MEAN-L-STABLE SYSTEMS

BY

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1. Introduction

Let X be a compact metric space with metric ρ, and let T be a homeomorphism of X onto itself. The pair (X, T) will be called a *compact system.*

In this paper we shall be concerned with compact systems which are mean-L-stable, as defined in Section 4. The definition of mean-L-stable systems is due to Fomin [3]. Mean-L-stable systems were also discussed briefly by Oxtoby in [7]. The theorems he obtained will be quoted at appropriate places in this paper.

We adopt the following notations. If E is a set, χ_E denotes its characteristic function, and E' denotes its complement (when the containing space is understood.) If E is a subset of X, its closure is denoted by $\bar{E}$.

If E is a set of integers, let

$$\delta_k(E) = (2k + 1)^{-1}\sum_{j=-k}^{k} \chi_E(j).$$

The *upper density* of E, $\delta^*(E)$, is defined by

$$\delta^*(E) = \limsup_{k\to\infty} \delta_k(E),$$

and the *lower density of* E, $\delta_*(E)$, is defined by

$$\delta_*(E) = \liminf_{k\to\infty} \delta_k(E).$$

If $\delta_*(E) = \delta^*(E)$, their common value is called the *density of* E, and is denoted by $\delta(E)$.

2. Measure theoretic preliminaries. The theory of Kryloff and Bogoliouboff

A *Borel measure* on X is a finite measure on the algebra of all Borel subsets of X. A Borel measure μ is *normalized* if $\mu(X) = 1$. An *invariant Borel measure* on (X, T) is a Borel measure μ on X such that if E is a Borel subset of X, then $\mu(E) = \mu(ET)$. It is known [7, (2.1)] that any compact system admits at least one normalized invariant Borel measure. A Borel subset E on X is said to have *invariant measure zero* (*invariant measure one*) provided $\mu(E) = 0$ $(\mu(E) = 1)$ for every normalized invariant Borel measure μ on (X, T).

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Let f be a real valued function on X, and let k be a positive integer. Let

$$M(f, k, x) = f_k(x) = (2k+1)^{-1}\sum_{i=-k}^{k} f(xT^i),$$

and let

$$M(f, x) = f^*(x) = \lim_{k\to\infty} M(f, k, x),$$

if this limit exists. If μ is a normalized invariant Borel measure on (X, T), and f is integrable with respect to μ, then f^* exists for almost all x, by the Birkhoff ergodic theorem.

We now summarize, without giving proofs, those concepts and results from the theory of Kryloff and Bogoliouboff which are needed in this paper. A concise exposition of the theory, which includes proofs, may be found in [7].

Let $C(X)$ denote the set of continuous real valued functions on X. A point $x \in X$ is called *quasi-regular* if $M(f, x)$ exists for every $f \in C(X)$. The set of quasi-regular points has invariant measure one.

For each quasi-regular point x, $M(f, x)$ is a bounded linear functional, so, by the Riesz theorem, there corresponds a unique normalized Borel measure μ_x such that

$$M(f, x) = \int f\, d\mu_x$$

for every $f \in C(x)$.

A measure is called *ergodic* if X cannot be split into two disjoint T-invariant sets, each of positive μ-measure.

A quasi-regular point is called *transitive* if μ_x is an ergodic measure, and it is called a *point of density* if $\mu_x(U) > 0$ for every open set U containing x. If a quasi-regular point is both transitive and a point of density, it is said to be *regular*.

Let Q, Q_T, Q_D, and R denote respectively the set of quasi-regular points, transitive points, points of density, and regular points. Q_D and Q_T (and therefore R) are Borel sets of invariant measure one.

If f is a bounded Borel measurable function on X, $\int f\, d\mu_x$ is a Borel measurable function of x on Q, and

$$\int f\, d\mu = \int_Q \left(\int f\, d\mu_x\right) d\mu(x),$$

for every finite invariant Borel measure μ. From this it follows that a Borel set E has invariant measure zero if and only if $\mu(E) = 0$ for every ergodic measure μ.

For any ergodic measure μ, $\mu_x = \mu$ for all x except a set of μ-measure zero. The set of all such x is called the *quasi-ergodic set* corresponding to μ. The intersection of the quasi-ergodic set with R is called the *ergodic set* corresponding to μ. The ergodic sets constitute a partition of R, and are in one-to-one correspondence with the ergodic measures. Each ergodic measure vanishes outside the corresponding ergodic set.

A system (X, T) is called *uniquely ergodic* if it has a unique normalized invariant Borel measure, or, what is the same thing, if X contains only one ergodic set. (X, T) is called *strictly ergodic* if X consists of a single ergodic set. Clearly, any strictly ergodic system is uniquely ergodic.

3. Proximal and persistently proximal pairs of points

Let (X, T) be a compact system. The points x and y of X are said to be *proximal* provided, for any $\varepsilon > 0$, there exists an integer n such that

$$\rho(xT^n, yT^n) < \varepsilon.$$

If x and y are not proximal, they are said to be *distal.* The system (X, T) is called distal if, for any pair of points x and y with $x \neq y$, x and y are distal.

The easy proof of the following lemma is omitted.

LEMMA 1. *Let (X, T) and (X^*, T^*) be compact systems. Let φ be a continuous mapping of X onto X^* such that, for $x \in X$, $\varphi(x)T^* = \varphi(xT)$. Let x and y be proximal in X. Then $\varphi(x)$ and $\varphi(y)$ are proximal in X^*.*

The points x and y of X are said to be *persistently proximal* provided, for any $\varepsilon > 0$, $\rho(xT^n, yT^n) < \varepsilon$ for $n \in E$, where E is a set of integers of density one. It is clear that "persistently proximal" is a T-invariant equivalence relation.

For x and y in X and k a positive integer, let

$$\rho_k(x, y) = (2k + 1)^{-1}\textstyle\sum_{i=-k}^{k} \rho(xT^i, yT^i),$$

and let

$$\rho'(x, y) = \limsup_{k\to\infty} \rho_k(x, y).$$

LEMMA 2. *The function ρ' is a T-invariant pseudometric on X, and $\rho'(x, y) = 0$ if and only if x and y are persistently proximal.*

Proof. For each positive integer k, ρ_k is a metric on X. It follows that ρ' is a pseudometric. That ρ' is T-invariant, that is, that

$$\rho'(xT, yT) = \rho'(x, y),$$

for x, y in X, follows easily from the definition of ρ'.

For any $\varepsilon > 0$ let $J = [i \mid \rho(xT^i, yT^i) \geqq \varepsilon]$. Then

$$\varepsilon\chi_J(i) \leqq \rho(xT^i, yT^i) \leqq \operatorname{diam}(X)\chi_J(i) + \varepsilon$$

for all i. Averaging over $[-k, k]$ and taking the lim sup as $k \to \infty$, we get $\varepsilon\delta^*(J) \leqq \rho'(x, y) \leqq \operatorname{diam}(X)\delta^*(J) + \varepsilon$. Hence $\rho'(x, y) = 0$ if and only if $\delta^*(J) = 0$ for every $\varepsilon > 0$.

Let $\tilde{X}$ be the set whose points are the equivalence classes of mutually persistently proximal points of X, and let π be the natural projection of X onto $\tilde{X}$. We define a metric $\tilde{\rho}$ for $\tilde{X}$ by $\tilde{\rho}(\pi x, \pi y) = \rho'(x, y)$, for $x, y \in X$.

Let $\tilde{T}$ be the mapping of $\tilde{X}$ onto itself defined by $(\pi x)\tilde{T} = \pi(xT)$. It follows from Lemma 2 that $\tilde{T}$ is an isometry on $\tilde{X}$.

4. Mean-L-stable systems

The compact system (X, T) is said to be *mean-L-stable* (*"stable in the mean in the sense of Liapounov"*) if for every pair of positive numbers ε_1 and ε_2, there is a positive number δ such that $x, y \in X$ with $\rho(x, y) < \delta$ implies $\rho(xT^n, yT^n) < \varepsilon_1$, for all n except in a set E with $\delta^*(E) < \varepsilon_2$. We say that δ *corresponds to* ε_1 *and* ε_2 above. If $\varepsilon_1 = \varepsilon_2 = \varepsilon$, we say that δ *corresponds to* ε.

If, for any $\varepsilon > 0$, δ can be chosen so that the set E is vacuous, (that is, if the powers of T are uniformly equicontinuous) the system (X, T) is called *uniformly-L-stable.* It is easily proved that if a compact system (X, T) is mean-L-stable (uniformly-L-stable) with respect to the metric ρ, it is mean-L-stable (uniformly-L-stable) with respect to any equivalent metric ρ_1.

The proofs of the following two theorems are immediate, using the definition of mean-L-stability and elementary properties of upper density.

THEOREM 1 (Inheritance Theorem). *Let n be an integer different from zero. Then (X, T) is mean-L-stable if and only if (X, T^n) is mean-L-stable.*

THEOREM 2. *Let $\hat{T}$ be the self homeomorphism of $X \times X$ defined by*

$$(x, y)\hat{T} = (xT, yT).$$

Then (X, T) is mean-L-stable if and only if $(X \times X, \hat{T})$ is mean-L-stable.

The following theorem is proved in [7].

THEOREM 3. *In a mean-L-stable system (X, T) every point is quasi-regular and transitive. For each f in $C(X)$, the sequence $f_n(x)$ is equi-uniformly continuous and uniformly convergent on X.*

THEOREM 4. *If (X, T) is mean-L-stable, $\lim_{k\to\infty} \rho_k(x, y) = \rho^*(x, y)$ exists for all $x, y \in X$, and therefore $\rho'(x, y) = \rho^*(x, y)$.*

Proof. By Theorem 2, $(X \times X, T)$ is mean-L-stable, and so by Theorem 3 every point $z = (x, y) \in X \times X$ is quasi-regular. Hence the above limit exists. Moreover, $\rho'(x, y)$ is continuous on $X \times X$, since by Theorem 3 the convergence is uniform.

THEOREM 5. *The following statements are equivalent:*

(i) *The system (X, T) is mean-L-stable.*

(ii) *ρ' is continuous on $X \times X$.*

(iii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(x, y) < \delta$ implies $\rho'(x, y) < \varepsilon$.*

(iv) *The projection π of X onto $\tilde{X}$ is continuous.*

Proof. That (i) implies (ii) has just been shown.

Suppose (ii) holds. Then ρ' is uniformly continuous on $X \times X$. Let $\hat{\rho}$ be the metric on $X \times X$ defined by

$$\hat{\rho}((x_1, y_1), (x_2, y_2)) = (\rho(x_1, x_2)^2 + \rho(y_1, y_2)^2)^{1/2}$$

for (x_1, y_1), (x_2, y_2) in $X \times X$. Let $\varepsilon > 0$, and let δ correspond to ε in the uniform continuity of ρ'. Let $\rho(x, y) < \delta$. Then

$$\hat{\rho}((x, x), (x, y)) = \rho(x, y) < \delta.$$

Hence $| \rho'(x, x) - \rho'(x, y)| < \varepsilon$. Since $\rho'(x, x) = 0$, we have $\rho'(x, y) < \varepsilon$, which proves (iii).

Now, suppose (iii) is true. We prove (i). Let $\varepsilon > 0$, and choose $\delta > 0$ so that $\rho(x, y) < \delta$ implies $\rho'(x, y) < \varepsilon^2$. We show that $\rho(xT^n, yT^n) < \varepsilon$, except for a set of upper density less than ε. For suppose

$$\rho(xT^n, yT^n) \geqq \varepsilon,$$

for n in a set F of upper density $\geqq \varepsilon$. Then

$$\rho'(x, y) \geqq \limsup_{n\to\infty} (2n+1)^{-1} \textstyle\sum_{j=-n}^{n} \chi_F(j)\rho(xT^j, yT^j)$$
$$\geqq \limsup_{n\to\infty} (2n+1)^{-1} \textstyle\sum_{j=-n}^{n} \varepsilon\chi_F(j) \geqq \varepsilon^2,$$

contradicting $\rho'(x, y) < \varepsilon^2$.

Finally, recall that for $x, y \in X$, $\tilde{\rho}(\pi x, \pi y) = \rho'(x, y)$. It follows that (iii) and (iv) are equivalent.

THEOREM 6. *Let (X, T) be mean-L-stable. Then $\tilde{X}$ is compact, and the system $(\tilde{X}, \tilde{T})$ is uniformly-L-stable and distal.*

Proof. By Theorem 5, π is continuous, so $\tilde{X}$ is compact. Since $\tilde{T}$ is an isometry on $\tilde{X}$, $(\tilde{X}, \tilde{T})$ is uniformly-L-stable and therefore distal.

COROLLARY 1. *Let (X, T) be mean-L-stable. Then the points x and y of X are proximal if and only if they are persistently proximal.*

Proof. If x and y are persistently proximal, they are obviously proximal. Suppose x and y are proximal. Then, by Lemma 1, πx and πy are proximal in $\tilde{X}$. Since $(\tilde{X}, \tilde{T})$ is distal, $\pi x = \pi y$. That is, x and y are persistently proximal.

COROLLARY 2. *If (X, T) is mean-L-stable and distal, it is uniformly-L-stable. A mean-L-stable system is not uniformly-L-stable if and only if there exists a pair of distinct points which are persistently proximal.*

Proof. Since (X, T) is distal, the mapping π is one-to-one. By Theorem 5, π is continuous, and is therefore a homeomorphism. Since, for $x \in X$, $(\pi x)\tilde{T} = \pi(xT)$, we may identify (X, T) and $(\tilde{X}, \tilde{T})$. Therefore (X, T) is uniformly-L-stable.

Let P be the subset of $X \times X$ consisting of (x, y) such that x and y are proximal. Let $P^* = [(x, y) \in P \mid x \neq y]$, $P(x) = [y \in X \mid (x, y) \in P]$, and $P^*(x) = [y \in X \mid (x, y) \in P^*]$. It is clear that P and P^* are $\hat{T}$-invariant sets, that $P(x)T^n = P(xT^n)$, and that $P^*(x)T^n = P^*(xT^n)$.

THEOREM 7. *Let (X, T) be mean-L-stable. Then*
(i) *P is closed in $X \times X$.*
(ii) *If E is closed in X, then $\bigcup_{x \in E} P(x)$ is closed in X.*
(iii) *$P(x)$ is closed in X.*

Proof. (i) By Lemma 2 and Corollary 1, $P = [(x, y) \mid \rho'(x, y) = 0]$. Since ρ' is continuous, P is closed in $X \times X$.

(ii) Let π_2 denote the second projection of $X \times X$ onto X; that is, if $(x, y) \in X \times X$, then $\pi_2(x, y) = y$. Then $\bigcup_{x \in E} P(x) = \pi_2((E \times X) \cap P)$. Since E and P are closed, $(E \times X) \cap P$ is closed. It follows that $\bigcup_{x \in E} P(x)$ is closed.

(iii) Apply (ii) with $E = \{x\}$.

If $x \in X$, the orbit of x, denoted by $O(x)$, is defined by

$$O(x) = [xT^n \mid -\infty < n < \infty].$$

The orbit closure of x is defined to be the set $\bar{O}(x)$.

A nonempty subset M of X is called a *minimal set* or a *minimal orbit closure* if M is the orbit closure of each of its points. A compact system always contains at least one minimal set [5, 2.22].

THEOREM 8. *Let (X, T) be a mean-L-stable system. Then*
(i) *If (X, T) has at least one dense orbit, it is uniquely ergodic.*
(ii) *If (X, T) is minimal, it is strictly ergodic.*
(iii) *Every ergodic set and every quasi-ergodic set is closed.*
(iv) *The family of minimal sets and the family of ergodic sets coincide.*

For the proof, see [7, (6.2), (6.3), (6.4), and (6.5)].

COROLLARY 3. *Let (X, T) be mean-L-stable, and let A denote the set of almost periodic points of X. Then $A = Q_D = R$.*

Proof. If $x \in Q_D$, then $x \in R$, since in a mean-L-stable system all points are transitive. Since $R \subset Q_D$ for any compact system, this proves that $Q_D = R$. If $x \in A$, $\bar{O}(x)$ is minimal. Hence $\bar{O}(x)$ is an ergodic set, and $x \in R$. Finally, if $x \in R$, then x is contained in an ergodic set M. Therefore M is a minimal set, so $\bar{O}(x) = M$, and $x \in A$.

The following lemma, which is a corollary to the ergodic theorem, is proved in [7].

LEMMA 3. *Let (X, μ) be a measure space such that $\mu(X) = 1$. Let T be a one-to-one measure-preserving transformation of X onto itself. Let f be a nonnegative function defined on X which is integrable with respect to μ. Then, for almost all $x \in X$, $f^*(x) > 0$ or $f(x) = 0$.*

THEOREM 9. *Let (X, T) be mean-L-stable. Let ν be a $\hat{T}$-invariant measure on $X \times X$. Then $\nu(P^*) = 0$.*

Proof. By Lemma 2 and Theorem 4, $(x, y) \in P$ implies that $\rho^*(x, y) = 0$. It follows from Lemma 3 that $\rho(x, y) = 0$ for almost all $(x, y) \in P$. That is, the set of $(x, y) \in P$ with $x \neq y$ has measure zero. But this set is precisely P^*.

THEOREM 10. *Let (X, T) be mean-L-stable, and let $x \in X$. Let μ be an invariant Borel measure on X. Then*

(i) *For all x except a set of μ-measure 0, $\mu(P^*(x)) = 0$.*

(ii) *If, for some $x_0 \in X$, $\mu(P^*(x_0)) > 0$, there exists $y_0 \in P^*(x_0)$ such that $\mu(\{y_0\}) = \mu(P^*(x_0))$.*

Proof. (i) Let $\nu = \mu \times \mu$, the product measure of μ with itself on $X \times X$. Then ν is a $\hat{T}$-invariant measure on $X \times X$. Hence

$$\nu(P^*) = \int_X \mu(P^*(x))\, d\mu(x).$$

But $\nu(P^*) = 0$, by Theorem 9, so $\mu(P^*(x)) = 0$ for all x except a set of μ-measure zero.

(ii) Since $\mu(P^*(x_0)) = \alpha > 0$, $P^*(x_0)$ is nonvacuous. Moreover, since $\mu(P^*(x)) = 0$ for almost all $x \in X$, there exists $y_0 \in P^*(x_0)$ such that

$$\mu(P^*(y_0)) = 0.$$

Let $F = P^*(x_0) - \{y_0\}$. Since y_0 and x_0 are proximal, $F \subset P^*(y_0)$, and therefore $\mu(F) = 0$. Now $P^*(x_0) = F \cup \{y_0\}$, and

$$\alpha = \mu(P^*(x_0)) = \mu(\{y_0\}).$$

COROLLARY 4. *Let (X, T) be mean-L-stable, and let μ be an invariant Borel measure on X. Then*

(i) *If $\mu(\{x\}) = 0$ for every $x \in X$, then $\mu(P(x)) = 0$ for every $x \in X$.*

(ii) *If every point of X has an infinite orbit, then $\mu(P(x)) = 0$ for every $x \in X$.*

Proof. (i) If $\mu(P(x)) = \alpha > 0$ for some $x \in X$, then there exists $y \in P(x)$ such that $\mu(\{y\}) = \alpha$, by Theorem 10 (ii).

(ii) Since $\mu(X) = 1$, $\mu(\{x\}) = 0$ for every $x \in X$. By (i), $\mu(P(x)) = 0$ for every $x \in X$.

5. Recursive properties of mean-L-stable systems

We now define several recursive concepts which we shall discuss in connection with mean-L-stability. These notions have been extensively studied, in the more general setting of transformation groups, by Gottschalk and Hedlund in [5].

The system (X, T) is called *almost periodic* provided that for any $\varepsilon > 0$, there exists a relatively dense set A of integers such that $\rho(x, xT^n) < \varepsilon$ for every $x \in X$ and every $n \in A$.

The system (X, T) is called *weakly almost periodic* provided that for every $\varepsilon > 0$ there is an integer N such that $x \in X$ implies the existence of a relatively dense set A of integers with maximum gap at most N such that $\rho(x, xT^n) < \varepsilon$ for every $n \in A$.

The system (X, T) is said to be *locally almost periodic* at $x \in X$, and x is called a *locally almost periodic point*, provided that for every $\varepsilon > 0$ there exist $\delta > 0$ and a relatively dense set A of integers such that $\rho(x, y) < \delta$ implies $\rho(x, yT^n) < \varepsilon$ for every $n \in A$.

The system (X, T) is called locally almost periodic if it is locally almost periodic at every $x \in X$.

(X, T) is said to be *almost periodic* at $x \in X$, and x is called an *almost periodic point* provided that for every $\varepsilon > 0$ there exists a relatively dense set A of integers such that $\rho(x, xT^n) < \varepsilon$ for every $n \in A$.

If every $x \in X$ is almost periodic, then (X, T) is said to be *pointwise almost periodic.*

THEOREM 11. (i) *The system (X, T) is almost periodic if and only if it is uniformly-L-stable.*

(ii) *(X, T) is weakly almost periodic if and only if the class of orbit closures constitutes a star closed decomposition of X.*

(iii) *A point $x \in X$ is almost periodic if and only if $\bar{O}(x)$ is a minimal set.*

(iv) *If x is a locally almost periodic point, the system $(\bar{O}(x), T)$ is locally almost periodic.*

For the proofs, see [5, 4.38], [5, 4.24], [5, 4.05 and 4.07], and [5, 4.31], respectively.

LEMMA 4. *Let (X, T) be mean-L-stable and pointwise almost periodic. Let $\{x_j\}$ and $\{y_j\}$ $(j = 1, 2, \cdots)$ be sequences in X such that $\bar{O}(x_j) = \bar{O}(y_j)$. Suppose $x_j \to x$ and $y_j \to y$, as $j \to \infty$. Then $\bar{O}(x) = \bar{O}(y)$.*

Proof. If the conclusion were not true, $\bar{O}(x) \cap \bar{O}(y) = \phi$ and

$$\rho(\bar{O}(x), \bar{O}(y)) = \varepsilon > 0.$$

Therefore it is sufficient to show that for any $\varepsilon > 0$ there exist integers m and n such that $\rho(xT^m, yT^n) < \varepsilon$.

Let δ correspond to $\frac{1}{4}\varepsilon$ $(< \frac{1}{4})$ in the definition of mean-L-stable. Choose j so that $\rho(x, x_j) < \delta$ and $\rho(y, y_j) < \delta$. Choose k so that $\rho(x_j, y_j T^k) < \delta$.

There exist sets of integers E_1, E_2, and E_3, each of upper density less than $\frac{1}{4}\varepsilon$, such that

$$\rho(xT^n, x_j T^n) < \tfrac{1}{4}\varepsilon \quad \text{for} \quad n \in E_1', \qquad \rho(yT^n, y_j T^n) < \tfrac{1}{4}\varepsilon \quad \text{for} \quad n \in E_2',$$

and

$$\rho(x_j T^n, y_j T^{k+n}) < \tfrac{1}{4}\varepsilon \quad \text{for} \quad n \in E_3'.$$

Let $E_4 = [n \,|(k + n) \in E_2)$. Clearly $\delta^*(E_4) = \delta^*(E_2) < \frac{1}{4}\varepsilon$.

Now, choose $n \in E_1' \cap E_3' \cap E_4'$. Then

$$\rho(xT^n, yT^{k+n}) \leqq \rho(xT^n, x_j T^n) + \rho(x_j T^n, y_j T^{k+n}) + \rho(y_j T^{k+n}, yT^{k+n}) < \tfrac{3}{4}\varepsilon < \varepsilon.$$

THEOREM 12. *If (X, T) is pointwise almost periodic and mean-L-stable, then (X, T) is weakly almost periodic.*

Proof. By Theorem 11 (ii) it is sufficient to show that the class of minimal sets of X constitutes a star closed decomposition of X. Let $\{M_\alpha\}$ denote the class of minimal sets of X. Since X is pointwise almost periodic, $\{M_\alpha\}$ constitutes a decomposition of X. Let R be closed in X, and let $R^* = \bigcup_{M_\alpha \cap R \neq \phi} M_\alpha$. We must show that R^* is closed.

Let $x_j \in R^*$, and suppose $x_j \to x$, as $j \to \infty$. To show that $x \in R^*$, it is sufficient to show $\bar{O}(x) \cap R$ is nonvacuous. Since $x_j \in R^*$, $\bar{O}(x_j) \cap R \neq \phi$. Let $y_j \in \bar{O}(x_j) \cap R$. Since every orbit closure in X is minimal,

$$\bar{O}(x_j) = \bar{O}(y_j).$$

Let $y_j \to y$, as $j \to \infty$. Since R is closed, $y \in R$. Therefore $\bar{O}(y) \cap R \neq \phi$. By Lemma 4, $\bar{O}(x) = \bar{O}(y)$, and $\bar{O}(x) \cap R \neq \phi$.

If (X, T) is minimal and mean-L-stable, it is clear that $(\tilde{X}, \tilde{T})$ is minimal. The next theorem is in the converse direction.

THEOREM 13. *Let (X, T) be mean-L-stable, and suppose $(\tilde{X}, \tilde{T})$ is minimal. Then*

(i) *There exists precisely one minimal set M in X.*

(ii) *If $y \in X$, there exists $y' \in P(y) \cap M$; that is, $\pi M = \tilde{X}$.*

(iii) *If $x \in X$, $M \subset \bar{O}(x)$.*

(iv) *The set M has invariant measure one.*

(v) *The system (X, T) is uniquely ergodic, with ergodic set M and quasi-ergodic set X.*

Proof. (i) X contains at least one minimal set M. Suppose M_1 and M_2 were distinct minimal sets contained in X. Let $x_1 \in M_1$ and $x_2 \in M_2$. Then x_1 and x_2 are distal, since M_1 and M_2 are disjoint closed invariant sets. Hence πM_1 and πM_2 are disjoint minimal sets in $\tilde{X}$. But this contradicts the assumed minimality of $\tilde{X}$.

(ii) Let $x \in M$. Then $\pi M = \pi \bar{O}(x) = \bar{O}(\pi x) = \tilde{X}$, since $\tilde{X}$ is minimal.

(iii) Since $\bar{O}(x)$ is a closed invariant set, it contains a minimal set. But M is the only minimal set contained in X, so $M \subset \bar{O}(x)$.

(iv) Since M is the only minimal set contained in X, M is the only ergodic set, by Theorem 8 (iv), so $M = R$, and hence M has invariant measure one.

(v) M is the only ergodic set contained in X, so there exists only one ergodic measure μ on X; that is, (X, T) is uniquely ergodic. Since every point of X is quasi-regular and transitive, μ_x exists for all $x \in X$, and μ_x is an ergodic measure. But (X, T) is uniquely ergodic, so $\mu_x = \mu$, for all $x \in X$. That is, X is the quasi-ergodic set corresponding to μ.

THEOREM 14 (Decomposition Theorem). *Let (X, T) be mean-L-stable. Then*

(o) $X = \bigcup_\alpha N_\alpha$, *where the N_α are disjoint, each N_α is closed invariant, and $\{N_\alpha\}$ is a star closed decomposition of X.*

(i) *Each N_α contains precisely one minimal set M_α.*

(ii) *If $x \in N_\alpha$, then $P(x) \subset N_\alpha$, and $P(x) \cap M_\alpha \neq \phi$.*

(iii) *If $x \in N_\alpha$, $M_\alpha \subset \bar{O}(x)$.*

(iv) $X - \bigcup_\alpha M_\alpha = \bigcup_\alpha (N_\alpha - M_\alpha)$ *has invariant measure zero; that is, measure is concentrated entirely on the minimal sets.*

(v) *If μ_α is the ergodic measure corresponding to M_α, then N_α is the quasi-ergodic set corresponding to μ_α.*

Proof. (o) $(\tilde{X}, \tilde{T})$ is uniformly-L-stable, and therefore is almost periodic. Hence if $\{\tilde{M}_\alpha\}$ denotes the class of minimal sets in $\tilde{X}$, $\{\tilde{M}_\alpha\}$ is a star closed decomposition of $\tilde{X}$. Let $N_\alpha = \pi^{-1}\tilde{M}_\alpha$. It follows immediately that N_α is closed invariant, and that $\{N_\alpha\}$ is a star closed decomposition of X.

Parts (i), (ii), (iii), and (v) are immediate consequences of the corresponding parts of Theorem 13.

To prove (iv), note that $\bigcup_\alpha M_\alpha = R$, by Theorem 8 (iv).

THEOREM 15. *Let (X, T) be mean-L-stable, and let $x \in X$ be distal to all other points of X. Then $\bar{O}(x)$ is minimal and locally almost periodic.*

Proof. Let M be the unique minimal set contained in $\bar{O}(x)$. By Theorem 14 there exists $x' \in M$ such that $x' \in P(x)$. But $P(x) = \{x\}$, so $x \in M$, and $\bar{O}(x)$ is minimal.

To show that $\bar{O}(x)$ is locally almost periodic, it is sufficient, by Theorem 11 (iv), to show that x is a locally almost periodic point. Let $\varepsilon > 0$, let $\tilde{x} = \pi x$, and let $\tilde{\rho}$ be a metric for $\tilde{X}$. Since $P(x) = \{x\}$, the mapping π is open at x. That is, there exists $\eta > 0$ such that $\tilde{\rho}(\tilde{x}, \tilde{y}) < \eta$ implies $\rho(x, y) < \varepsilon$ for all $y \in \pi^{-1}\tilde{y}$.

Now $(\tilde{X}, \tilde{T})$ is an almost periodic system, so in particular $\tilde{x}$ is a locally almost periodic point. Thus there exist $\delta > 0$ and a relatively dense set A of integers such that $\tilde{\rho}(\tilde{x}, \tilde{y}) < \delta$ implies $\tilde{\rho}(\tilde{x}, \tilde{y}\tilde{T}^n) < \eta$, for $n \in A$. Now, by the continuity of π, there exists $\delta' > 0$ such that $\rho(x, y) < \delta'$ implies $\tilde{\rho}(\pi x, \pi y) < \delta$. For such y, $\tilde{\rho}(\tilde{x}, (\pi y)\tilde{T}^n) < \eta$ for $n \in A$. Hence

$$\rho(x, yT^n) < \varepsilon$$

for $n \in A$, and x is a locally almost periodic point.

COROLLARY 5. *Let (X, T) be mean-L-stable and minimal. If there exists a point of X which is distal to all other points of X, then (X, T) is locally almost periodic.*

It is not known whether a minimal mean-L-stable system always contains a point distal to all other points, or whether mean-L-stability implies local almost periodicity. It is known (cf. [7]) that a minimal locally almost periodic system is not necessarily mean-L-stable.

Let (X, T) be a compact system, and let $x \in X$. A set $M \subset X$ is called a *center of attraction of* x [1] if, for every neighborhood U of M, $xT^n \in U$, for $n \in E$, a set of integers of density one. The set M is called a *minimal center of attraction of* x if it is a closed center of attraction of x and contains no proper subset with the same property. It is proved in [1] that for each $x \in X$ there exists a unique minimal center of attraction of x, and this center of attraction is a T-invariant set.

Now suppose (X, T) is mean-L-stable, and let $x \in X$. Let M be the unique minimal set contained in $\bar{O}(x)$, and let U be a neighborhood of M. Let $\rho(M, X - U) = \varepsilon > 0$. By Theorem 14 there exists $x' \in M$ such that $x' \in P(x)$. Then for $n \in E$, a set of density one, $\rho(xT^n, x'T^n) < \varepsilon$. That is, $xT^n \in U$, for $n \in E$. Thus we have proved

THEOREM 16. *If (X, T) is mean-L-stable and $x \in X$, the minimal center of attraction of x is the unique minimal set contained in $\bar{O}(x)$.*

6. Examples

If (X, T) is mean-L-stable, Theorems 1 and 2 provide methods for constructing new mean-L-stable systems. Another method is as follows. Let I denote the closed unit interval. We extend (X, T) to a system (Y, U) where Y is a subset of $X \times I$. Let x' be an arbitrary point of X, and let $y = (x', 1) \in Y$. Define $yU^n = (x'T^n, 1/2^{|n|})$, and $(x, 0)U^n = (xT^n, 0)$, for $x \in X$. The space Y thus consists of X and an additional orbit approaching X asymptotically. The system (Y, U) is mean-L-stable and is not uniformly-L-stable, even if (X, T) is uniformly-L-stable. The quotient system is $(\tilde{X}, \tilde{T})$.

Less trivial examples of mean-L-stable systems are furnished by the Sturmian minimal sets, studied by Hedlund in [6]. Let X denote the bisequence space based on two symbols. The space X is a self-dense zero-dimensional compact metrizable space which is homeomorphic to the Cantor discontinuum. Let T denote the shift transformation of X onto itself.

The Sturmian minimal sets $M(\beta)$ (where β is a positive irrational number) are compact T-invariant subsets of X. The systems $(M(\beta), T)$ are mean-L-stable and not uniformly-L-stable. Each system $(M(\beta), T)$ contains a pair of doubly asymptotic points; that is, points x and y such that

$$\rho(xT^n, yT^n) \to 0$$

as $n \to \pm\infty$. The quotient space $\tilde{M}(\beta)$ is a 1-sphere and the induced homeomorphism $\tilde{T}: \tilde{M}(\beta) \to \tilde{M}(\beta)$ is a rotation through the angle $2\pi\beta$.

In [2] Floyd gives an example of a minimal set which is of dimension zero

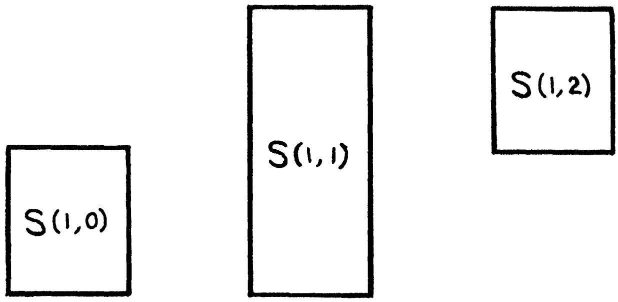

Figure 1

at some points and of dimension one at the others. This example was studied further by Gottschalk in [4], where it was shown that it is locally almost periodic, not uniformly-L-stable, and does not possess a pair of asymptotic points.

We now construct a mean-L-stable system (X, T) which is a modification of the Floyd example. The space X is a subset of $I \times I$ and will be defined as the intersection of a decreasing sequence of closed sets X_n. Each set X_n will consist of the disjoint union of 3^n closed rectangles, $S(n, 0), S(n, 1), \cdots, S(n, 3^n - 1)$.

To define X_1 we omit from I the two "middle fifths" namely the open intervals $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$. Over the remaining closed intervals $[0, \frac{1}{5}]$, $[\frac{2}{5}, \frac{3}{5}]$, and $[\frac{4}{5}, 1]$, we construct rectangles $S(1, 0)$, $S(1, 1)$ and $S(1, 2)$ of height $\frac{1}{2}$, 1, and $\frac{1}{2}$ respectively, as in Figure 1.

More precisely, define

$$S(1, 0) = [(x, t) \mid 0 \leqq x \leqq \tfrac{1}{5}, 0 \leqq t \leqq \tfrac{1}{2}],$$

$$S(1, 1) = [(x, t) \mid \tfrac{2}{5} \leqq x \leqq \tfrac{3}{5}, 0 \leqq t \leqq 1],$$

$$S(1, 2) = [(x, t) \mid \tfrac{4}{5} \leqq x \leqq 1, \tfrac{1}{2} \leqq t \leqq 1].$$

Let $X_1 = \bigcup_{j=0}^{2} S(1, j)$.

To obtain X_2, we perform the same operation on $S(1, 0)$, $S(1, 1)$, and $S(1, 2)$ as we did on $I \times I$. The situation is as indicated in Figure 2, where the rectangles $S(2, 0), \cdots, S(2, 8)$ are labelled $0, \cdots, 8$. Let

$$X_2 = \bigcup_{j=0}^{8} S(2, j).$$

Clearly $S(2, j) \subset S(1, j \pmod 3)$, $j = 0, 1, \cdots, 8$. Therefore $X_2 \subset X_1$.

Continue this process to obtain $X_3, X_4, \cdots$. We have $X_n \supset X_{n+1}$, $n = 1, 2, \cdots$; indeed $S(n + 1, j) \subset S(n, j \pmod{3^n})$.

Let $X = \bigcap_{n=1}^{\infty} X_n$. X is compact, since all the X_n are compact. X consists of vertical line segments, some of which are degenerate.

To define $T: X \to X$ we proceed as follows. For each positive integer n, let $G_n = [S(n, j) \mid 0 \leqq j \leqq 3^n - 1]$, and let T_n be the mapping of G_n onto

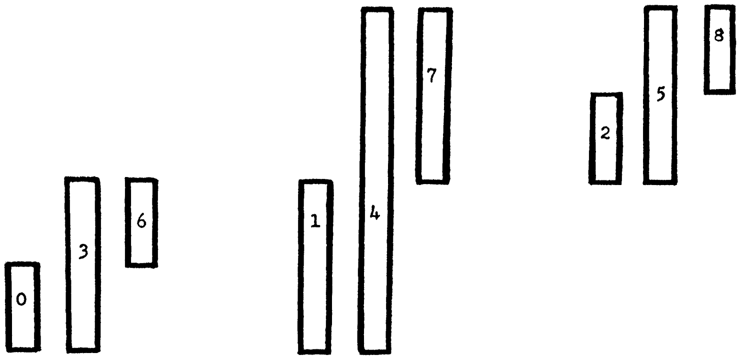

Figure 2

itself defined by $S(n, j)T_n = S(n, (j + 1) \pmod{3^n})$. Note that if

$$S(n, j) \subset S(m, k),$$

then $S(n, j)T_n \subset S(m, k)T_m$, and that $S(n, j)T_n^{3^n} = S(n, j)$.

Let $A = \bigcap_{n=1}^{\infty} S(n, j_n)$, where $0 \leqq j_n \leqq 3^n - 1$, and let

$$B = \bigcap_{n=1}^{\infty} S(n, j_n)T_n .$$

It is clear that $A \neq \phi$ if and only if $B \neq \phi$. If $A \neq \phi$, it consists of either a single point or a vertical line segment. Moreover, since the height of the rectangle $S(n, j_n)T_n$ is either half or twice the height of $S(n, j_n)$, B is a point or a nondegenerate line segment according as A is a point or a nondegenerate segment.

Now, if A consists of a single point x, we define xT to be the single point contained in B. If A consists of a nondegenerate segment, we define T on A so that A is mapped linearly onto B.

We show that (X, T) is minimal and mean-L-stable. To show minimality, let z_1 and z_2 be points of X, and let $\varepsilon > 0$. Choose n sufficiently large so that the width of the rectangles $S(n, j)$ is less than $\frac{1}{2}\varepsilon$. Let $z_2 = (x_2, t_2) \in S(n, j')$. If $z = (x, t) \in S(n, j')$, then $| x - x_2 | < \frac{1}{2}\varepsilon$. Now there exist $m \geqq n$ and j, $0 \leqq j \leqq 3^m - 1$, such that $S(m, j) \subset S(n, j')$, and if $z = (x, y) \in S(m, j)$, then $| t - t_2 | < \frac{1}{2}\varepsilon$. Hence for such a z, $\rho(z, z_2) < \varepsilon$.

Now $z_1 \in S(m, k)$, for some k. Hence there exists r such that $z_1 T^r \in S(m, j)$. Therefore $\rho(z_2, z_1 T^r) < \varepsilon$. Since ε is arbitrary, this proves $z_2 \in \bar{O}(z_1)$, and consequently (X, T) is minimal.

To see that (X, T) is mean-L-stable, let $\varepsilon > 0$, and let

$$E_n = [k \mid \operatorname{diam} S(n, k \pmod{3^n}) \geqq \varepsilon], \qquad n = 1, 2, \cdots .$$

It is easy to see that n may be chosen so large that $\delta^*(E_n) < \varepsilon$. Choose $\delta > 0$ so that if $\rho(z_1, z_2) < \delta$, then z_1 and z_2 are in the same $S(n, j)$. In this case, $\rho(z_1 T^k, z_2 T^k) < \varepsilon$, for $k \in E'_n$, which proves that (X, T) is mean-L-stable.

All points of a given nondegenerate segment are mutually proximal. The quotient space $\tilde{X}$ is homeomorphic to the Cantor discontinuum.

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