mathbb{D})$. So by Theorem 2.1 we have

$$\int_G |h(w)|^2 n_\varphi(w) dA(w) = \int_D |h(\varphi(z))|^2 \left|\varphi'(z)\right|^2 dA(z) < \infty.$$

Thus, $h \in L^2(\mu)$ and so μ is a Carleson measure for $H^2(G)$. □

This theorem has a very nice corollary.

COROLLARY 2.8. *Suppose φ is an analytic function on $\mathbb{D}$ and $G = \varphi(\mathbb{D})$. If φ' is a multiplier, then $H^2(G) \subseteq L^2_a(G)$.*

Proof. Since $n_\varphi \geq 1$ on G the theorem gives $H^2(G) \subseteq L^2_a(G, n_\varphi dA) \subseteq L^2_a(G)$. □

Theorem 2.3 and Corollary 2.4 give some examples of multipliers to which Corollary 2.8 may be applied. Also, it will be shown in Theorem 3.3 that if f is any function in $H^2(\mathbb{D})$ or $L^4_a(\mathbb{D})$, then f is a multiplier. Hence if φ is a primitive of f, then Corollary 2.8 applies to give $H^2(G) \subseteq L^2_a(G)$ for $G = \varphi(\mathbb{D})$. See Corollary 3.4.

3. Examples

In this section some examples of multipliers are given. The main tool used to show functions are multipliers is the Carleson measure condition of Theorem 1.3.

Since $H^2(\mathbb{D}) \subseteq L^2_a(\mathbb{D})$ it is clear that every bounded analytic function on $\mathbb{D}$ is a multiplier. We next show that much more is true. Recall that Theorem 1.3 shows that if $f \in M(H^2, L^2_a)$, then $|f(z)| \leq \frac{C}{(1-|z|^2)^{1/2}}$. In Theorem 3.1 below we show that a slight improvement on this condition implies that f is a compact multiplier.

THEOREM 3.1. *If f is analytic on $\mathbb{D}$ and $|f(z)| \leq \rho(|z|)$, where $\rho \in L^2(0,1)$, then f induces a compact multiplier from $H^2(\mathbb{D})$ to $L^2_a(\mathbb{D})$.*

Proof. Let $\{g_n\} \subseteq H^2(\mathbb{D})$ and suppose $g_n \to 0$ weakly. So $\{g_n\}$ is norm bounded in $H^2(\mathbb{D})$ and $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. We must show that $\|fg_n\|_{L^2} \to 0$ as $n \to \infty$. So

$$\begin{aligned}\|fg_n\|^2_{L^2} &= \int_0^1 \int_0^{2\pi} \left|f(re^{i\theta})g_n(re^{i\theta})\right|^2 d\theta r\, dr \leq \int_0^1 \int_0^{2\pi} \left|g_n(re^{i\theta})\right|^2 d\theta \rho(r)^2 r\, dr \\ &= \int_0^1 M_2(g_n, r)^2 \rho(r)^2 r\, dr,\end{aligned}$$

where $M_2(g_n, r)^2 = \int_0^{2\pi} |g_n(re^{i\theta})|^2 d\theta$. But, $M_2(g_n, r)^2\rho(r)^2 r \to 0$ as $n \to \infty$ for each r, because $g_n \to 0$ uniformly on compact sets in $\mathbb{D}$. Also, there is a constant

C so that $M_2(g_n, r)^2 \leq C$ for all n and r, because the set $\{g_n\}$ is norm bounded in $H^2(\mathbb{D})$. So the integrand of the last integral above is dominated by an integrable function, namely $C\rho^2$. Thus the Lebesgue Dominated Convergence Theorem says that the last integral goes to zero as n tends to infinity. $\square$

COROLLARY 3.2. *If f is analytic on $\mathbb{D}$ and $|f(z)| \leq \frac{C}{(1-|z|^2)^{1/2-\varepsilon}}$ for some $\varepsilon, C > 0$ and all z in $\mathbb{D}$, then f induces a compact multiplier.*

The previous results give a large number of examples of compact multipliers and Theorem 3.3 below even gives more. However, not all multipliers are compact because $f(z) = \frac{1}{(1-z)^{1/2}}$ is a multiplier that is not compact, see Theorem 1.6.

THEOREM 3.3. *(a) $H^2(\mathbb{D}) \subseteq L_a^4(\mathbb{D})$.*
(b) Each function in $L_a^4(\mathbb{D})$ induces a compact multiplier.

Proof. (a) It is a classic result due to Hardy and Littlewood that $H^p(\mathbb{D}) \subseteq L_a^{2p}(\mathbb{D})$; see Duren [7], p. 87.
(b) If $f \in L_a^4(\mathbb{D})$ and $g \in H^2(\mathbb{D})$, then $\|fg\|_{L^2} \leq \|f\|_{L^4}\|g\|_{L^4} \leq c\|f\|_{L^4}\|g\|_{H^2}$. Thus we see that $f \in M(H^2, L_a^2)$ and $\|M_f\| \leq C\|f\|_{L^4}$ for an absolute constant C. It follows from Theorem 3.1 that every polynomial induces a compact multiplication operator. Since the polynomials are dense in $L_a^4(\mathbb{D})$, we may choose $p_n \to f$ in $L_a^4(\mathbb{D})$. Thus from the above estimate we see that $M_{p_n} \to M_f$ in operator norm, hence M_f is compact. $\square$

The following result follows from Theorem 3.4 and Corollary 2.8.

COROLLARY 3.4. *If φ is an analytic map on the disk $\mathbb{D}$, $G = \varphi(\mathbb{D})$ and $\varphi' \in L_a^4(\mathbb{D})$, then $H^2(G) \subseteq L_a^2(G)$.*

A function φ in $L_a^2(\mathbb{D})$ is a *Bergman inner function* if $\int_D u(z)|\varphi(z)|^2 dA = u(0)$ for every bounded harmonic function u on $\mathbb{D}$; see [1] or [8]. It is known that if $\mathcal{M}$ is an invariant subspace of the Bergman shift and $\varphi \in \mathcal{M} \cap (z\mathcal{M})^\perp$ has norm one, then φ is a Bergman inner function. Next we show that every Bergman inner function is a multiplier. This also appears in Hedenmalm's [10] work, although this proof is easier and more direct.

THEOREM 3.5. *Every Bergman inner function induces a contractive multiplier.*

Proof. Let φ be a Bergman inner function. Thus $\int_D u(z)|\varphi(z)|^2 dA = u(0)$ for every bounded harmonic function u on $\mathbb{D}$. If u is a positive harmonic function on $\mathbb{D}$ and $u_r(z) = u(rz)$, then $u_r \to u$ pointwise as $r \to 1$, so Fatou's Lemma easily implies that $\int_D u(z)|\varphi(z)|^2 dA \leq u(0)$ for all positive harmonic functions on $\mathbb{D}$. If $f \in H^2(\mathbb{D})$ and u_f is the least harmonic majorant for $|f|^2$, then we have

$$\int_D |f(z)\varphi(z)|^2\, dA \leq \int_D u_f(z)\, |\varphi(z)|^2\, dA \leq u_f(0) = \|f\|_{h^2}^2 .$$

Thus, φ is a multiplier and $\|M_\varphi\| \leq 1$. □

We close with a few natural questions that remain open.

Question 1. If G is a region such that $H^2(G) \subseteq L_a^2(G)$, then must G be bounded?

Question 2. If G is a region such that $H^2(G) \subseteq L_a^2(G)$, then must there exist an analytic function φ on $\mathbb{D}$ with φ' a multiplier and $\varphi(\mathbb{D}) = G$? That is, is the converse of Corollary 2.8 true?

Question 3. If φ is an analytic function on $\mathbb{D}$ that is Lipschitz of order 1/2 and $n_\varphi\, dA$ is a Carleson measure for $H^2(G)$, $G = \varphi(\mathbb{D})$, then must φ' be a multiplier?

Notice that Questions 1 and 2 both have an affirmative answer when G is simply connected.

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