INDECOMPOSABLE REPRESENTATIONS

BY

A. HELLER AND I. REINER[1]

1. Introduction

Let Λ be a finite-dimensional algebra over a field K. By a Λ-module we shall mean always a finitely generated left Λ-module on which the unity element of Λ acts as identity operator. It is well known that the Krull-Schmidt theorem holds for Λ-modules: each module is a direct sum of indecomposable Λ-modules, and these summands are uniquely determined up to order of occurrence and Λ-isomorphism. Thus the problem of classifying Λ-modules is reduced to that of finding the isomorphism classes of indecomposable Λ-modules. We denote the set of these by $M(\Lambda)$.

A central problem in the theory of group representations is that of determining a set of representatives of $M(\Lambda)$ for the special case where $\Lambda = KG$, the group algebra of a finite group G over the field K. A definitive answer can be given when the characteristic of K does not divide the group order $[G:1]$; in this case KG is semisimple, all indecomposable modules over KG are irreducible, and a full set of non-isomorphic minimal left ideals of KG constitute a set of representatives of $M(KG)$. For the case where the characteristic of K is p ($p \neq 0$), Higman [6] has proved the following remarkable result: *$M(KG)$ is finite if and only if the p-Sylow subgroups of G are cyclic.* If such is the case, Higman obtained an upper bound on the number of elements of $M(KG)$. A best possible upper bound was later obtained by Kasch, Kupisch, and Kneser [5].

We shall attempt to elucidate Higman's theorem by considering in detail the special case where G is an abelian p-group, and K a field of characteristic p. We shall exhibit some new classes of indecomposable modules. However we shall show that the problem of computing $M(KG)$, in case G is not cyclic, is at least as difficult as a classical unsolved problem in matrix theory.

It should be pointed out that the question of determining all representations of a p-group in a field of characteristic p has been extensively treated by Brahana [1, 2, 3] from a somewhat different viewpoint. There is consequently a certain amount of overlapping between his results and ours, but we have thought it best to make this paper completely self-contained.

2. C-algebras

Inasmuch as we shall need to consider, together with modules over an algebra Λ, also modules over sub- and quotient-algebras of Λ, we cannot re-

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strict our attention only to group algebras. Instead we shall work with a special type of commutative completely primary algebras.

DEFINITION. A *C-algebra* Λ over a field K of arbitrary characteristic is a finite-dimensional commutative algebra over K with a unity element, such that

$$\Lambda/R(\Lambda) \cong K,$$

where $R(\Lambda)$ denotes the radical of Λ. Any quotient algebra of a C-algebra is easily seen to be a C-algebra. Likewise any subalgebra Λ' of a C-algebra Λ, which contains the unity element of Λ, is a C-algebra.

We may describe a C-algebra Λ explicitly as follows. Let

$$u_1, \cdots, u_n \in R(\Lambda)$$

map onto a K-basis of $R(\Lambda)/R(\Lambda)^2$. From the nilpotency of $R(\Lambda)$ it follows readily that

$$\Lambda = K[u_1, \cdots, u_n], \tag{1}$$

though of course there are polynomial relations connecting the $\{u_i\}$. Let $x_1, \cdots, x_n$ be indeterminates over K, and define a K-homomorphism

$$\phi : K[x_1, \cdots, x_n] \to \Lambda \tag{2}$$

by means of

$$\phi(1) = 1, \quad \phi(x_1) = u_1, \quad \cdots, \quad \phi(x_n) = u_n. \tag{3}$$

Then ϕ is an algebra epimorphism; its kernel J has the property that

$$\sqrt{J} = (x_1, \cdots, x_n), \tag{4}$$

where as usual

$$\sqrt{J} = \{F \in K[x_1, \cdots, x_n] : F^r \in J \text{ for some } r\},$$

and where $(x_1, \cdots, x_n)$ denotes the ideal generated by the $\{x_i\}$. We have

$$K[x_1, \cdots, x_n]/J \cong \Lambda. \tag{5}$$

Conversely if J is an ideal in $K[x_1, \cdots, x_n]$ for which (4) holds, then equation (5) defines a C-algebra Λ. The integer n given by

$$n = \dim_K R(\Lambda)/R(\Lambda)^2$$

we shall call the *rank* of Λ.

In particular let G be an abelian p-group, and write

$$G = G_1 \times \cdots \times G_n,$$

where for each i, G_i is cyclic generated by an element g_i of order $r_i = p^{\alpha_i}$. Let K be any field of characteristic p. Then the K-homomorphism

$$\phi : K[x_1, \cdots, x_n] \to KG$$

defined by

$$\phi(1) = 1, \quad \phi(x_1) = g_1 - 1, \quad \cdots, \quad \phi(x_n) = g_n - 1,$$

is an algebra epimorphism with kernel

$$J = (x_1^{r_1}, \cdots, x_n^{r_n}).$$

Thus KG is a C-algebra of rank n.

3. Quotient algebras; the height of a module

Let Λ be a finite-dimensional K-algebra, and let $\Lambda' = \Lambda/W$ be a quotient algebra of Λ, where W is a two-sided ideal in Λ. Then each Λ'-module M may be made into a Λ-module by defining

$$\lambda \cdot m = (\lambda + W)m, \qquad \lambda \in \Lambda, \quad m \in M.$$

The Λ-modules obtained in this way are precisely those which are annihilated by W.

Moreover if a Λ-module is annihilated by W, then so is each sub- or quotient-module. In particular the direct sum of two Λ-modules is annihilated by W if and only if each summand is. Thus a Λ'-module is indecomposable if and only if it is indecomposable when considered as a Λ-module. This immediately implies the following result.

PROPOSITION 1. *If Λ' is a quotient algebra of Λ, then $M(\Lambda')$ may be canonically identified with a subset of $M(\Lambda)$.*

Now suppose that $R = R(\Lambda)$ is the radical of Λ; then for some integer m, $R^m = 0$. Thus for any Λ-module A there is a smallest integer h such that $R^h A = 0$. We call this h the *height* of A, and clearly $h \leqq m$.

Thus a module is of height $\leqq h$ if and only if it is annihilated by R^h, and so by Proposition 1 we may identify $M(\Lambda/R^h)$ with the subset of $M(\Lambda)$ consisting of the isomorphism classes of Λ-modules of height $\leqq h$.

If A is of height h, we have the upper Loewy series

$$A \supset RA \supset \cdots \supset R^{h-1}A \supset R^h A = 0,$$

and all inclusions are proper. On the other hand R annihilates each quotient of two successive terms, and so each quotient is semisimple. This establishes

PROPOSITION 2. *A Λ-module of height h is an $(h - 1)$-fold successive extension of semisimple modules. In particular a module of height 1 is semisimple, while a module of height 2 is an extension of one semisimple module by another.*

4. Height two modules over C-algebras

Let Λ be a C-algebra over K, and let R be its radical. Then $\Lambda/R \cong K$ shows that a semisimple Λ-module is just a vector space over K, so that $M(\Lambda/R)$ has just one element, namely, the class containing K.

As we have seen, the set of isomorphism classes of indecomposable Λ-modules of height $\leqq 2$ may be identified with $M(\Lambda/R^2)$. But Λ/R^2 depends only upon the rank of Λ, since we have

PROPOSITION 3. *Set $\Delta_n = K[x_1, \cdots, x_n]/(x_1, \cdots, x_n)^2$, where the $\{x_i\}$ are indeterminates over K. If Λ is any C-algebra over K of rank n, then*

$$\Lambda/R^2 \cong \Delta_n .$$

Proof. Let $u_1, \cdots, u_n \in R$ map onto a K-basis of R/R^2. For each $\lambda \in \Lambda$ let $\bar{\lambda}$ denote its image in Λ/R^2. Then we have at once

$$\Lambda/R^2 = K\bar{1} \oplus K\bar{u}_1 \oplus \cdots \oplus K\bar{u}_n .$$

On the other hand let $x \in K[x_1, \cdots, x_n]$ map onto $\tilde{x} \in \Delta_n$. Then

$$\Delta_n = K\tilde{1} \oplus K\tilde{x}_1 \oplus \cdots \oplus K\tilde{x}_n . \tag{6}$$

The map $\bar{1} \to \tilde{1}$, $\bar{u}_i \to \tilde{x}_i$ $(1 \leqq i \leqq n)$ thus gives the desired isomorphism.

COROLLARY. *The set of isomorphism classes of indecomposable Λ-modules of height $\leqq 2$ may be identified with $M(\Delta_n)$, where n = rank of Λ.*

We remark that (6) determines the structure of Δ_n, since $\tilde{1}$ is its unity element, and $\tilde{x}_i \tilde{x}_j = 0$ for all i, j. Set

$$S = K\tilde{x}_1 \oplus \cdots \oplus K\tilde{x}_n = \text{radical of } \Delta_n .$$

If A is any Δ_n-module, the sequence

$$0 \to SA \to A \to A/SA \to 0$$

is exact. Both SA and A/SA are annihilated by S, hence are semisimple Δ_n-modules, that is, they are vector spaces over K which are annihilated by S, and upon which K acts by scalar multiplication. For each i we define a K-homomorphism

$$\zeta_i : A/SA \to SA$$

by means of

$$a + SA \to \tilde{x}_i a, \qquad a \in A.$$

Then A is Δ_n-isomorphic to the space

$$A/SA \oplus SA,$$

the action on Δ_n on this space being given by

$$\tilde{x}_i(a + SA, b) = (0, \zeta_i a), \qquad a \in A, b \in SA, 1 \leqq i \leqq n. \tag{7}$$

We have thus shown that to each module A there corresponds a pair of vector spaces A/SA and SA, and an n-tuple of homomorphisms of the first space into the second. This pair of spaces, and the set of homomorphisms, completely determines A up to isomorphism.

Conversely let V, W be any pair of K-spaces, and let

$$\zeta_1, \cdots, \zeta_n \in \mathrm{Hom}_K(V, W)$$

be arbitrary. Define the action of Δ_n on $V \oplus W$ by letting K act by scalar multiplication, and using (7) to define the action of S. Then $V \oplus W$ becomes a Δ_n-module which we denote by

$$[V, W; \zeta_1, \cdots, \zeta_n],$$

and it is clear that the preceding construction associates with this module precisely the spaces V and W, and the homomorphisms $\zeta_1, \cdots, \zeta_n$.

Clearly $[V, W; \zeta_1, \cdots, \zeta_n] \cong [V', W'; \zeta_1', \cdots \zeta_n']$ if and only if there exist K-isomorphisms

$$\theta: V \cong V', \qquad \eta: W \cong W'$$

such that

$$\zeta_i' \theta = \eta \zeta_i, \qquad 1 \leqq i \leqq n.$$

We note further that the direct sum of the modules $[V, W; \zeta_1, \cdots, \zeta_n]$ and $[V', W'; \zeta_1', \cdots, \zeta_n']$ is just

$$[V \oplus V', W \oplus W'; \zeta_1 \oplus \zeta_1', \cdots, \zeta_n \oplus \zeta_n'].$$

We have thus introduced the concepts of isomorphism and decomposability for arrays $[V, W; \zeta_1, \cdots, \zeta_n]$, and have proved

PROPOSITION 4. *The elements of $M(\Delta_n)$ are in one-to-one correspondence with the set $S(n)$ of isomorphism classes of indecomposable arrays.*

(We have in fact constructed functors which connect the category of Δ_n-modules with that of arrays, and which provide a weak equivalence of these categories.)

The problem of determining a complete set of non-isomorphic indecomposable arrays is a classical problem of matrix theory, namely that of equivalence of matrix n-tuples. (In matrix terminology, we seek a complete set of non-equivalent indecomposable n-tuples of matrices, where "equivalence" is given by

$$(\mathbf{T}_1, \cdots, \mathbf{T}_n) \sim (\mathbf{P}\mathbf{T}_1\mathbf{Q}, \cdots, \mathbf{P}\mathbf{T}_n\mathbf{Q}),$$

$\mathbf{P}$ and $\mathbf{Q}$ nonsingular.) The problem has been solved for $n \leqq 2$ (see [4], [7]), and is unsolved for $n > 2$. We shall use the solution for the case $n = 2$ to compute $M(\Delta_2)$, and hence to give a set of representatives for the isomorphism classes of indecomposable Λ-modules of height $\leqq 2$.

Since we are dealing with a C-algebra Λ of rank 2, we may write $\Lambda = K[u_1, u_2]$, where u_1 and $u_2 \in R(\Lambda)$ are such that their images form a K-basis for $R(\Lambda)/R(\Lambda)^2$. Then we have

PROPOSITION 5. *Up to isomorphism, there is only one indecomposable Λ-module of height* 1, *namely the space K on which K acts by scalar multiplication, and which is annihilated by u_1 and u_2. There are infinitely many non-isomorphic* *indecomposable Λ-modules of height 2, and a full set of these is given by the spaces $V \oplus W$, where*

$$V = Ka_1 \oplus \cdots \oplus Ka_r, \qquad W = Kb_1 \oplus \cdots \oplus Kb_s,$$

the action of K being scalar multiplication, and the action of u_1, u_2 given by

$$u_m \cdot a_i = \textstyle\sum_{j=1}^{s} t_{ij}^{(m)} b_j, \qquad 1 \leqq i \leqq r, \quad m = 1, 2,$$

where

$$\mathbf{T}^{(1)} = (t_{ij}^{(1)}), \qquad \mathbf{T}^{(2)} = (t_{ij}^{(2)})$$

are $r \times s$ matrices over K given by the following choices:

(i)
$$\mathbf{T}^{(1)} = \mathbf{I}_{em}, \qquad \mathbf{T}^{(2)} = \mathbf{C}_e(p(x))$$

where $\mathbf{I}_{em}$ denotes the em-rowed identity matrix, e is an arbitrary positive integer, $p(x) = x^m - a_{m-1}x^{m-1} - \cdots - a_0$ is an arbitrary irreducible polynomial in $K[x]$, and $\mathbf{C}_e(p(x))$ is defined as

$$\mathbf{C}_e(p(x)) = \begin{bmatrix} \mathbf{B} & \mathbf{U} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{U} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \cdots & \mathbf{0} \\ \cdot & \cdot & \cdot & \cdots & \mathbf{U} \\ \cdot & \cdot & \cdot & \cdots & \mathbf{B} \end{bmatrix}, \qquad e \text{ } \mathbf{B}\text{'s occur},$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{m-1} \end{bmatrix} = \textit{companion matrix of } p(x),$$

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

(ii)
$$\mathbf{T}^{(1)} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ & & & 0 \\ & \mathbf{I}_m & & \vdots \\ & & & 0 \end{bmatrix}^{(m+1)\times(m+1)}, \qquad \mathbf{T}^{(2)} = \mathbf{I}_m,$$

(iii)
$$\mathbf{T}^{(1)} = \begin{bmatrix} 0 & \cdots & 0 \\ & \mathbf{I}_m & \end{bmatrix}^{(m+1)\times m}, \qquad \mathbf{T}^{(2)} = \begin{bmatrix} & \mathbf{I}_m & \\ 0 & \cdots & 0 \end{bmatrix}^{(m+1)\times m},$$

and

(iv)

$$\mathbf{T}^{(1)} = \begin{bmatrix} 0 & \\ \vdots & \mathbf{I}_m \\ 0 & \end{bmatrix}^{m\times(m+1)}, \qquad \mathbf{T}^{(2)} = \begin{bmatrix} & 0 \\ \mathbf{I}_m & \vdots \\ & 0 \end{bmatrix}^{m\times(m+1)},$$

where in (ii), (iii), *and* (iv) *m is an arbitrary positive integer.*

Remark. If K is algebraically closed, then $p(x) = x - \alpha$ for some $\alpha \in K$ and $\mathbf{C}_e(p(x))$ takes the simpler form

$$\mathbf{C}_e(p(x)) = \begin{bmatrix} \alpha & 1 & \cdots & 0 \\ 0 & \alpha & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & \alpha \end{bmatrix}.$$

Corollary. *Let $G = G_1 \times G_2$, where for $i = 1, 2$, G_i is a cyclic group with generator g_i of order p^{α_i}, $\alpha_i > 0$. Let K be any field of characteristic p. Then there are infinitely many indecomposable KG-modules. A complete set of non-isomorphic indecomposable modules of height 2 is given by the above spaces $V \oplus W$, where the action of G is given as follows:*

$$(g_1 - 1)a_i = \sum t_{ij}^{(1)} b_j, \qquad (g_2 - 1)a_i = \sum t_{ij}^{(2)} b_j, \quad 1 \leqq i \leqq r,$$

and where

$$(g_1 - 1)W = (g_2 - 1)W = 0.$$

Finally we note that for $n \geqq 2$, Δ_2 is a quotient algebra of Δ_n, and hence by Proposition 1 we may conclude that $M(\Delta_n)$ is infinite. Thus $M(KG)$ is infinite whenever G is a non-cyclic abelian p-group, and K has characteristic p.

5. C-algebras of rank two

We have seen that if an abelian p-group G is a direct product of r cyclic groups, and K is a field of characteristic p, then KG is a C-algebra of rank r, and consequently $M(KG)$ contains a subset in one-to-one correspondence with $S(r)$, the set of isomorphism classes of indecomposable arrays $[V, W; \zeta_1, \cdots, \zeta_r]$. This shows that for $r > 2$ we cannot hope to find a complete system of non-isomorphic indecomposable KG-modules. We might expect, however, that this could be done for the special case where $r = 2$. The aim of this section is to show that even this special case leads to the problem of computing $S(p)$, and hence cannot be solved explicitly as soon as $p > 2$.

Let $G = G_1 \times G_2$, where for $i = 1, 2$, G_i is generated by an element g_i of order $r_i = p^{\alpha_i}$, $\alpha_i > 0$. Then we have seen that

$$KG \cong K[x_1, x_2]/(x_1^{r_1}, x_2^{r_2}),$$

and so surely

$$(x_1^{r_1}, x_2^{r_2}) \subset (x_1, x_2)^p.$$

We now prove generally

PROPOSITION 6. *Let* $\Lambda = K[x, y]/J$ *be a C-algebra of rank* 2 *such that for some* $n > 2$,

$$J \subset (x, y)^n.$$

Then $M(\Lambda)$ *contains a subset in one-to-one correspondence with* $S(n)$.

Proof. We begin by observing that

$$\Lambda_n = K[x, y]/(x, y)^n$$

is a quotient of Λ, so that $M(\Lambda_n) \subset M(\Lambda)$, and it suffices to prove the result for $M(\Lambda_n)$. Let x and y map onto X and Y, respectively, in Λ_n; then

$$\Lambda_n = K[X, Y], \qquad (X, Y)^n = 0.$$

Using formula (6) for Δ_n, we embed Δ_n in Λ_n by the mapping

$$\psi(\tilde{1}) = 1, \quad \psi(\tilde{x}_1) = X^{n-1}, \quad \psi(\tilde{x}_2) = X^{n-2}Y, \quad \cdots, \quad \psi(\tilde{x}_n) = Y^{n-1},$$

which is easily seen to be an algebra isomorphism of Δ_n into Λ_n. By means of this embedding we may regard Δ_n as a subalgebra of Λ_n.

If A is a Δ_n-module, define

$$A^* = \Lambda_n \otimes_{\Delta_n} A, \tag{8}$$

which is a Λ_n-module. The correspondence $A \to A^*$ preserves isomorphisms and direct sums. In the other direction we proceed as follows. Let

$$R = (X, Y) = \text{radical of } \Lambda_n.$$

Then (as a subalgebra of Λ_n)

$$\Delta_n = K\cdot 1 \oplus R^{n-1}, \tag{9}$$

and $R^{n-1} = S$ is the radical of Δ_n. If B is a Λ_n-module, then for $1 \leqq i \leqq n$ we have

$$\tilde{x}_i B = X^{n-i}Y^{i-1}B \subset R^{n-1}B,$$

$$\tilde{x}_i \cdot RB \subset R^n B = 0,$$

and so there exists a K-homomorphism $\theta_i : B/RB \to R^{n-1}B$ given by

$$\theta_i(b + RB) = \tilde{x}_i b, \qquad b \in B.$$

Setting

$$B' = B/RB \oplus R^{n-1}B,$$

we may therefore make B' into a Δ_n-module by defining for each i,

$$\tilde{x}_i(\bar{b}, b_1) = (0, \theta_i \bar{b}), \qquad \bar{b} \in B/RB, \quad b_1 \in R^{n-1}B.$$

The correspondence $B \to B'$ maps Λ_n-modules onto Δ_n-modules and clearly preserves isomorphisms and direct sums.

We shall prove that for any Δ_n-module A, we have

$$(A^*)' \cong A, \tag{10}$$

so that each class in $M(\Delta_n)$ determines a class in $M(\Lambda_n)$, and the result follows from Proposition 4.

We have shown in Section 4 that

$$A \cong A/SA \oplus SA,$$

the action of Δ_n on the right-hand side being given by

$$\tilde{x}_i(a + SA, a_1) = (0, \tilde{x}_i a), \qquad a \in A, \quad a_1 \in SA.$$

On the other hand every element of A^* is expressible as a sum

$$\sum\nolimits_{0 \leqq i+j \leqq n-1} X^i Y^j \otimes a_{ij}, \qquad a_{ij} \in A.$$

But we have

$$X^{n-i} Y^{i-1} \otimes a = 1 \otimes \tilde{x}_i a, \qquad a \in A,$$

and so every element of A^* is expressible as

$$a^* = 1 \otimes a_0 + \sum\nolimits_{0<i+j<n-1} X^i Y^j \otimes a_{ij}, \qquad a_0 \in A, \quad \{a_{ij}\} \in A.$$

To compute $(A^*)'$, we determine RA^*:

$$Xa^* = X \otimes a_0 \oplus \sum\nolimits_{0<i+j<n-2} X^{i+1} Y^j \otimes a_{ij} \oplus \sum\nolimits_{i+j=n-2} 1 \otimes \tilde{x}_{n-i-1} a_{ij},$$

and likewise for Ya^*. Thus

$$A^*/RA^* \cong (1 \otimes A)/(1 \otimes SA) \cong A/SA.$$

Furthermore

$$R^{n-1} A^* = 1 \otimes SA \cong SA.$$

Thus

$$(A^*)' \cong A/SA \oplus SA,$$

where the action of each $\tilde{x}_i$ is given by

$$\begin{aligned} \tilde{x}_i(a + SA, a_1) &= \tilde{x}_i(1 \otimes a + 1 \otimes SA, 1 \otimes SA) \\ &= 1 \otimes \tilde{x}_i a = (0, \tilde{x}_i a), \qquad a \in A, \quad a_1 \in SA. \end{aligned}$$

This completes the proof of (10), and establishes the proposition.

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UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS