RANDOM WALKS WITH ABSORBING BARRIERS AND TOEPLITZ FORMS

BY
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1. Introduction

In a recent paper Spitzer and Stone [11] considered some asymptotic properties of the Toeplitz matrices

$$\| T(N)_{k,j} \| = \| c_{j-k} \| \qquad (k, j = 0, 1, \cdots, N)$$

where the c_k satisfied

$$c_k = c_{-k} \geqq 0, \qquad k = 0, 1, \cdots, \tag{1.1}$$

$$\textstyle\sum_{k=-\infty}^{+\infty} c_k = 1, \tag{1.2}$$

$$\text{g.c.d. } [k \mid k > 0,\ c_k > 0] = 1, \tag{1.3}$$

$$0 < \textstyle\sum_{k=-\infty}^{+\infty} k^2 c_k < \infty. \tag{1.4}$$

By (1.1) and (1.2)[1]

$$c_k = P\{X = k\} \tag{1.5}$$

defines a probability distribution of a random variable X, and consequently most of the results in [11] have an easy probability interpretation. Putting

$$S_n = X_0 + \textstyle\sum_{k=1}^{n} X_k ,$$

where $X_1, X_2, \cdots$ is a sequence of independent random variables, each distributed as X in (1.5), and X_0 an unspecified integer, it was shown in [11] that

(1.6) $H(N)_{k,j} = [I - T(N)]_{k,j}^{-1} =$ Expected number of visits to j of the S_n process with $S_0 = X_0 = k$ before leaving the interval $[0, N]$.[2]

One also has ([11])

$$H(N)_{k,j} = \textstyle\sum_{r=\max(k,j)}^{N} p_{r,k}\, p_{r,j} ,$$

where the $p_{r,k}$ are the coefficients of the orthogonal polynomials corresponding to the weight function

$$1 - \phi(t) = 1 - \textstyle\sum_{k=-\infty}^{+\infty} c_k \exp(ikt).$$

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[1] $P\{A\}$ = the probability of the event A,
$P\{A|B\}$ = the conditional probability of A, given B,
EX = expectation of the random variable X,
$E\{X|B\}$ = the conditional expectation of X, given B.

[2] $[a]$ = the largest integer $\leqq a$,
$[b, c]$ is the closed interval $b \leqq \xi \leqq c$.
This double use of square brackets is not likely to lead to confusion. (b, c) is the open interval $b < \xi < c$.

The author, in [7], proved some probabilistic results which by means of (1.5) can be interpreted as results for Toeplitz matrices. The sequences $\{c_k\}$ in [7] satisfied (1.1)–(1.3), but instead of (1.4) it was assumed that for some $1 \leqq \alpha \leqq 2$

$$(1.7)\quad \begin{aligned} 0 &< \lim_{t\to 0} |t|^{-\alpha}(1 - \phi(t)) \\ &= \lim_{t\to 0} |t|^{-\alpha}\cdot 2 \textstyle\sum_{k=1}^{\infty} c_k(1 - \cos kt) = Q < \infty. \end{aligned}$$

It seems that for $\alpha < 2$ the probabilistic methods more easily lead to results on Toeplitz matrices than the direct methods used in [11] for $\alpha = 2$. The present paper completes the results of [7] and [11].

We obtain the limit (as $N \to \infty$) of the probability that the S_n process with $S_0 = [xN]$ leaves the interval[2] $[0, N]$ for the first time at the left and the probability that at the first departure of $[0, N]$, it jumps to a point in $[-yN, 0]$ (Theorem 1). This is of course related to the classical problem of gambler's ruin. In addition, for $1 < \alpha < 2$

$$(1.8)\qquad \lim_{N\to\infty} N^{1-\alpha} H(N)_{[xN],[yN]}$$

and

$$(1.9)\qquad \lim_{N\to\infty} N^{1-\alpha/2} p_{N,[xN]}$$

are obtained (Theorems 3 and 6). The result (1.8) is applied to find the inverse of a certain infinitesimal generator (Theorem 5). This infinitesimal generator was found by Elliott [4] and Getoor [5] as corresponding to the stable process of index α with absorbing barriers. Another form of the inverse of this operator was found by Widom [13].

Under assumption (1.7) one can expect several asymptotic formulae to be the same as their analogues for stable processes with independent increments of index α. Our paper rests indeed heavily on the recent results of Blumenthal, Getoor, Ray, Widom ([2], [5], and [13]) concerning these processes. It would even be possible to prove Theorem 1 concerning the distribution of the first sum S_n outside a given interval from the above results by using a general invariance theorem of Skorohod [10]. For the quantities $p_\alpha(y)$, $q_\alpha(y; c)$ (cf. (2.8) and (2.29)), and $\lim_{N\to\infty} N^{1-\alpha} H(N)_{[xN],[yN]}$ one first has to prove continuity (Lemmas 3 and 4) before an approach via the invariance principle becomes feasible. In the proofs appearing below we follow the method of [7] which does not use the invariance principle, especially since we need Lemma 1 anyway for Corollary 2 of Theorem 1.

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2. Absorption probabilities and probabilities of visits before absorption

Unfortunately, we worked in [7] always with the interval $[-cN, N]$ and took $S_0 = 0$, while in [11] the interval $[0, N]$ and $S_0 = [xN]$ was considered. Since this section is a sequel to [7], we shall here still work with $[-cN, N]$ and $S_0 = 0$, but when applying the results in the next section to Toeplitz matrices we shall always follow the notation of [11].

Let $X_1, X_2, \cdots$ be a sequence of integer-valued independent random variables each with the distribution

$$P\{X_i = k\} = c_k, \qquad k = 0, \pm 1, \pm 2, \cdots, \tag{2.1}$$

where the c_k satisfy (1.1) and (1.2). (1.3) is not required for most results in this section. If it is required, we shall mention it explicitly. In addition, putting[1]

$$\phi(t) = Ee^{itX_1} = \textstyle\sum_{k=-\infty}^{+\infty} c_k e^{itk}, \tag{2.2}$$

we assume

$$0 < \lim_{t\to 0} |t|^{-\alpha}(1 - \phi(t)) = Q < \infty \tag{2.3}$$

for some $0 < \alpha \leqq 2$. As before, we define

$$S_n = X_0 + \textstyle\sum_{k=1}^{n} X_k, \tag{2.4}$$

where X_0 is an unspecified integer. Following [11], we also introduce

$$\begin{aligned} &P(k \xrightarrow{n} j; [a, b]) \\ &\qquad = P\{S_n = j; a \leqq S_i \leqq b \quad \text{for} \quad i = 0, 1, \cdots, n \mid S_0 = k\}, \end{aligned} \tag{2.5}$$

and similarly

$$\begin{aligned} &\tilde{P}(k \to j; [a, b]) \\ &\quad = P\{\text{there exists an } n \geqq 0 \text{ with } S_n = j \text{ and } a \leqq S_i \leqq b \\ &\qquad\qquad\qquad \text{for } i = 0, 1, \cdots, n \mid S_0 = k\} \\ &\quad = \text{probability of visiting } j \text{ before leaving } [a, b] \text{ when } S_0 = k. \end{aligned} \tag{2.6}$$

(Note that $\tilde{P}(k \to k; [a, b]) = 1$ when $k \in [a, b]$ since we required only $n \geqq 0$ in the definition of $\tilde{P}$.) In [7] we introduced the following quantities:

$$F_\alpha(x) = \pi^{-1} \sin \frac{\pi\alpha}{2} \int_0^x t^{-\alpha/2}(1 + t)^{-1}\, dt \tag{2.7}$$

and (for $1 < \alpha < 2, 0 < y < 1$)

$$\begin{aligned} p_\alpha(y) &= \lim_{\substack{N\to\infty \\ kN^{-1}\to y}} \tilde{P}(k \to 0; [-\infty, N]) \\ &= (\alpha - 1)y^{\alpha-1} \int_y^1 w^{-\alpha}(1 - w)^{\alpha/2-1}\, dw \end{aligned} \tag{2.8}$$

(cf. Lemma 9 in [7]) as well as

$$G_\alpha(x; N, c) = P\{\text{the first } S_n \text{ outside of } [-cN, N] \text{ lies in } [-(x+c)N, -cN) \mid S_0 = 0\} \tag{2.9}$$

and

$$H_\alpha(x; N, c) = P\{\text{the first } S_n \text{ outside of } [-cN, N] \text{ lies in } (N, (1+x)N] \mid S_0 = 0\}. \tag{2.10}$$

For $1 < \alpha < 2$ it was proved in [7] that

$$\lim_{N\to\infty} G_\alpha(x; N, c) = G_\alpha(x; c)$$

and

$$\lim_{N\to\infty} H_\alpha(x; N, c) = H_\alpha(x; c)$$

exist and are the unique nondecreasing and bounded solution of

$$\begin{aligned} G_\alpha(x; c) &= F_\alpha(xc^{-1}) - \int_0^\infty dF_\alpha(x_1)F_\alpha(x(1+c+x_1)^{-1}) \\ &+ \int_0^\infty dG_\alpha(x_1; c) \int_0^\infty dF_\alpha\,(x_2\,(1+c+x_1)^{-1})F_\alpha(x(1+c+x_2)^{-1}) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} H_\alpha(x; c) &= F_\alpha(x) - \int_0^\infty dF_\alpha(x_1 c^{-1})F_\alpha(x(1+c+x_1)^{-1}) \\ &+ \int_0^\infty dH_\alpha(x_1; c) \int_0^\infty dF_\alpha(x_2(1+c+x_1)^{-1})F_\alpha(x(1+c+x_2)^{-1}) \end{aligned} \tag{2.12}$$

under the conditions

$$\lim_{x\to\infty} G_\alpha(x; c) + \lim_{x\to\infty} H_\alpha(x; c) = 1 \tag{2.13}$$

and

$$\begin{aligned} 1 - c^{\alpha-1} &= \int_0^\infty dG_\alpha(x_1; c) \int_0^1 p_\alpha(x_2)\, dF_\alpha(x_2(c+x_1)^{-1}) \\ &- c^{\alpha-1} \int_0^\infty dH_\alpha(x_1; c) \int_0^c p_\alpha(x_2 c^{-1})\, dF_\alpha(x_2(1+x_1)^{-1}). \end{aligned} \tag{2.14}$$

For $0 < \alpha \leqq 1$, G_α and H_α are uniquely determined by (2.11) and (2.12) alone (cf. Lemma 10 in [7]). It is easily seen that (2.11) and (2.12) are implied by

$$G_\alpha(x; c) = F_\alpha(xc^{-1}) - \int_0^\infty dH_\alpha(x_1; c)F_\alpha(x(1+c+x_1)^{-1}) \tag{2.15}$$

together with

$$H_\alpha(x; c) = F_\alpha(x) - \int_0^\infty dG_\alpha(x_1; c)F_\alpha(x(1+c+x_1)^{-1}). \tag{2.16}$$

Actually (2.15) is the limiting relation of the first equality of (4.34) in [7].

In addition it is an immediate consequence of the definitions (2.9) and (2.10) and the symmetry condition (1.1) that

$$H_\alpha(x; c) = G_\alpha(xc^{-1}; c^{-1}). \tag{2.17}$$

Replacing c by c^{-1} in (2.15) and using (2.17) one gets (2.16). We therefore conclude that there is at most one pair of bounded, nondecreasing solutions G and H of (2.13), (2.14), (2.15), and (2.17). The solutions which we shall exhibit are therefore the G_α and H_α looked for. As already indicated in the introduction, the solutions are the expressions found by Blumenthal, Getoor, Ray, Widom ([2], [13]) for the stable processes with independent increments.

THEOREM 1. *If* (1.1), (1.2), *and* (2.3) *are satisfied for some* $0 < \alpha < 2$, *then*

$$G_\alpha(x; c) = \pi^{-1} \sin \frac{\pi\alpha}{2} c^{\alpha/2} \int_0^x z^{-\alpha/2}(z + c + 1)^{-\alpha/2}(z + c)^{-1}\, dz, \tag{2.18}$$

and

$$H_\alpha(x; c) = \pi^{-1} \sin \frac{\pi\alpha}{2} c^{\alpha/2} \int_0^x z^{-\alpha/2}(z + c + 1)^{-\alpha/2}(z + 1)^{-1}\, dz. \tag{2.19}$$

Proof. (2.17) is obviously satisfied. (2.15), rewritten for densities says

$$\pi^{-1} \sin \frac{\pi\alpha}{2} c^{\alpha/2}x^{-\alpha/2}(x + c + 1)^{-\alpha/2}(x + c)^{-1} = \pi^{-1} \sin \frac{\pi\alpha}{2} c^{\alpha/2}x^{-\alpha/2}(x + c)^{-1}$$
$$- \pi^{-2}\left(\sin \frac{\pi\alpha}{2}\right)^2 c^{\alpha/2}x^{-\alpha/2}\int_0^\infty z^{-\alpha/2}(z + 1)^{-1}(x + z + c + 1)^{-1}\, dz$$

which immediately follows by rewriting the last integral as

$$\int_0^\infty z^{-\alpha/2}(z + 1)^{-1}(x + z + c + 1)^{-1}\, dz$$
$$= (x + c)^{-1}\left\{\int_0^\infty z^{-\alpha/2}(z + 1)^{-1}\, dz - \int_0^\infty z^{-\alpha/2}(x + z + c + 1)^{-1}\, dz\right\}$$
$$= \pi\left(\sin \frac{\pi\alpha}{2}\right)^{-1}(x + c)^{-1}\{1 - (x + c + 1)^{-\alpha/2}\}.$$

This completes the proof when $0 < \alpha \leqq 1$. For $1 < \alpha < 2$ one still has to verify (2.13) and (2.14), which will be done after the next two lemmas.

LEMMA 1. *For* $0 < \alpha < 2$

$$\pi^{-1} \sin \frac{\pi\alpha}{2} c^{\alpha/2} \int_0^\infty z^{-\alpha/2}(z + c + 1)^{-\alpha/2}(z + c)^{-1}\, dz = \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma(\alpha/2)} \int_0^{(c+1)^{-1}} y^{\alpha/2-1}(1 - y)^{\alpha/2-1}\, dy. \tag{2.20}$$

Proof. After substituting $(z + c) = cu^{-1}$ in the left-hand side of (2.20) it is easily verified that the two sides of (2.20) are equal to 1 for $c = 0$ and in addition have equal derivatives with respect to c.

LEMMA 2. *For* $1 < \alpha < 2$

$$\begin{aligned}\pi^{-1} \sin \frac{\pi\alpha}{2} c^{\alpha/2} \int_0^\infty & z^{-\alpha/2}(z + c + 1)^{-\alpha/2}(z + c)^{-1}\, dz \\ & \cdot \int_0^1 p_\alpha(y)\, dF_\alpha(y(z + c)^{-1}) = 1 - \left(\frac{c}{c+1}\right)^{\alpha-1}.\end{aligned} \tag{2.21}$$

Proof. Substituting P_α and F_α from (2.7) and (2.8) we see that we have to compute

$$\begin{aligned}\pi^{-2}\left(\sin\frac{\pi\alpha}{2}\right)^2 (\alpha - 1)c^{\alpha/2}\int_0^\infty & z^{-\alpha/2}(z + c + 1)^{-\alpha/2}(z + c)^{\alpha/2-1} \\ & \cdot \int_0^1 y^{\alpha/2-1}(z + c + y)^{-1}\, dy \int_y^1 w^{-\alpha}(1 - w)^{\alpha/2-1}\, dw.\end{aligned} \tag{2.22}$$

Interchanging the order of integration of y and w and then introducing the new variable $x = (z + c)w((z + c)w + y)^{-1}$ gives

$$\begin{aligned}\pi^{-2}&\left(\sin\frac{\pi\alpha}{2}\right)^2(\alpha - 1)c^{\alpha/2}\int_0^\infty z^{-\alpha/2}(z + c + 1)^{-\alpha/2}(z + c)^{\alpha-2} \\ &\cdot\int_0^1 w^{-\alpha/2}(1 + w)^{\alpha/2-1}\, dw \\ &\cdot\int_{(z+c)(z+c+1)^{-1}}^1 x^{-\alpha/2}(1 - x)^{\alpha/2-1}(x + w(1 - x))^{-1}\, dx \\ &= \pi^{-2}\left(\sin\frac{\pi\alpha}{2}\right)^2(\alpha - 1)c^{\alpha/2}\int_{c(c+1)^{-1}}^1 x^{-\alpha/2}(1 - x)^{\alpha/2-1}\, dx \\ &\cdot\int_0^{x(1-x)^{-1}-c} z^{-\alpha/2}(z + c + 1)^{-\alpha/2}(z + c)^{\alpha-2}\, dz \\ &\cdot\int_0^1 w^{-\alpha/2}(1 - w)^{\alpha/2-1}(x + w(1 - x))^{-1}\, dw.\end{aligned}$$

However, the introduction of $u = w(x + w(1 - x))^{-1}$ shows that

$$\int_0^1 w^{-\alpha/2}(1 - w)^{\alpha/2-1}(x + w(1 - x))^{-1}\, dw = \pi\left(\sin\frac{\pi\alpha}{2}\right)^{-1} x^{-\alpha/2}.$$

Hence, after introducing $v = (z + c)(z + c + 1)^{-1}(c + 1)c^{-1}x^{-1}$ we obtain that the expression (2.22) equals

$$\pi^{-1} \sin \frac{\pi\alpha}{2} (\alpha - 1)c^{\alpha-1}(c + 1)^{1-\alpha} \cdot \int_{c(c+1)^{-1}}^{1} x^{-1}(1 - x)^{\alpha/2-1} \int_{x^{-1}}^{(c+1)c^{-1}} v^{\alpha-2}(xv - 1)^{-\alpha/2}\, dv$$

$$= \pi^{-1} \sin \frac{\pi\alpha}{2} (\alpha - 1)c^{\alpha-1}(c + 1)^{1-\alpha} \cdot \int_{1}^{(c+1)c^{-1}} v^{\alpha-2}\, dv \int_{v^{-1}}^{1} x^{-1}(1 - x)^{\alpha/2-1}(xv - 1)^{-\alpha/2}\, dx.$$

Finally, introducing $t = (1 - x)x^{-1}(v - 1)^{-1}$, the last expression is seen to equal

$$1 - \left(\frac{c}{c + 1}\right)^{\alpha-1}.$$

This proves the lemma.

The proof of Theorem 1 for the case $1 < \alpha < 2$ is now easily completed. Since (2.17) is satisfied, it follows from Lemma 1 (replacing c by c^{-1}) that

$$\begin{aligned}\pi^{-1} \sin \frac{\pi\alpha}{2} c^{\alpha/2} \int_0^{\infty} z^{-\alpha/2}(z + c + 1)^{-\alpha/2}(z + 1)^{-1} &= \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma(\alpha/2)} \int_0^{c(c+1)^{-1}} y^{\alpha/2-1}(1 - y)^{\alpha/2-1}\, dy \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma(\alpha/2)} \int_{1-c(c+1)^{-1}}^{1} y^{\alpha/2-1}(1 - y)^{\alpha/2-1}\, dy,\end{aligned}$$

which, together with Lemma 1, implies (2.13). Similarly, by changing c into c^{-1} in Lemma 2

$$\pi^{-1} \sin \frac{\pi\alpha}{2} c^{\alpha/2} \int_0^{\infty} z^{-\alpha/2}(z + c + 1)^{-\alpha/2}(z + 1)^{-1}\, dz \cdot \int_0^{c} p_\alpha(yc^{-1})\, dF_\alpha(y(1 + z)^{-1}) = 1 - \left(\frac{1}{c + 1}\right)^{\alpha-1}.$$

This together with Lemma 2 proves (2.14). Hence G_α and H_α in (2.18) and (2.19) are indeed bounded, nondecreasing solutions of (2.13)–(2.17), which proves the theorem.

COROLLARY 1. *If* (1.1), (1.2), *and* (2.3) *are satisfied for some* $0 < \alpha \leqq 2$, *then*

$$\begin{aligned}&\lim_{N\to\infty} P\{\textit{the first } S_n \textit{ outside } [-cN, N] \textit{ is less than } -cN \mid S_0 = 0\} \\ &\qquad = \lim_{x\to\infty} G_\alpha(x; c) = \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma(\alpha/2)} \int_0^{(c+1)^{-1}} y^{\alpha/2-1}(1 - y)^{\alpha/2-1}\, dy.\end{aligned} \tag{2.23}$$

This is of course nothing but Lemma 1 when $0 < \alpha < 2$. For $\alpha = 2$ it was already proved in [7] or [11].

COROLLARY 2. *If* (1.1), (1.2), *and* (2.3) *are satisfied for some* $1 < \alpha \leqq 2$, *then*

$$
\begin{aligned}
\lim_{N\to\infty} \underline{\rho}(N, -cN) &= \lim_{N\to\infty} P\{\text{the } S_n \text{ process leaves } [-cN, +N] \text{ at the left and returns to } 0 \text{ before it ever crosses } +N \mid S_0 = 0\} \\
&= \lim_{N\to\infty} P\{\text{there exist } n, m \text{ such that } n < m,\ S_n < -cN,\ S_m = 0, \text{ and } S_i \leqq N \text{ for } i = 1, 2, \cdots, m \mid S_0 = 0\} \\
&= 1 - \left(\frac{c}{c+1}\right)^{\alpha-1}.
\end{aligned} \tag{2.24}
$$

Consequently, if I *is a set of* μ *integers and*

$$N_I(A, -B) = \text{number of terms } S_k \text{ in the infinite sequence } S_1, S_2, \cdots \text{ with } S_k \in I \text{ and } -B \leqq S_i \leqq A \text{ for } i = 1, 2, \cdots, k \text{ when } S_0 = 0, \tag{2.25}$$

and if (1.3) *is satisfied, then*

$$\lim_{A\to\infty} A^{1-\alpha} EN_I(A, -cA) = \left(\frac{c}{c+1}\right)^{\alpha-1} C(\alpha, Q)\mu, \tag{2.26}$$

and

$$\lim_{A\to\infty} P\left\{N_I(A, -cA) \leqq A^{\alpha-1}\left(\frac{c}{c+1}\right)^{\alpha-1} C(\alpha, Q)\mu x \mid S_0 = 0\right\} = 1 - e^{-x}, \tag{2.27}$$

where

$$
\begin{aligned}
C(\alpha, Q) &= (\pi Q 2^{1-\alpha}\Gamma(\alpha))^{-1} \int_0^\infty (1 + y^2)^{-\alpha/2}\, dy \\
&= (Q(\alpha - 1)\Gamma(\alpha/2)\Gamma(\alpha/2))^{-1}.
\end{aligned} \tag{2.28}
$$

(Q *is defined in* (2.3).)

Proof. It was proved in equation (4.45) of [7] that for $1 < \alpha < 2$

$$\lim_{N\to\infty} \underline{\rho}(N, -cN) = \int_0^\infty dG_\alpha(x_1; c) \int_0^1 p_\alpha(x_2)\, dF_\alpha(x_2(c + x_1)^{-1}),$$

which implies (2.24) in virtue of Lemma 2. (2.26) and (2.27) were also proved under the additional condition (1.3) in Theorem 4 of [7] except for the explicit expression of $1 - \underline{\rho}$. For $\alpha = 2$, (2.24), (2.26), (2.27) are contained in Theorem 2 of [7] ((2.24) not explicitly, but it follows easily from the proof there). The first expression for $C(\alpha, Q)$ was obtained in [7], while the last equality follows from p. 240 in [12].

In order to find the asymptotic behaviour of $[I - T(N)]_{k,j}$ we shall need the analogue of $p_\alpha(y)$ for the case of two absorbing barriers, i.e.,

$$q_\alpha(y; c) = \lim_{\substack{N\to\infty \\ kN^{-1}\to y}} \tilde{P}(k \to 0; [-cN, N]) \qquad (0 < y < 1). \tag{2.29}$$

In [7] we left out the proof of the *existence* of the limit in (2.8). Since a similar argument is needed here a few times, we shall give a complete proof of the existence of the limit in (2.29) and compute the limit. The missing detail for (2.8) (or Lemma 9 in [7]) can then be proved in the same way.

LEMMA 3. *Let* (1.1), (1.2), *and* (2.3) *be satisfied for some* $1 < \alpha \leqq 2$. *Then for each* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that*

$$\tilde{P}(k \to 0; [-\infty, N]) \geqq 1 - \varepsilon$$

whenever $|k| \leqq \delta(\varepsilon)N$.

Proof.

$$\textstyle\sum_{n=1}^{M} P(k \xrightarrow{\;n\;} 0; [-\infty, +\infty])$$
$$\leqq P\{\text{there exists an } n \leqq M \text{ with } S_n = 0 \mid S_0 = k\}$$
$$\cdot \textstyle\sum_{n=0}^{M} P(0 \xrightarrow{\;n\;} 0; [-\infty, +\infty]).$$

However,

$$\sum_{n=1}^{M} P(k \xrightarrow{\;n\;} 0; [-\infty, +\infty]) = \frac{1}{2\pi} \sum_{n=1}^{M} \int_{-\pi}^{+\pi} e^{ikt}(\phi(t))^n \, dt, \tag{2.30}$$

and one can show, using (2.3), that for fixed k

$$\lim_{M\to\infty} M^{1/\alpha-1} \cdot \frac{1}{2\pi} \sum_{n=1}^{M} \int_{-\pi}^{+\pi} e^{ikt}(\phi(t))^n = C_1 > 0. \tag{2.31}$$

One can even find for each $\varepsilon > 0$ a $\delta_1(\varepsilon) > 0$ such that, for sufficiently large M,

$$\sum_{n=1}^{M} P(k \xrightarrow{\;n\;} 0; [-\infty, +\infty]) \left\{ \sum_{n=0}^{M} P(0 \xrightarrow{\;n\;} 0; [-\infty, +\infty]) \right\}^{-1} \geqq 1 - \varepsilon/3$$

whenever $|k| \leqq \delta_1(\varepsilon)M^{1/\alpha}$. Consequently

$$P\{\text{there exists an } n \leqq M \text{ with } S_n = 0 \mid S_0 = k\} \geqq 1 - \varepsilon/3 \tag{2.32}$$

when

$$|k| \leqq \delta_1(\varepsilon)M^{1/\alpha} \quad \text{and} \quad M \text{ is sufficiently large.} \tag{2.33}$$

Thus under condition (2.33)

$$\begin{aligned} &P\{\text{there exists an } n \leqq M \text{ for which } S_n = 0 \text{ while} \\ &\qquad\qquad S_i \leqq N \text{ for } i = 0, 1, \cdots, n \mid S_0 = k\} \end{aligned} \tag{2.34}$$
$$\geqq 1 - \varepsilon/3 - P\{\text{there exists an } n \leqq M \text{ with } S_n > N \mid S_0 = k\}.$$

But

$$\begin{aligned}1 - P\{&\text{there exists an } n \leqq M \text{ with } S_n > N \mid S_0 = k\}\\ &= P\{S_n \leqq N - k \text{ for all } n \leqq M \mid S_0 = 0\}\\ &\geqq P\{\text{the } S_n \text{ process returns to 0 at least } m \text{ times before leaving } (-\infty, N-k] \mid S_0 = 0\}\\ &\quad - P\{\text{there do exist } m \text{ integers } 0 < n_1 < n_2 < \cdots < n_m \leqq M \text{ with } S_{n_i} = 0 \mid S_0 = 0\}\\ &\geqq (\tilde{P}(0 \to 0; [-\infty, N-k]))^m - C_2 m^{-1} M^{1-1/\alpha}\end{aligned}$$

for some $C_2 > 0$. In the last step we used Chebychev's inequality and the conclusion

$$E\{\text{number of indices } n \leqq M \text{ with } S_n = 0 \mid S_0 = 0\} = O(M^{1-1/\alpha}) \tag{2.35}$$

of (2.30) and (2.31).

We know from Theorem 1 in [7], that

$$1 - \tilde{P}(0 \to 0; [-\infty, N-k]) = O(N-k)^{1-\alpha} \qquad ((N-k) \to \infty).$$

Hence, there exists a $\delta_2(\varepsilon) > 0$ such that

$$(\tilde{P}(0 \to 0; [-\infty, N-k]))^m \geqq 1 - \varepsilon/3$$

when $m \leqq \delta_2(\varepsilon) N^{\alpha-1}$ and $k \leqq \delta_2(\varepsilon) N$. Taking

$$m = [\delta_2(\varepsilon) N^{\alpha-1}] \quad \text{and then} \quad M = [\varepsilon m / 3C_2]^{\alpha/(\alpha-1)},$$

we see that

$$1 - P\{\text{there exists an } n \leqq M \text{ with } S_n > N \mid S_0 = k\} \geqq 1 - 2\varepsilon/3$$

when $k \leqq \delta_2(\varepsilon) N$. For

$$|k| \leqq \delta_1(\varepsilon) M^{1/\alpha} = \delta_3(\varepsilon) N.$$

(2.33) is also fulfilled; hence if one takes

$$\delta_4(\varepsilon) = \min(\delta_2(\varepsilon), \delta_3(\varepsilon)),$$

one has in virtue of (2.34)

$$\begin{aligned}\tilde{P}(k \to 0; [-\infty, N]) &\geqq P\{\text{there exists an } n \leqq M \text{ with } S_n = 0 \text{ and } S_i \leqq N \text{ for } i = 0, \cdots, n \mid S_0 = k\}\\ &\geqq 1 - \varepsilon\end{aligned} \tag{2.36}$$

whenever $|k| \leqq \delta_4(\varepsilon) N$ for all sufficiently large N, say $N \geqq N_0(\varepsilon)$. (2.36) will be valid for all N, if

$$|k| \leqq \delta(\varepsilon) N = \tfrac{1}{2} \min(\delta_4(\varepsilon), N_0^{-1}(\varepsilon)) N.$$

This completes the proof.

Put now for $k \geqq 0$

$$q_{\alpha,N}(kN^{-1}, c) = \tilde{P}(k \to 0; [-cN, N])$$

and define, for $kN^{-1} \leqq y \leqq (k+1)N^{-1}$, $q_{\alpha,N}(y, c)$ by linear interpolation between $q_{\alpha,N}(kN^{-1}, c)$ and $q_{\alpha,N}((k+1)N^{-1}, c)$. One has then, for fixed $c > 0$,

LEMMA 4. *If* (1.1), (1.2), *and* (2.3) *are satisfied for some* $1 < \alpha \leqq 2$, *and* $c > 0$, *then the functions* $q_{\alpha,N}(y, c)$, $N = 1, 2, \cdots$, *are equicontinuous* (*in* y) *on* $[0, 1)$, *and consequently from each sequence of integers* $\{N_i\}$ *one can select a subsequence* $\{N_{i_r}\}$ *such that*

$$\lim_{r\to\infty} q_{\alpha,N_{i_r}}(y, c) = q_\alpha(y, c)$$

exists and is a continuous function on $[0, 1)$.

Proof.

$$\begin{aligned} q_{\alpha,N}(k_1 N^{-1}, c) &= \tilde{P}(k_1 \to 0; [-cN, N]) \\ &\geqq \tilde{P}(k_1 \to k_2\,; [-cN, N])\tilde{P}(k_2 \to 0; [-cN, N]) \\ &= \tilde{P}(k_1 \to k_2\,, [-cN, N])q_{\alpha,N}(k_2 N^{-1}, c). \end{aligned}$$

It follows immediately from Lemma 3, that for each $\varepsilon > 0$ there exists a $\delta_5(\varepsilon) > 0$ such that

$$\tilde{P}(k_1 \to k_2\,; [-cN, N]) \geqq 1 - \varepsilon$$

whenever

$$0 \leqq k_1\,, k_2 \leqq N(1 - \varepsilon) \quad \text{and} \quad |k_1 - k_2| \leqq \delta_5(\varepsilon)N.$$

Hence, there exists a $\delta_6(\varepsilon) > 0$ such that

$$q_{\alpha,N}(y_1\,, c) \geqq (1 - \varepsilon)q_{\alpha,N}(y_2\,, c) \tag{2.37}$$

whenever

$$0 \leqq y_1\,, y_2 \leqq 1 - \varepsilon \quad \text{and} \quad |y_1 - y_2| = \delta_6(\varepsilon).$$

Since $0 \leqq q_{\alpha,N}(y, c) \leqq 1$, the equicontinuity follows immediately from (2.37). The remainder of the lemma is a well-known consequence of the equicontinuity.

THEOREM 2. *If* (1.1), (1.2), *and* (2.3) *are satisfied for some* $1 < \alpha < 2$, *and* $c > 0$, *then*

$$\lim_{N\to\infty} q_{\alpha,N}(y, c) = q_\alpha(y, c) \tag{2.38}$$

exists for $0 \leqq y < 1$, *and*

$$\begin{aligned} q_\alpha(y, c) = (\alpha - 1)c^{1-\alpha/2}(c+1)^{\alpha-1}(y+c)^{\alpha/2}y^{\alpha-1} \\ \cdot \int_y^1 (y + cv)^{-\alpha}(1 - v)^{\alpha/2-1}\,dv. \end{aligned} \tag{2.39}$$

Remark. By the equicontinuity of $q_{\alpha,N}(y, c)$ one has also for $0 \leqq y < 1$

$$\lim_{\substack{N\to\infty \\ y_N \to y}} q_{\alpha,N}(y_N, c) = q_\alpha(y, c).$$

Proof. Let $N_1 < N_2 < \cdots$ be any sequence of integers such that

$$\lim_{i\to\infty} q_{\alpha,N_i}(y, c) = \tilde{q}_\alpha(y, c) \tag{2.40}$$

exists and is continuous on [0, 1). Such a sequence may be selected from any infinite set of positive integers by Lemma 4. One has, for $0 < z < 1$ (cf. (2.25) for definition of $N_{\{0\}}(A, -B)$; cf. also proof of Lemma 9 in [7])

$$\begin{aligned} EN_{\{0\}}(N, -cN) = EN_{\{0\}}(zN, -cN) \\ + \textstyle\sum_{zN<k\leqq N} P\{\text{the first } S_n \text{ outside } [-cN, zN] \text{ equals } k \mid S_0 = 0\} \\ \cdot q_{\alpha,N}(kN^{-1}, c) EN_{\{0\}}(N, -cN). \end{aligned}$$

Dividing by $EN_{\{0\}}(N, -cN)$ and letting $N \to \infty$ through the sequence N_j one gets

$$\begin{aligned} 1 = \lim_{N\to\infty} EN_{\{0\}}(zN, -cN)\{EN_{\{0\}}(N, -cN)\}^{-1} \\ + \int_z^1 d_y H_\alpha((y - z)z^{-1}, cz^{-1})\tilde{q}_\alpha(y, c). \end{aligned} \tag{2.41}$$

By (2.26)

$$\lim_{N\to\infty} EN_{\{0\}}(zN, -cN)\{EN_{\{0\}}(N, -cN)\}^{-1} = z^{\alpha-1}(c + 1)^{\alpha-1}(z + c)^{1-\alpha}.$$

Substituting this and the expression for H_α into (2.41) one obtains

$$\begin{aligned} 1 = z^{\alpha-1}(c + 1)^{\alpha-1}(z + c)^{1-\alpha} \\ + \pi^{-1} \sin\frac{\pi\alpha}{2} c^{\alpha/2} z^{\alpha/2} \int_z^1 (y - z)^{-\alpha/2}(y + c)^{-\alpha/2} y^{-1}\tilde{q}_\alpha(y, c)\, dy, \end{aligned}$$

or

$$\begin{aligned} \pi\left(\sin\frac{\pi\alpha}{2}\right)^{-1} c^{-\alpha/2}[z^{-\alpha/2} - z^{\alpha/2-1}(c + 1)^{\alpha-1}(z + c)^{1-\alpha}] \\ = \int_z^1 \{\tilde{q}_\alpha(y, c)(y + c)^{-\alpha/2} y^{-1}\}(y - z)^{-\alpha/2}\, dy. \end{aligned}$$

This is again Abel's integral equation for $\tilde{q}_\alpha(y, c)(y + c)^{-\alpha/2}y^{-1}$. Solving it according to the standard formula [1], we find that the unique continuous solution is

$$\begin{aligned} &\tilde{q}_\alpha(y, c) \\ &= -(y + c)^{\alpha/2} y c^{-\alpha/2} \frac{d}{dy} \int_y^1 \{z^{-\alpha/2} - z^{\alpha/2-1}(c + 1)^{\alpha-1}(z + c)^{1-\alpha}\}\, (z - y)^{\alpha/2-1}\, dz \end{aligned}$$

(putting $u = zy^{-1}$)

$$= -(y+c)^{\alpha/2} y c^{-\alpha/2} \frac{d}{dy} \int_1^{y^{-1}} \{u^{-\alpha/2} - u^{\alpha/2-1}(c+1)^{\alpha-1}(u+cy^{-1})^{1-\alpha}\} \cdot (u-1)^{\alpha/2-1}\, du$$

$$= (\alpha-1)(c+1)^{\alpha-1} c^{1-\alpha/2} (y+c)^{\alpha/2} y^{-1} \int_1^{y^{-1}} u^{\alpha/2-1}(u+cy^{-1})^{-\alpha} \cdot (u-1)^{\alpha/2-1}\, du$$

By putting $v = u^{-1}$, the last expression goes over into (2.39), showing that $\tilde{q}_\alpha(y, c)$ is given by (2.39) independently of the sequence $\{N_i\}$. This shows that the ordinary limit of $q_{\alpha,N}(y, c)$ as $N \to \infty$ exists and is given by (2.39).

3. Applications to Toeplitz matrices

We consider again a sequence $\{c_n\}$ satisfying (1.1), (1.2). The random variables X_n and S_n are defined as in (2.1) and (2.4). (2.2) and (2.3) are assumed valid. The case where α in (2.3) equals 2 has been extensively treated in [11], so we shall restrict ourselves to $0 < \alpha < 2$.

The corresponding sequence of $(N+1) \times (N+1)$ Toeplitz matrices $(N = 0, 1, \cdots)$ is defined by

$$\| T(N)_{k,j} \| = \| c_{j-k} \|, \qquad k, j = 0, 1, \cdots, N. \tag{3.1}$$

The inverse of $I - T(N)$ is denoted by

$$H(N) = [I - T(N)]^{-1}, \tag{3.2}$$

where I stands for the $(N+1) \times (N+1)$ identity matrix. It was proved in [11] that this inverse exists, and that

$$\begin{aligned} H(N)_{k,j} &= \textstyle\sum_{n=0}^{\infty} P(k \xrightarrow{n} j;\ [0, N]) \\ &= E\{\text{number of indices } n \text{ for which } S_n = j \\ &\qquad \text{while } 0 \leqq S_i \leqq N \text{ for } i = 0, 1, \cdots, n \mid S_0 = k\}. \end{aligned} \tag{3.3}$$

The proof was based on the fact that

$$T^n(N)_{k,j} = \text{the } k, j \text{ entry of the } n^{\text{th}} \text{ power of } T(N) = P(k \xrightarrow{n} j;\ [0, N]).$$

Using these facts, it becomes quite easy to translate the results of the last section into results on Toeplitz matrices.

Following [11] we put for $j \notin [0, N]$

$$\begin{aligned} \bar{H}(N)_{k,j} &= \textstyle\sum_{r=0}^{N} H(N)_{k,r}\, c_{j-r} \\ &= P\{\text{the first } S_n \text{ outside } [0, N] \text{ equals } j \mid S_0 = k\}. \end{aligned}$$

THEOREM 3. *If* (1.1), (1.2), *and* (2.3) *are satisfied for some* $0 < \alpha < 2$, *then for* $0 < x < 1$

$$\lim_{\substack{N\to\infty \\ kN^{-1}\to x}} \sum_{-yN \leqq j < 0} \bar{H}(N)_{k,j} = G_\alpha(y(1-x)^{-1}, x(1-x)^{-1})$$

$$(3.4) \qquad = \pi^{-1} \sin\frac{\pi\alpha}{2} x^{\alpha/2}(1-x) \int_0^{y(1-x)^{-1}} z^{-\alpha/2} \cdot (z(1-x)+1)^{-\alpha/2}(z(1-x)+x)^{-1}\,dz,$$

$$(3.5) \qquad \lim_{\substack{N\to\infty \\ kN^{-1}\to x}} \sum_{j=-\infty}^{-1} \bar{H}(N)_{k,j} = \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma(\alpha/2)} \int_0^{1-x} y^{\alpha/2-1}(1-y)^{\alpha/2-1}\,dy.$$

If (1.1), (1.2), (1.3), *and* (2.3) *are satisfied for some* $1 < \alpha < 2$, *then for* $0 < x, y < 1$

$$(3.6) \qquad \lim_{\substack{N\to\infty \\ k_1N^{-1}\to x \\ k_2N^{-1}\to y}} N^{1-\alpha} H(N)_{k_1,k_2} = Q^{-1}(\Gamma(\alpha/2))^{-2}\,|x-y|^{\alpha-1} \int_0^{\frac{\min(x(1-x)^{-1},\,y(1-y)^{-1})}{\max(x(1-x)^{-1},\,y(1-y)^{-1})}} w^{\alpha/2-1}(1-w)^{-\alpha}\,dw.$$

For $x = y$ *the expression in the right-hand side of* (3.6) *has to be read as*

$$Q^{-1}(\alpha-1)^{-1}(\Gamma(\alpha/2))^{-2}x^{\alpha-1}(1-x)^{\alpha-1}.$$

Proof. (3.4) and (3.5) follow from Corollaries 1 and 2 to Theorem 1 by a change of scale. Only (3.6) requires further proof. But

$$\begin{aligned} H(N)_{k_1,k_2} &= E\{\text{number of indices } n \text{ for which } S_n = k_2 \text{ while } 0 \leqq S_i \leqq N \text{ for } i = 0, 1, \cdots, n \mid S_0 = k_1\} \\ (3.7) \qquad &= \tilde{P}(k_1 \to k_2\,;\,[0, N]) \cdot E\{\text{number of indices } n \text{ for which } S_n = k_2 \text{ while } 0 \leqq S_i \leqq N \text{ for } i = 0, 1, \cdots, n \mid S_0 = k_2\} \\ &= \tilde{P}(k_1 \to k_2\,;\,[0, N]) E N_{\{0\}}(N - k_2, -k_2). \end{aligned}$$

By (2.26)

$$(3.8) \qquad \lim_{\substack{N\to\infty \\ k_2N^{-1}\to y}} N^{1-\alpha} E N_{\{0\}}(N-k_2, -k_2) = C(\alpha, Q) y^{\alpha-1}(1-y)^{\alpha-1}$$

whereas, for $0 < y < x < 1$, by Theorem 2

$$(3.9) \qquad \lim_{\substack{N\to\infty \\ k_1N^{-1}\to x \\ k_2N^{-1}\to y}} \tilde{P}(k_1 \to k_2; [0, N]) = q_\alpha\left(\frac{x-y}{1-y}, \frac{y}{1-y}\right).$$

By combining (3.7)–(3.9) and using the expressions for q_α and $C(\alpha, Q)$, (3.6) now follows for $0 < y < x < 1$. For $0 < x < y < 1$ it follows from the fact that $H(N)$ is symmetric, being the inverse of a symmetric matrix.

For $y = x$ it is even easier, since $\tilde{P}(k_1 \to k_2\,;\,[0, N])$ tends to 1 as $N \to \infty$, and $k_1 N^{-1} \to x$, $k_2 N^{-1} \to x$, $0 < x < 1$, by Lemma 3.

Remark. For $0 < \alpha < 1$ we know

$$\lim_{N\to\infty} H(N)_{k,j} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{e^{-it(j-k)}}{1-\phi(t)}\,dt \tag{3.10}$$

since then the S_n process is not recurrent and the integral in (3.10) converges (cf. [3]). In this case

$$\lim_{N\to\infty} H(N)_{k,j} = E\{\text{number of indices } n \text{ for which } S_n = j \mid S_0 = k\},$$

which is exactly the integral in (3.10).

For $\alpha = 1$, the problem is more delicate. In [7] we proved that for $\alpha = 1$, $0 < x < 1$,

$$\lim_{\substack{N\to\infty\\ kN^{-1}\to x}} (\log N)^{-1} H(N)_{k,k} = \pi^{-1}Q^{-1}.$$

One can also show that for $x \neq y$

$$\lim_{\substack{N\to\infty\\ k_1N^{-1}\to x\\ k_2N^{-1}\to y}} (\log N)^{-1}\, H(N)_{k,j} = 0, \tag{3.11}$$

but it is not clear what will happen in (3.11) when $x = y$ or if $\log N$ is replaced by another appropriate factor. There are good reasons to believe that for $\alpha = 1$, $x \neq y$

$$\lim_{\substack{N\to\infty\\ k_1N^{-1}\to x\\ k_2N^{-1}\to y}} H(N)_{k_1,k_2}$$

exists and is given by the right-hand side of (3.33) with $\alpha = 1$.

Let us now assume $Q = 1$, i.e.,

$$\lim_{t\to 0} |\, t\,|^{-\alpha}(1 - \phi(t)) = 1. \tag{3.12}$$

It is clear that in this case

$$\lim_{N\to\infty} E \exp\,(itN^{-1/\alpha}S_N) = \exp\,(-\,|\, t\,|^{\alpha}), \tag{3.13}$$

or, in other words, $N^{-1/\alpha}S_N$ behaves like $X(1)$, where $X(s)$ is a stable process with independent increments with

$$E \exp\,(it\,X(s)) = \exp\,(-s\,|\, t\,|^{\alpha}).$$

It seems reasonable therefore, to expect that

$$-N^{\alpha}[I - T(N)] \tag{3.14}$$

will approximate the infinitesimal generator corresponding to a stable process with absorbing barriers. For $0 < \alpha < 1$ this infinitesimal generator was found by Elliott in [4]. Elliott's results were extended for all $\alpha < 2$ by Getoor [5]. We shall show in the next theorem that (3.14) indeed approaches the same operator.

THEOREM 4. *If* (1.1), (1.2), *and* (3.12) *are satisfied for some* $0 < \alpha < 2$, *and if* $f(x)$ *is twice continuously differentiable on* [0, 1], *then for* $0 < x < 1$

$$\lim_{\substack{N\to\infty\\ kN^{-1}\to x}} N^\alpha \sum_{j=0}^{N} [I - T(N)]_{k,j} f(jN^{-1}) = -\Gamma(\alpha)\pi^{-1} \sin\frac{\pi\alpha}{2} \frac{d}{dx} \text{P.V.} \int_0^1 f(y)\,|\,y - x\,|^{-\alpha} \operatorname{sgn}(y - x)\, dy. \tag{3.15}$$

$\left(\text{P.V.} \int_0^1\right.$ *stands for* $\left.\lim_{\varepsilon\downarrow 0}\left\{\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^1\right\}.\right)$

Proof.

$$\begin{aligned}\textstyle\sum_{j=0}^N [I - T(N)]_{k,j} f(j\,N^{-1}) &= f(kN^{-1}) - \textstyle\sum_{j=0}^N c_{j-k} f(j\,N^{-1})\\ &= f(kN^{-1})\left(\textstyle\sum_{j=-\infty}^{-k-1} c_j + \sum_{j=N-k+1}^{\infty} c_j\right)\\ &\quad + \textstyle\sum_{j=0}^N c_{j-k}\,(f(kN^{-1}) - f(jN^{-1})).\end{aligned}$$

Putting

$$\gamma_m = \textstyle\sum_{j=m}^{\infty} c_j$$

one obtains by partial summation (taking into account (1.1))

$$\begin{aligned}&\textstyle\sum_{j=0}^N [I - T(N)]_{k,j} f(jN^{-1})\\ &\quad = f(0)\gamma_{k+1} + f(1)\gamma_{N-k+1}\\ &\qquad + \textstyle\sum_{j=1}^{N-k} \gamma_j(f((k + j - 1)N^{-1}) - f((k + j)N^{-1}))\\ &\qquad + \textstyle\sum_{j=1}^{k} \gamma_j(f((k - j + 1)N^{-1}) - f((k - j)N^{-1}))\\ &\quad = f(0)\gamma_{k+1} + f(1)\gamma_{N-k+1} - N^{-1} \textstyle\sum_{j=1}^{N-k} \gamma_j f'((k + j)N^{-1})\\ &\qquad + N^{-1} \textstyle\sum_{j=1}^{k} \gamma_j f'((k - j)N^{-1}) + O(N^{-2} \sum_{j=1}^N \gamma_j).\end{aligned} \tag{3.16}$$

By (3.12), (3.13), and Theorem 5, p. 181 in [6]

$$\lim_{k\to\infty} k^\alpha \gamma_k = c > 0. \tag{3.17}$$

One easily derives from (3.12) that one must have

$$c = \Gamma(\alpha)\pi^{-1} \sin(\pi\alpha/2). \tag{3.18}$$

Since

$$\text{P.V.} \int_0^1 f'(y)\,|\,y - x\,|^{-\alpha} \operatorname{sgn}(y - x)\, dy$$

exists, it is easily seen from (3.16)–(3.18) that

$$\lim_{\substack{N\to\infty\\ kN^{-1}\to x}} N^\alpha \sum_{j=0}^{N} [I - T(N)]_{k,j} f(jN^{-1}) = \Gamma(\alpha)\pi^{-1} \sin\frac{\pi\alpha}{2}\left\{f(0)x^{-\alpha} + f(1)(1 - x)^{-\alpha}\right.$$

$$\left. - \text{P.V.} \int_0^1 f'(y) \,|\, y - x \,|^{-\alpha} \operatorname{sgn}(y - x)\, dy \right\}$$

$$= -\Gamma(\alpha)\pi^{-1} \sin \frac{\pi\alpha}{2} \frac{d}{dx} \text{P.V.} \int_0^1 f(y) \,|\, y - x \,|^{-\alpha} \operatorname{sgn}(y - x)\, dy.$$

In Theorem 3 the asymptotic behavior of $H(N)$ was found. We shall show how this can be used to find the inverse of the infinitesimal generator just found.

LEMMA 5. *Let* (1.1), (1.2), (1.3), *and* (3.12) *be satisfied for some* $1 < \alpha < 2$. *If*[3] $g(x) \in C[0, 1]$, *then*

$$\lim_{\substack{N\to\infty \\ kN^{-1}\to x}} N^{-\alpha} \sum_{j=0}^{N} H(N)_{k,j}\, g(jN^{-1})$$

$$= (\Gamma(\alpha/2))^{-2} \int_0^x (x - z)^{\alpha-1} g(z)\, dz \int_0^{\frac{z(1-x)}{x(1-z)}} w^{\alpha/2-1}(1 - w)^{-\alpha}\, dw$$

$$(3.19) \qquad + (\Gamma(\alpha/2))^{-2} \int_x^1 (z - x)^{\alpha-1} g(z)\, dz \int_0^{\frac{x(1-z)}{z(1-x)}} w^{\alpha/2-1}(1 - w)^{-\alpha}\, dw$$

$$= (\Gamma(\alpha/2))^{-2} \int_0^1 w^{\alpha/2-1}\, dw \left\{ \int_0^{x(1-x)(1-x+wx)^{-1}} u^{\alpha-1} g(x - u(1 - w))\, du \right.$$

$$\left. + \int_0^{x(1-x)((1-x)w+x)^{-1}} u^{\alpha-1} g(x + u(1 - w))\, du \right\}.$$

Proof. From (3.7) and (3.8) one concludes that $N^{1-\alpha}H(N)_{k,j}$ is bounded, uniformly in k, j. The first equality in (3.19) is now an immediate consequence of (3.6) as

$$N^{-\alpha}\textstyle\sum_{j=0}^{N} H(N)_{k,j}\, g(jN^{-1}) = \sum_{j=0}^{N} N^{1-\alpha}H(N)_{k,j}\, g(jN^{-1})N^{-1}.$$

The second equality in (3.19) follows after a simple transformation.

Define now for $0 < \alpha < 2$

$$Tg(x) = (\Gamma(\alpha/2))^{-2} \int_0^1 w^{\alpha/2-1}\, dw$$

$$(3.20) \qquad \cdot \left\{ \int_0^{x(1-x)(1-x+wx)^{-1}} u^{\alpha-1} g(x - u(1 - w))\, du \right.$$

$$\left. + \int_0^{x(1-x)((1-x)w+x)^{-1}} u^{\alpha-1} g(x + u(1 - w))\, du \right\}$$

whenever the integrals exist. Since for $0 < \alpha < 2$

$$\int_0^1 w^{\alpha/2-1}\, dw \int_0^1 u^{\alpha-1}\, du < \infty,$$

[3] $C[0, 1]$ is the class of functions continuous on $[0, 1]$.

it is easy to check the following properties:

(i) If $g(x) \in C[0, 1]$, then also $Tg(x) \in C[0, 1]$.

(ii) If $g(x) \in C[0, 1]$, and if $g(x)$ is twice continuously differentiable in $[0, 1]$, then $Tg(x)$ is twice continuously differentiable in $(0, 1)$.

(iii) Under the conditions of (ii) there exists a function $K(x)$, bounded in each interval $[\varepsilon, 1 - \varepsilon]$ for $\varepsilon > 0$, such that for $i = 0, 1, 2$, $0 < x_0 < 1$

$$| [(d/dx)^i Tg(x)]_{x=x_0} | \leqq K(x_0)\{\sup_{0 \leqq x \leqq 1} | g(x) | + \sup_{0 \leqq x \leqq 1} | g'(x) | + \sup_{0 \leqq x \leqq 1} | g''(x) |\}.$$

Theorems 3 and 4 suggest that a solution of

$$-\Gamma(\alpha)\pi^{-1} \sin \frac{\pi\alpha}{2} \frac{d}{dx} \text{P.V.} \int_0^1 f(y) \mid y - x \mid^{-\alpha} \operatorname{sgn} (y - x) \, dy = g(x) \tag{3.21}$$

is given by

$$f(x) = Tg(x).$$

For certain g's we shall indeed prove this in

THEOREM 5. *If $0 < \alpha < 2$, and if $g(x)$ is twice continuously differentiable in $[0, 1]$, then*

$$f(x) = Tg(x)$$

satisfies (3.21) *for* $0 < x < 1$.

Proof. Take first

$$g(x) = (1 - tx)^{-\alpha-1}, \tag{3.22}$$

where t is a complex number with $| t | < 1$. Substituting this $g(x)$ and introducing $v = u^{-1}$ in (3.20) one obtains easily

$$\begin{aligned} f(x) = Tg(x) &= (\Gamma(\alpha/2))^{-2} x^{\alpha/2}(1 - x)^{\alpha/2}\alpha^{-1}(1 - tx)^{-1}(1 - t)^{-\alpha/2} \\ &\quad \cdot \left\{ \int_0^{x(1-x)^{-1}(1-t)} y^{\alpha/2-1}(1 + y)^{-\alpha} \, dy + \int_{x(1-x)^{-1}(1-t)}^{\infty} y^{\alpha/2-1}(1 + y)^{-\alpha} \, dy \right\} \\ &= (\Gamma(\alpha + 1))^{-1} x^{\alpha/2}(1 - x)^{\alpha/2}(1 - tx)^{-1}(1 - t)^{-\alpha/2}. \end{aligned} \tag{3.23}$$

In order to check (3.21) we have to compute

$$-\text{P.V.} \int_0^1 y^{\alpha/2}(1 - y)^{\alpha/2}(1 - ty)^{-1} \mid y - x \mid^{-\alpha} \operatorname{sgn} (y - x) \, dy \qquad (0 < x < 1, | t | < 1). \tag{3.24}$$

This can be done by contour integration, exactly as in [8], pp. 23, 24.

The result for the expression in (3.24) is

$$\pi(\sin (\pi\alpha/2))^{-1} t^{-1}\{(1 - t)^{\alpha/2}(1 - tx)^{-\alpha} - 1\}, \tag{3.25}$$

from which (3.21) immediately follows for the particular function $g(x)$ of

(3.22). Since both (3.22) and (3.23) can be expanded as a power series in t, we deduce that $f(x) = Tg(x)$ indeed satisfies (3.21) whenever $g(x)$ is a polynomial in x. If now $g(x)$ is twice continuously differentiable on $[0\ ,\ 1]$, then for each $\varepsilon > 0$ we can find a polynomial $g_\varepsilon(x)$ such that

$$\sup_{0 \leqq x \leqq 1} | g(x) - g_\varepsilon(x) | + \sup_{0 \leqq x \leqq 1} | g'(x) - g'_\varepsilon(x) | + \sup_{0 \leqq x \leqq 1} | g''(x) - g''_\varepsilon(x) | \leqq \varepsilon.$$

Putting

$$f(x) = Tg(x) \quad \text{and} \quad f_\varepsilon(x) = Tg_\varepsilon(x),$$

we see from Lemma 5 that, for $i = 0, 1, 2,$

$$\left[\left(\frac{d}{dx}\right)^i (f(x) - f_\varepsilon(x))\right]_{x_0} \to 0 \qquad \text{as } \varepsilon \downarrow 0 \tag{3.26}$$

for any $x_0 \in (0, 1)$. Writing now for $2\delta \leqq x \leqq 1 - 2\delta$

$$\begin{aligned} -\text{P.V.} \int_0^1 & f(y) \, | y - x |^{-\alpha} \operatorname{sgn} (y - x) \, dy \\ &= \int_0^{x-\delta} f(y)(x - y)^{-\alpha} \, dy - \int_{x+\delta}^1 f(y)(y - x)^{-\alpha} \, dy \\ &\qquad + \int_0^\delta (f(x - y) - f(x + y))y^{-\alpha} \, dy, \end{aligned} \tag{3.27}$$

we can easily differentiate the three terms on the right, obtaining

$$\begin{aligned} -\alpha \int_0^{x-\delta} & f(y)(x - y)^{-\alpha-1} \, dy + f(x - \delta)\delta^{-\alpha} \\ & - \alpha \int_{x+\delta}^1 f(y)(y - x)^{-\alpha-1} \, dy + f(x + \delta)\delta^{-\alpha} \\ &\qquad + \int_0^\delta (f'(x - y) - f'(x + y))y^{-\alpha} \, dy. \end{aligned} \tag{3.28}$$

The same formula holds with $f_\varepsilon(x)$ replacing $f(x)$. This together with (3.26) shows that

$$-\Gamma(\alpha)\pi^{-1} \sin \frac{\pi\alpha}{2} \frac{d}{dx} \text{P.V.} \int_0^1 (f(y) - f_\varepsilon(y)) \, | y - x |^{-\alpha} \operatorname{sgn} (y - x) \, dy$$

tends to zero as $\varepsilon \downarrow 0$. Since (3.21) is valid with $f_\varepsilon(x)$ and $g_\varepsilon(x)$ instead of $f(x)$ and $g(x)$, the theorem follows.

Remark. The condition that $g(x)$ be twice continuously differentiable on $[0, 1]$ is most likely stronger than necessary. E.g., we know from [4] and the corollary below that for $0 < \alpha < 1$, $g(x) \in C[0, 1]$ already suffices. In [4] it is also proved that the solution in $C[0, 1]$ is unique. Similar uniqueness properties can be proved here.

COROLLARY. *Let $X(s)$ be a stationary stable process with independent increments such that for some $0 < \alpha < 1$*

$$Ee^{itX(s)} = e^{-s|t|^\alpha},$$

and let

$$A_\alpha(\lambda; x, y) = \int_0^\infty e^{-\lambda s}\mathcal{A}_\alpha(s, x, y)\, ds$$

where $\mathcal{A}_\alpha(s, x, y)$ is the density (at y, $0 < y < 1$) of

$$X_{abs}(s) = \begin{cases} 0 & \text{if } \inf_{0 \leqq \xi \leqq s} (x + X(\xi)) \leqq 0, \\ 1 & \text{if } \sup_{0 \leqq \xi \leqq s} (x + X(\xi)) \geqq 1, \\ x + X(s) & \text{otherwise;} \end{cases}$$

then, for $0 < \alpha < 1$ and $0 < x, y < 1$, $x \neq y$

$$\begin{aligned} A_\alpha(0; x, y) &= \int_0^\infty \mathcal{A}_\alpha(s, x, y)\, ds \\ &= (\Gamma(\alpha/2))^{-2} |x - y|^{\alpha-1} \int_0^{\frac{\min(x(1-x)^{-1}, y(1-y)^{-1})}{\max(x(1-x)^{-1}, y(1-y)^{-1})}} w^{\alpha/2-1}(1 - w)^{-\alpha}\, dw. \end{aligned} \tag{3.33}$$

Proof. It was shown in [4], that for $0 < \alpha < 1$ and $g(x) \in C[0, 1]$, the unique solution in $C[0, 1]$ of (3.21) is given by

$$f(x) = \int_0^1 A_\alpha(0; x, y)g(y)\, dy.$$

Together with (3.19) and Theorem 5 this implies (3.33). Using the expressions in [9] one can show that (3.33) remains valid, even for $1 \leqq \alpha \leqq 2$. Another form of (3.33) was round by Widom [13].

One may also consider the analogue of (3.33) for a process with one absorbing barrier only; i.e., let $\mathcal{B}(s, x, y)$ be the density of

$$\tilde{X}_{abs}(s) = \begin{cases} 1 & \text{if } \sup_{0 \leqq \xi \leqq s} (x + X(\xi)) \geqq 1, \\ x + X(s) & \text{otherwise,} \end{cases}$$

and

$$B(\lambda; x, y) = \int_0^\infty e^{-\lambda s}\mathcal{B}(s, x, y)\, ds.$$

Seeing that for $1 < \alpha < 2$, $y < x < 1$,

$$\begin{aligned} \lim_{\substack{N\to\infty \\ k_1N^{-1}\to x \\ k_2N^{-1}\to y}} & N^{1-\alpha}E\,\{\text{number of indices } n \text{ for which } S_n = k_2 \text{ while } S_i \leqq N \\ &\qquad \text{for } i = 0, 1, \cdots, n \mid S_0 = k_1\} \\ &= p_\alpha((x - y)(1 - y)^{-1}))(1 - y)^{\alpha-1}C(\alpha, 1) \\ &= (\Gamma(\alpha/2))^{-2}(x - y)^{\alpha-1} \int_{(x-y)(1-y)^{-1}}^1 w^{-\alpha}(1 - w)^{\alpha/2-1}\, dw, \end{aligned}$$

one is lead to expect

$$B_\alpha(0; x, y) = \int_0^\infty \mathcal{B}_\alpha(s, x, y)\, ds$$
$$= (\Gamma(\alpha/2))^{-2} |x - y|^{\alpha-1} \int_{\frac{|x-y|}{1-\min(x,y)}}^{1} w^{-\alpha}(1 - w)^{\alpha/2-1}\, dw$$
$$= (\Gamma(\alpha/2))^{-2} | x - y |^{\alpha-1} \int_0^{\frac{1-\max(x,y)}{1-\min(x,y)}} w^{\alpha/2-1}(1 - w)^{-\alpha}\, dw$$
$$(0 < \alpha < 1; x, y < 1; x \neq y).$$

This result was indeed proved in [9].

Finally we shall derive an analogue of Theorem 2.2 in [11]. This concerns the coefficients $p_{n,k}$ of the orthogonal polynomials $p_n(z)$, defined by

(i) For $n \geqq 0$,
$$p_n(z) = \sum_{k=0}^{n} p_{n,k}\, z^k \qquad \text{with } p_{n,n} > 0,$$

(ii) $$\frac{1}{2\pi}\int_{-\pi}^{+\pi} p_n(e^{it})\overline{p_m}(e^{it})(1 - \phi(t))\, dt = \delta_{n,m}, \qquad n, m = 0, 1, \cdots.$$

It was shown in [11], Theorem 1.1 that

$$(3.34) \quad H(N)_{k,j} = [I - T(N)]_{k,j}^{-1} = \sum_{r=\max(k,j)}^{N} p_{r,k}\, p_{r,j} \qquad (0 \leqq j, k \leqq N),$$

and in Theorem 1.5

$$(3.35) \qquad p_{N,k}^2 = H^2(N)_{N,k}(H(N)_{N,N})^{-1}.$$

We interpret this probabilistically in

LEMMA 6. *If* (1.1) *and* (1.2) *are satisfied, then*

$$(3.36) \quad p_{N,k}^2 = \tilde{P}(k \rightarrow N; [0, N])H(N)_{N,k}$$
$$= E\{\textit{number of indices } n \textit{ for which } S_n = k \textit{ while } 0 \leqq S_i \leqq N \textit{ for } i = 0, 1, \cdots, n \textit{ and for some } i_0 \leqq n,\ S_{i_0} = N \mid S_0 = k\}.$$

Proof. From (1.6) and (2.6) it is clear that

$$(3.37) \qquad H(N)_{k,j} = \tilde{P}(k \rightarrow j; [0, N])H(N)_{j,j}\,.$$

Furthermore

$$(3.38) \qquad H(N)_{N,k} = H(N)_{k,N}\,.$$

(3.35), (3.37), and (3.38) show

$$p_{N,k}^2 = H(N)_{k,N}\, H(N)_{N,k}(H(N)_{N,N})^{-1}$$
$$= \tilde{P}(k \rightarrow N; [0, N])H(N)_{N,k}\,.$$

This proves the first equality in (3.36); the second is again clear from (1.6) and (2.6).

COROLLARY 1.

$$p^2_{N,k} \leqq E\{\textit{number of indices } n \textit{ for which } S_n = 0 \quad \textit{while } \max_{0 \leqq i \leqq n} S_i = N - k \mid S_0 = 0\} = u^2_{N-k}, \tag{3.39}$$

where

$$\sum_{k=0}^{\infty} e^{-sk} u^2_k = \frac{1}{2\pi} \int_{-\pi}^{+\pi} dy \exp\left(-\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{s}{(t-y)^2 + s^2/4} \log(1 - \phi(t))\right) dt. \tag{3.40}$$

Proof. The inequality is clear from the interpretation (3.36) of $p^2_{N,k}$. The generating function (3.40) of u^2_k was proved in Lemma 2 of [7].

Lemma 6 and this corollary explain why $\lim_{N\to\infty} p_{N,N-k} = u_k$ as proved in Theorem 1.6 of [11].

COROLLARY 2. *Let* (1.1), (1.2), *and* (2.3) *be satisfied for some* $1 < \alpha \leqq 2$. *Then one can find for each* $\varepsilon > 0$ *a* $\delta_7(\varepsilon) > 0$ *such that*

$$(p^2_{N_1,k})/(p^2_{N_2,k}) \geqq 1 - \varepsilon \tag{3.41}$$

whenever

$$\varepsilon \leqq kN_2^{-1} \leqq 1 - \varepsilon \quad \textit{and} \quad N_2 \leqq N_1 \leqq N_2(1 + \delta_7(\varepsilon)).$$

Similarly, there exists a $\delta_8(\varepsilon) > 0$ *such that*

$$(u^2_{N_1})/(u^2_{N_2}) \geqq 1 - \varepsilon \tag{3.42}$$

whenever $N_2 \leqq N_1 \leqq N_2(1 + \delta_8(\varepsilon))$.

Proof. (3.36) and (3.37) show that for $N_1 \leqq N_2$

$$\frac{p^2_{N_1,k}}{p^2_{N_2,k}} = \frac{\tilde{P}(k \to N_1; [0, N_1])\tilde{P}(N_1 \to k; [0, N_1])H(N_1)_{k,k}}{\tilde{P}(k \to N_2; [0, N_2])\tilde{P}(N_2 \to k; [0, N_2])H(N_2)_{k,k}} \geqq \frac{\tilde{P}(k \to N_1; [0, N_1])\tilde{P}(N_1 \to k; [0, N_1])}{\tilde{P}(k \to N_2; [0, N_2])\tilde{P}(N_2 \to k; [0, N_2])}.$$

In addition,

$$\begin{aligned}\tilde{P}(k \to N_1; [0, N_1]) &\geqq \tilde{P}(k \to k + N_1 - N_2; [0, N_1])\tilde{P}(k + N_1 - N_2 \to N_1; [0, N_1]) \\ &\geqq \tilde{P}(k \to k + N_1 - N_2; [0, N_1])\tilde{P}(k \to N_2; [0, N_2]).\end{aligned}$$

Since

$$1 - \tilde{P}(k \to k + N_1 - N_2; [0, N_1]) \leqq 1 - \tilde{P}(k \to k + N_1 - N_2; (-\infty, N_1]) + 1 - \tilde{P}(k \to k + N_1 - N_2; [0, \infty)),$$

it follows from Lemma 3 that for

$$(N_1 - N_2)/N_1 \leqq \delta_9(\varepsilon), \quad \text{say}$$

$$\tilde{P}(k \to k + N_1 - N_2; [0, N_1]) \geqq (1 - \varepsilon)^{1/2},$$

and consequently

$$\frac{\tilde{P}(k \to N_1;\, [0, N_1])}{\tilde{P}(k \to N_2;\, [0, N_2])} \geqq (1 - \varepsilon)^{1/2}.$$

In a similar way one treats the factor

$$\frac{\tilde{P}(N_1 \to k;\, [0, N_1])}{\tilde{P}(N_2 \to k;\, [0, N_2])}.$$

This proves (3.41). (3.42) is proved in the same way taking into account that by the probabilistic interpretation of u_N (cf. (3.39))

$$u_N = \tilde{P}(0 \to N;\, (-\infty, N])\tilde{P}(N \to 0;\, (-\infty, N])EN_{\{0\}}(N, -\infty)$$

(cf. (2.25) for definition of $N_{\{0\}}(A, -B)$).

THEOREM 6. *If* (1.1), (1.2), *and* (2.3) *are satisfied for some* $1 < \alpha < 2$, *then*

$$p_{N,k} \to 0 \quad as \quad N - k \to \infty, \tag{3.43}$$

and for $0 < x < 1$,

$$\lim_{\substack{N\to\infty \\ kN^{-1}\to x}} N^{1-\alpha/2}p_{N,k} = Q^{-1/2}(\Gamma(\alpha/2))^{-1}x^{\alpha/2}(1 - x)^{\alpha/2-1}, \tag{3.44}$$

$$\lim\nolimits_{N\to\infty} N^{1-\alpha/2}u_N = Q^{-1/2}(\Gamma(\alpha/2))^{-1}. \tag{3.45}$$

Proof. By (3.39), $p_{N,k}^2 \leqq u_{N-k}^2$, so that (3.43) is a consequence of (3.45). We shall now prove (3.44). By Theorem 3

$$\begin{aligned} H(N)_{k,k} &= N^{\alpha-1}Q^{-1}(\alpha - 1)^{-1}(\Gamma(\alpha/2))^{-2}(kN^{-1})^{\alpha-1}(1 - kN^{-1})^{\alpha-1} + o(N^{\alpha-1}) \\ &= Q^{-1}(\alpha - 1)^{-1}(\Gamma(\alpha/2))^{-2}k^{\alpha-1}(1 - kN^{-1})^{\alpha-1} + o(N^{\alpha-1}) \end{aligned}$$

as $N \to \infty$, $kN^{-1} \to x$. Recalling (3.34) and (3.41) one has, for $\varepsilon < x$

$$\begin{aligned} \delta_7(\varepsilon)N\, p_{N,k}^2(1 - \varepsilon) &\leqq \textstyle\sum_{r=N+1}^{N(1+\delta_7(\varepsilon))} p_{r,k}^2 \\ &= H(N(1 + \delta_7(\varepsilon)))_{k,k} - H(N)_{k,k} \\ &= Q^{-1}(\alpha - 1)^{-1}(\Gamma(\alpha/2))^{-2}k^{\alpha-1} \\ &\quad \cdot \{(1 - kN^{-1}(1 + \delta_7(\varepsilon))^{-1})^{\alpha-1} - (1 - kN^{-1})^{\alpha-1}\} \\ &\quad + o(N^{\alpha-1}) \end{aligned}$$

or

$$N^{2-\alpha}p_{N,k}^2 \leqq (1 - \varepsilon)^{-1}Q^{-1}(\Gamma(\alpha/2))^{-2}(kN^{-1})^{\alpha}(1 - kN^{-1})^{\alpha-2} + o(1).$$

As $p_{N,N}\, p_{N,k} = H(N)_{N,k} > 0$ and $p_{N,N} > 0$, also $p_{N,k} > 0$. Consequently

$$\limsup_{\substack{N\to\infty \\ kN^{-1}\to x}} N^{1-\alpha/2}P_{N,k} \leqq Q^{-1/2}(\Gamma(\alpha/2))^{-1}x^{\alpha/2}(1 - x)^{\alpha/2-1}.$$

Similarly, considering the sum

$$\sum_{r=N(1+\delta_7(\varepsilon))^{-1}}^{N} p_{r,k}^2$$

one proves

$$\liminf_{\substack{N\to\infty \\ kN^{-1}\to x}} N^{1-\alpha/2} p_{N,k} \geqq Q^{-1/2}(\Gamma(\alpha/2))^{-1} x^{\alpha/2}(1-x)^{\alpha/2-1},$$

completing the proof of (3.44). (3.45) is proved in the same way, taking into account (3.42), Theorem 1 of [7], saying

$$\lim_{N\to\infty} N^{1-\alpha} \sum_{k=0}^{N} u_k^2 = C(\alpha, Q),$$

and the fact that $u_k = \lim_{N\to\infty} p_{N,N-k} \geqq 0$ (Theorem 1.6 in [11]).

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