HAUSDORFF DIMENSION IN PROBABILITY THEORY II

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1. Introduction and definitions

Let $\{x_1, x_2, \cdots\}$ be a stochastic process, with finite or countable state space σ, defined on a probability measure space $(\Omega, \mathcal{B}, \mu)$. In [1] a Hausdorff dimension $\dim_\mu M$ was defined for each set $M \subset \Omega$, in the following way.[1] A cylinder (of rank n) is defined to be a set of the form $\{\omega : x_k(\omega) = a_k, k = 1, 2, \cdots, n\}$, where $a_k \in \sigma$. If $M \subset \Omega$ and $\rho > 0$, a μ-ρ-covering of M is a finite or countable collection $\{v_i\}$ of cylinders such that $M \subset \bigcup_i v_i$ and $\mu(v_i) < \rho$ for each i. If $\rho, \alpha > 0$, put $L_\mu(M, \alpha, \rho) = \inf \sum_i \mu(v_i)^\alpha$, where the infimum extends over all μ-ρ-coverings $\{v_i\}$ of M, and let $L_\mu(M, \alpha) = \lim_{\rho \to 0} L_\mu(M, \alpha, \rho)$. If $L_\mu(M, \alpha) < \infty$, then $L_\mu(M, \alpha + \varepsilon) = 0$ for all $\varepsilon > 0$; hence we can define

$$(1.1) \qquad \dim_\mu M = \sup\{\alpha : L_\mu(M, \alpha) = \infty\} = \inf\{\alpha : L_\mu(M, \alpha) = 0\}.$$

It was shown in [1] that if Ω is the unit interval $(0, 1]$, if μ is Lebesgue measure, and if $\sum_{n=1}^\infty x_n(\omega) s^{-n}$ is, for each ω, the nonterminating base s expansion of ω, then this definition reduces to the classical one due to Hausdorff.

The dimension of M depends both on the measure μ and the process $\{x_n\}$. The dependence upon $\{x_n\}$ is not exhibited in the notation $\dim_\mu M$, since $\{x_n\}$ will remain fixed throughout the discussion. However, we will consider several measures μ simultaneously, and the main purpose of the paper is to investigate how $\dim_\mu M$ varies as μ varies. For $\omega \in \Omega$ and $n = 1, 2, \cdots$, put

$$u_n(\omega) = \{\omega' : x_k(\omega') = x_k(\omega), k = 1, 2, \cdots, n\}.$$

In other words, $u_n(\omega)$ is that cylinder of rank n which contains ω. In §2 we prove several refinements of the fact that if μ and ν are probability measures on $\mathcal{B}$, and if

$$(1.2) \qquad M \subset \left\{\omega : \lim_{n \to \infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} = \delta\right\},$$

then

$$(1.3) \qquad \dim_\mu M = \delta \dim_\nu M.$$

In §3, the results of §2 are used to extend and simplify some of the theorems of [1]. The essential idea here is to compute $\dim_\mu M$ for certain sets M by constructing a measure ν such that (1.2) holds and such that $\dim_\nu M = 1$. It then follows from (1.3) that $\dim_\mu M = \delta$. Finally, §4 contains some re-

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[1] In [1] the state space σ was assumed to be finite, but the definition applies to a countable σ as well.

marks on the connection between limits like the one in (1.2) and the Shannon-McMillan theorem.

2. Main results

To simplify the statements of the theorems of this section it will be convenient to define $\lg \xi/\lg \eta$ throughout the unit square $0 \leqq \xi, \eta \leqq 1$. We do this by making the convention that if $0 < \xi, \eta < 1$ then

$$\begin{aligned} &\lg \xi/\lg 0 = \lg 1/\lg \eta = \lg 1/\lg 0 = 0, \\ &\lg 0/\lg \eta = \lg \xi/\lg 1 = \lg 0/\lg 1 = \infty, \qquad (2.1) \\ &\lg 0/\lg 0 = \lg 1/\lg 1 = 1. \end{aligned}$$

(This is the only way to define $\lg \xi/\lg \eta$ in $0 \leqq \xi, \eta \leqq 1$ so that it is increasing in ξ, decreasing in η, and goes into its reciprocal upon interchange of ξ and η.)

Our first theorem, which is essentially a generalization of one due to Gillis [4], will not be used in the rest of this paper; it is given for comparison with later results and because it is useful in coding theory [2]. In all that follows, μ and ν are two probability measures on $\mathcal{B}$.

THEOREM 2.1. *If $\mu(\bigcap_{n=1}^{\infty} u_n(\omega)) = 0$ for each ω (i.e., if all cylinders of rank ∞ have μ-measure 0), and if $\delta \geqq 0$, then*

$$\dim_\mu \left\{\omega\colon \liminf_{n\to\infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} \leqq \delta\right\} \leqq \delta. \qquad (2.2)$$

Proof. By our convention (2.1), the set in brackets in (2.2) is contained in the set of ω for which, for any $\varepsilon > 0$,

$$\nu(u_n(\omega)) \geqq \mu(u_n(\omega))^{\delta+\varepsilon}$$

holds for infinitely many values of n. Call this set A; we will prove that $\dim_\mu A \leqq \delta$.

Given $\varepsilon, \rho > 0$, let $\mathcal{U}$ consist of those cylinders of the form $u = u_n(\omega)$ with

$$\mu(u_n(\omega)) < \rho, \qquad \nu(u_n(\omega)) \geqq \mu(u_n(\omega))^{\delta+\varepsilon}, \qquad \omega \in A.$$

Then $\mathcal{U}$ covers A by the definition of A and the fact that every ∞-cylinder has μ-measure 0. Let $\mathcal{V}$ consist of those elements of $\mathcal{U}$ which are not subsets of other elements of $\mathcal{U}$. Then $\mathcal{V}$ covers A, $\mathcal{V}$ is disjoint, and if $v \in \mathcal{V}$ then $\mu(v) < \rho$ and $\nu(v) \geqq \mu(v)^{\delta+\varepsilon}$. Hence $\mathcal{V}$ is a μ-ρ-covering of A, and[2]

$$1 \geqq \sum \nu(v)\colon \mathcal{V} \geqq \sum \mu(v)^{\delta+\varepsilon}\colon \mathcal{V} \geqq L_\mu(A, \delta+\varepsilon, \rho).$$

Since $\rho > 0$ was arbitrary, $L_\mu(A, \delta+\varepsilon) \leqq 1$, and hence $\dim_\mu A \leqq \delta + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\dim_\mu A \leqq \delta$.

Before proceeding to the next theorem, we give a heuristic proof of the

[2] If $\mathcal{V}$ is a collection of cylinders, $\sum \nu(v)^\alpha\colon \mathcal{V}$ denotes $\nu(v)^\alpha$ summed over $v \in \mathcal{V}$.

fact that (1.2) implies (1.3). Let us pretend that for each $\omega \in M$, not only does $\lg \nu(u_n(\omega))/\lg \mu(u_n(\omega))$ approach δ, but that it actually equals δ for all n. If $\mathcal{V}$ is any covering of M each element of which intersects M, then any element v of $\mathcal{V}$ has the form $v = u_n(\omega)$, with $\omega \in M$, so that $\nu(v) = \mu(v)^{\delta}$. Then

$$\sum \nu(v)^{\alpha}:\mathcal{V} = \sum \mu(v)^{\alpha\delta}:\mathcal{V}$$

for any covering $\mathcal{V}$ of M. It follows that $L_\nu(M, \alpha) = L_\mu(M, \alpha\delta)$, from which, by the definition (1.1), we easily obtain (1.3). In what follows, this argument is generalized and made precise.

THEOREM 2.2. *If*

$$M \subset \left\{\omega\colon \liminf_{n\to\infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} \geqq \delta\right\}, \tag{2.3}$$

then

$$\dim_\mu M \geqq \delta \dim_\nu M. \tag{2.4}$$

Proof. We may assume $\delta > 0$, in which case the set in brackets in (2.3) is contained in the set, call it B, where $\mu(u_n(\omega)) > 0$ for all n, and where for all ε with $0 < \varepsilon < \delta$ we have $\nu(u_n(\omega)) \leqq \mu(u_n(\omega))^{\delta-\varepsilon}$ for all n exceeding some integer $N(\omega, \varepsilon)$. This is because of our convention (2.1). We will show that if $M \subset B$, then (2.4) holds.

We may assume that $\beta = \dim_\nu M > 0$. Consider an ε with $0 < \varepsilon < \min(\delta, \beta)$. For $\rho > 0$ let B_ρ be the set of ω such that for all n

$$\mu(u_n(\omega)) < \rho \quad \text{implies} \quad \nu(u_n(\omega)) \leqq \mu(u_n(\omega))^{\delta-\varepsilon}. \tag{2.5}$$

Suppose that $\omega \in B$. Then by the definition of B, $\nu(u_n(\omega)) \leqq \mu(u_n(\omega))^{\delta-\varepsilon}$ for all n exceeding some $N(\omega, \varepsilon)$. Take $\rho = \mu(u_{N(\omega,\varepsilon)}(\omega))$. Then $\rho > 0$, since $\mu(u_n(\omega)) > 0$ for $\omega \in B$, and $\mu(u_n(\omega)) < \rho$ implies $n > N(\omega, \varepsilon)$, so that $\nu(u_n(\omega)) \leqq \mu(u_n(\omega))^{\delta-\varepsilon}$. Hence $\omega \in B_\rho$, and it follows that $B_\rho \uparrow B$ as $\rho \downarrow 0$. Since $M \subset B$, it follows that $M \cap B_\rho \uparrow M$ as $\rho \downarrow 0$, and hence, by Theorem 4.1 of [1], $\dim_\nu (M \cap B_\rho) \uparrow \beta$ as $\rho \downarrow 0$. Therefore $\dim_\nu (M \cap B_\rho) > \beta - \varepsilon$ for ρ sufficiently small, and hence

$$L_\nu(M \cap B_{\rho_0}, \beta - \varepsilon) = \infty$$

for some $\rho_0 > 0$. Since $\delta - \varepsilon > 0$, $L_\nu(M \cap B_{\rho_0}, \beta - \varepsilon, \rho^{\delta-\varepsilon})$ goes to infinity as ρ goes to 0 and, in particular, is positive if ρ is sufficiently small. If, in addition, $\rho < \rho_0$, then

$$L_\nu(M \cap B_\rho, \beta - \varepsilon, \rho^{\delta-\varepsilon}) > 0. \tag{2.6}$$

Fix a $\rho > 0$ for which this is true.

Let $\mathcal{V}$ be a μ-ρ-covering of $M \cap B_\rho$. If $\mathcal{V}_0$ consists of those elements of $\mathcal{V}$ which meet B_ρ, then $\mathcal{V}_0$ alone covers $M \cap B_\rho$. If $v \in \mathcal{V}_0$, then it has the form $v = u_n(\omega)$ for some n, and hence, since $\mu(v) < \rho$, $\nu(v) \leqq \mu(v)^{\delta-\varepsilon}$ by the definition (2.5) of B_ρ. Therefore

$$\sum \mu(v)^{(\delta-\varepsilon)(\beta-\varepsilon)} : \mathcal{V} \geqq \sum \mu(v)^{(\delta-\varepsilon)(\beta-\varepsilon)} : \mathcal{V}_0 \geqq \sum \nu(v)^{\beta-\varepsilon} : \mathcal{V}_0 .$$

Since $\nu(v) \leqq \mu(v)^{\delta-\varepsilon} < \rho^{\delta-\varepsilon}$ for $v \in \mathcal{V}_0$, $\mathcal{V}_0$ is a ν-$\rho^{\delta-\varepsilon}$-covering of $M \cap B_\rho$; hence

$$\sum \nu(v)^{(\delta-\varepsilon)(\beta-\varepsilon)} : \mathcal{V} \geqq L_\nu(M \cap B_\rho, \beta - \varepsilon, \rho^{\delta-\varepsilon}) > 0$$

by (2.6). Since $\mathcal{V}$ was an arbitrary μ-ρ-covering of $M \cap B_\rho$,

$$L_\mu(M \cap B_\rho, (\delta - \varepsilon)(\beta - \varepsilon)) \geqq L_\mu(M \cap B_\rho, (\delta - \varepsilon)(\beta - \varepsilon), \rho) > 0,$$

and hence $\dim_\mu M \geqq \dim_\mu(M \cap B_\rho) \geqq (\delta - \varepsilon)(\beta - \varepsilon)$. Since ε was arbitrary, $\dim_\mu M \geqq \delta\beta$, which completes the proof. Some of the techniques used in this proof are to be found in Kinney's paper [5].

If we interchange the roles of μ and ν in Theorem 2.2 and replace δ by its reciprocal, we have

THEOREM 2.3. *If*

$$M \subset \left\{\omega: \limsup_{n\to\infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} \leqq \delta\right\},$$

then

$$\dim_\mu M \leqq \delta \dim_\nu M. \tag{2.7}$$

Comparing the last two theorems with Theorem 2.1, one might conjecture the stronger result that (2.7) holds if

$$M \subset \left\{\omega: \liminf_{n\to\infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} \leqq \delta\right\}.$$

This conjecture is false.

Combining Theorems 2.2 and 2.3 we have the result announced in §1 and proved heuristically above.

THEOREM 2.4. *If*

$$M \subset \left\{\omega: \lim_{n\to\infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} = \delta\right\},$$

then

$$\dim_\mu M = \delta \dim_\nu M.$$

A result which is more useful for computing the dimensions of specific sets is the following one.

THEOREM 2.5. *Suppose that* $\nu(\bigcap_{n=1}^{\infty} u_n(\omega)) = 0$ *for all* ω, *that*

$$M \subset \left\{\omega: \limsup_{n\to\infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} \leqq \delta\right\}, \tag{2.8}$$

and that

$$M_0 \subset \left\{\omega: \lim_{n\to\infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} = \delta\right\} \tag{2.9}$$

for some subset M_0 of M. If $\nu(M_0) > 0$, then

$$\dim_\mu M = \delta. \tag{2.10}$$

Proof. By (2.8) and Theorem 2.3 we have $\dim_\mu M \leqq \delta \dim_\nu M$. Since every cylinder of rank ∞ has ν-measure 0, it follows that $\dim_\nu M \leqq 1$, and hence $\dim_\mu M \leqq \delta$. By (2.9) and Theorem 2.4 we have $\dim_\mu M_0 = \delta \dim_\nu M_0$. Since $\nu(M_0) > 0$, $\dim_\nu M_0 = 1$, so that $\dim_\mu M_0 = \delta$. But then $\dim_\mu M \geqq \dim_\mu M_0 = \delta$, from which (2.10) follows.

3. Applications

In this section we show how most of the results of [1] follow in a simple manner from Theorem 2.5.

Suppose that the state space σ is finite, say $\sigma = \{0, 1, \cdots, s-1\}$. Suppose further that under μ the process $\{x_n\}$ is independent with

$$\mu\{\omega : x_n(\omega) = i\} = p_i > 0, \qquad i = 0, 1, \cdots, s-1.$$

Suppose finally that if $\zeta = (\zeta_0, \zeta_1, \cdots, \zeta_{s-1})$ is any set of nonnegative numbers which sum to 1, then there is a measure $\nu = \nu_\zeta$ on $\mathcal{B}$ such that $\{x_n\}$ is independent under ν and $\nu\{\omega : x_n(\omega) = i\} = \zeta_i$. This last assumption holds, for example, if $\Omega = \sigma \times \sigma \times \cdots$ and if the x_n are the coordinate variables on Ω, or if Ω is the unit interval and $\sum_{n=1}^{\infty} x_n(\omega)/s^n$ is the nonterminating base s expansion of ω. (See the end of this section for further remarks along this line.)

Let A be the set of vectors $\zeta = (\zeta_0, \zeta_1, \cdots, \zeta_{s-1})$ of s-space such that $\zeta_i \geqq 0$ and $\sum_{i=0}^{s-1} \zeta_i = 1$. For each $\omega \in \Omega$, $n \geqq 1$, and $i = 0, \cdots, s-1$, let $\delta_i(\omega, n)$ be $1/n$ times the number of integers k such that $1 \leqq k \leqq n$ and $x_k(\omega) = i$. Let $\delta(\omega, n) = (\delta_0(\omega, n), \cdots, \delta_{s-1}(\omega, n)) \in A$. Finally, let

$$H(\zeta : p) = \sum_{i=0}^{s-1} \zeta_i \lg \zeta_i \Big/ \sum_{i=0}^{s-1} \zeta_i \lg p_i .$$

We first show that if

$$M(\zeta) = \{\omega : \lim_{n\to\infty} \delta(\omega, n) = \zeta\}, \tag{3.1}$$

then

$$\dim_\mu M(\zeta) = H(\zeta : p). \tag{3.2}$$

To do this, let ν be that measure on $\mathcal{B}$ under which $\{x_n\}$ is independent with $\nu\{\omega : x_n(\omega) = i\} = \zeta_i$. Since

$$\lg \mu(u_n(\omega)) = n \sum_{i=0}^{s-1} \delta_i(\omega, n) \lg p_i ,$$

$$\lg \nu(u_n(\omega)) = n \sum_{i=0}^{s-1} \delta_i(\omega, n) \lg \zeta_i ,$$

we have

$$\lim_{n\to\infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} = H(\zeta : p)$$

for any ω in $M(\zeta)$. Hence, by Theorem 2.4, $\dim_\mu M(\zeta) = H(\zeta:p) \dim_\nu M(\zeta)$. But since $\nu(M(\zeta)) = 1$ by the strong law of large numbers, $\dim_\nu M(\zeta) = 1$, and (3.2) follows.

An application of Theorem 2.5 in place of Theorem 2.4 yields a stronger result. Let S be a subset of A, and put

$$M(S) = \{\omega : \lim_{n\to\infty} \rho(\delta(\omega, n), S) = 0\},$$

where ρ denotes Euclidean distance on A. We will show that

$$\dim_\mu M(S) = \sup_{\zeta\in S} H(\zeta:p). \tag{3.3}$$

Since neither the set $M(S)$ nor the right-hand side of (3.3) is altered if S is replaced by its closure, we may assume that S is closed. Let ζ^0 be an element of S such that

$$H(\zeta^0:p) = \sup_{\zeta\in S} H(\zeta:p).$$

Take ν to be that measure on $\mathcal{B}$ under which $\{x_n\}$ is independent with

$$\nu\{\omega : x_n(\omega) = i\} = \zeta_i^0.$$

An application of Theorem 2.5 with $M(S)$ and $M(\zeta^0)$ (as defined by (3.1)) playing the roles of M and M_0 immediately yields (3.3). In [1] it was shown how a number of results in classical Hausdorff dimension theory follow directly from (3.3).

The very same technique suffices to prove the following result. Suppose that S has the property that for some i_0, $\zeta_{i_0} = 0$ for any $\zeta \in S$. Then (3.3) still holds if $M(S)$ is replaced by its intersection with the set of ω for which $\delta_{i_0}(\omega, n) = 0$, $n = 1, 2, \cdots$. From this it follows, for example, that the (classical) Hausdorff dimension of the Cantor set is lg 2/lg 3.

There is one result of §7 of [1] which cannot be obtained by the present methods, namely,

$$\dim_\mu \{\omega : \rho(\delta(\omega, n), S) = O(1/n)\} = \sup_{\zeta\in S} H(\zeta:p). \tag{3.4}$$

Just as rates of convergence for the strong law of large numbers have never been obtained without special assumptions and methods, presumably (3.4) cannot be proved without using techniques like the combinatorial ones of [1].

There remains the question of under what conditions on $(\Omega, \mathcal{B}, \mu)$ and $\{x_n\}$ one can construct a measure ν on $\mathcal{B}$ under which $\{x_n\}$ becomes a process with specified finite-dimensional distributions. In [1] this problem was taken care of by assuming that $(\Omega, \mathcal{B}, \mu)$ together with $\{x_n\}$ satisfies the following condition (see §4 of [1]).

Condition (C). *All but a countable number of sequences $(a_1, a_2, \cdots)$ of states have the property that either $\{\omega : x_k(\omega) = a_k,\ k = 1, 2, \cdots\}$ is nonempty, or else $\mu\{\omega : x_k(\omega) = a_k,\ k = 1, 2, \cdots, n\} = 0$ for some n.*

If $\{p(a_1, \cdots, a_n)\}$ is any consistent collection of finite-dimensional distributions, and if $\mu\{\omega : x_k(\omega) = a_k, k = 1, \cdots, n\} = 0$ implies

$$p(a_1, \cdots, a_n) = 0,$$

then there exists a probability measure ν on $\mathcal{B}$ such that

$$\nu\{\omega : x_k(\omega) = a_k, k = 1, \cdots, n\} = p(a_1, \cdots, a_n),$$

provided Condition (C) is satisfied. This result is contained in the following theorem.

THEOREM 3.1. *Let Ω' be the set of sequences $\omega' = (\omega_1', \omega_2', \cdots)$ with $\omega_k' \in \sigma$; let $\mathcal{B}'$ be the Borel field generated by sets of the form $\{\omega' : \omega_k' = a\}$. If $p(a_1, \cdots, a_n)$ is a consistent set of finite-dimensional distributions, let ν' be that measure on $\mathcal{B}'$ such that $\nu'\{\omega' : \omega_k' = a_k, k = 1, \cdots, n\} = p(a_1, \cdots, a_n)$. A necessary and sufficient condition that there exist a probability measure ν on the Borel field generated by the x_n such that*

$$\nu\{\omega : x_k(\omega) = a_k, k = 1, \cdots, n\} = p(a_1, \cdots, a_n)$$

is that the ω'-set $\{(x_1(\omega), x_2(\omega), \cdots) : \omega \in \Omega\}$ have ν'-outer-measure 1.

Since this result is not central to the theory, and is essentially the same as standard theorems, its proof is omitted.

In this section, attention has been restricted to the independent case for simplicity only. As in [1], analogous results can be proved under the assumption that $\{x_n\}$ is a Markov chain.

4. A connection with the Shannon-McMillan theorem

Suppose that the state space σ of the process contains s points and that μ and ν are probability measures under which $\{x_n\}$ is stationary and ergodic. In the preceding sections the limit

$$\lim_{n\to\infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} \tag{4.1}$$

played a basic role. It would be interesting to have conditions under which this limit exists $[\nu]$ (that is, except on a set of ν-measure zero). If under μ the process $\{x_n\}$ is independent with $\mu\{\omega : x_n(\omega) = i\} \equiv 1/s$, this becomes

$$\lim\nolimits_{n\to\infty} \{n^{-1} \lg_s \nu(u_n(\omega))\}. \tag{4.2}$$

It follows from Breiman's version [3] of the Shannon-McMillan theorem that this limit exists $[\nu]$ and equals the relative entropy of $\{x_n\}$ under ν. Hence (4.1) exists $[\nu]$ if and only if $\lim_n \{n^{-1} \lg_s \mu(u_n(\omega))\}$ does. But this is in turn equivalent to the existence $[\nu]$ of

$$\lim_{n\to\infty} \left\{\frac{1}{n} \lg \frac{\nu(u_n(\omega))}{\mu(u_n(\omega))}\right\}.$$

Therefore one asks whether *this* limit exists. In §8 of [1] it was shown that these limits all exist $[\nu]$ if $\{x_n\}$ is a Markov chain under both μ and ν. It is easy to remove the restriction that $\{x_n\}$ be a Markov chain under ν. Since $\nu(u_n(\omega))/\mu(u_n(\omega))$ is the Radon-Nikodym derivative f_n of ν with respect to μ when both measures are restricted to the Borel field generated by $x_1, \cdots, x_n$, one can remove the restriction that the x_n have a finite state space and still ask whether $\lim_n n^{-1} \lg f_n$ exists $[\nu]$. Some progress in this direction has been made by Moy [6] and Perez [7].

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