LINEAR SPACES WITH A COMPACT GROUP OF OPERATORS[1]

BY

K. deLeeuw

Introduction

This paper is devoted to the establishment of several results of a rather general nature concerning a class of locally convex topological linear spaces that have compact abelian groups of operators. The class includes many of the spaces that occur in analysis as function spaces on compact groups or as completions of such spaces.

Section 1 is devoted to some necessary preliminary results concerning Fourier analysis in the spaces in question. The results are hardly original but do not seem to be in the literature in the form needed here. In Section 2 the formal topology is introduced. For the spaces that occur in analysis this topology agrees with that of formal (i.e., termwise) convergence of Fourier or power series. Our main result is that in the spaces under consideration a convex, invariant subset, closed in the original topology, remains closed in the much weaker formal topology. Several applications are given. The next section is devoted to a result establishing the equivalence of several types of continuity for linear transformations that commute with the group operations. In Section 4 we associate to each space of the type under consideration its G-dual (which will in general be smaller than its ordinary dual space) and prove that the association is reflexive, that is, that a space is canonically isomorphic to the G-dual of its G-dual.

We have restricted ourselves to compact abelian groups for the sake of simplicity and in order to avoid as many computations as possible. All of our results, with the appropriate modifications, remain valid for arbitrary compact groups.

In the following we assume a knowledge of the rudiments of the theory of locally convex topological linear spaces as found for example in [1]. Also needed are some of the simplest facts concerning vector valued integration of continuous functions (contained for example on pp. 79–89 of [2]); the fact of which we make most crucial use is that the integral of a continuous function with respect to a positive measure of mass 1 lies in the convex closure of its range.

I wish to express my indebtedness to H. Mirkil. Most of the ideas in this paper grew out of conversation with him.

1. Fourier analysis in G-spaces

Let G be a compact abelian group and A a complete locally convex topological linear space over the complex numbers. Assume that there is associated

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to each σ in G a continuous linear transformation T_σ on A so that $T_{\sigma\tau} = T_\sigma T_\tau$ for all σ and τ in G and the identity transformation is associated to the unit of G. A subset X of A is called *G-invariant* if $T_\sigma(X) = X$ for all σ in G, and a function f with domain A will be called *G-invariant* if $f(T_\sigma(x)) = f(x)$ for all x in A and σ in G. A neighborhood U of the zero element of A will be called *admissible* if it is G-invariant, convex, closed and circular (i.e., αx is in U if $|\alpha| = 1$ and x is in U).

A will be called a G-space if it satisfies the two conditions:

A1. *There is a fundamental system of neighborhoods of the zero element all of which are admissible* (*or equivalently, every neighborhood of zero contains an admissible neighborhood*).

A2. *For each x in A, the map $\sigma \to T_\sigma(x)$ of G into A is continuous.*

If A is a Banach space, it is clear that A1 is equivalent to the existence of a G-invariant norm for A. The existence of such a norm is known (see [3], p. 7) to be a consequence of A2 in this case. In the general case it is possible to prove that A1 and A2 together are equivalent to: A3. *The map $G \times A \to A$ defined by $(\sigma, x) \to T_\sigma(x)$ is jointly continuous.* Since this equivalence is not needed in what follows we shall not prove it.

Several more definitions are needed before we are able to state another condition A4 which together with A1 is equivalent to A2 plus A1. For each character χ of G let A_χ be the closed linear subspace

$$\{x : T_\sigma(x) = \chi(\sigma)x, \text{ all } \sigma \text{ in } G\}$$

of A. Let A^0 be the subspace of A consisting of all finite sums $\sum x_\chi$ with x_χ in A_χ. It is easy to check that the A_χ are linearly independent so that A^0 is actually the vector space direct sum of the subspaces A_χ.

Our fourth condition is: A4. *A^0 is dense in A.*

The equivalence of A1 plus A2 with A1 plus A4 will be established in Corollary 1.3. For the G-spaces that occur in analysis, the truth of A1 is usually obvious, while the verification of A2 or A4 may be nontrivial.

For every complex valued continuous function ϕ on G, define the linear operator $T_\phi : A \to A$ by the vector valued integral

$$T_\phi(x) = \int_G T_\sigma(x)\, \overline{\phi(\sigma)}\, d\sigma, \tag{1.1}$$

where $d\sigma$ is normalized Haar measure on G. If V is any admissible neighborhood of zero in A and λ is a real number with $|\phi(\sigma)| \leqq \lambda$ for all σ in G, the integrand of (1.1) will be in λV if x is in V. Thus if x is in V, $T_\phi(x)$ will be in the convex closure of λV which is λV itself, so that $T_\phi(V) \subset \lambda V$ and T_ϕ must be continuous.

If χ is a character of G, it is easy to check that

$$T_\sigma(T_\chi(x)) = \chi(\sigma) T_\chi(x)$$

for all x in A and all σ in G, so $T_\chi(V)$ is a subset of A_χ. If x is an element of A_χ, (1.1) can be computed and $T_\chi(x) = x$. Thus we see that T_χ is a continuous projection of A onto A_χ.

An *approximate identity* on G is a directed set $\{\phi_\gamma\}$ of functions on G that satisfies the following conditions:

1. $\phi_\gamma \geqq 0$ for each γ.
2. $\int_G \phi_\gamma(\sigma)\, d\sigma = 1$ for each γ.
3. If U is any neighborhood of the unit of G, ϕ_γ converges to zero uniformly on the complement of U.
4. Each ϕ_γ is a finite linear combination of characters of G.

It is well known (see [3], p. 13) that approximate identities do exist.

THEOREM 1.1. *If A is a G-space and $\{\phi_\gamma\}$ an approximate identity on G,*

$$\operatorname*{Lim}_{\gamma} T_{\phi_\gamma}(x) = x$$

for all x in A.

Proof. Let V be any admissible neighborhood of zero in A. Since the map $\sigma \rightarrow T_\sigma(x)$ is continuous, there is a neighborhood W of the unit of G which is such that $T_\sigma(x) - x$ is in V if σ is in W. Then

$$T_{\phi_\gamma}(x) - x = \int_G (T_\sigma(x) - x)\phi_\gamma(\sigma)\, d\sigma,$$

which is

$$\int_W (T_\sigma(x) - x)\phi_\gamma(\sigma)\, d\sigma + \int_{G-W} (T_\sigma(x) - x)\phi_\gamma(\sigma)\, d\sigma. \tag{1.2}$$

If σ is in W, $T_\sigma(x) - x$ is in V and so the first integral of (1.2) is in the convex closure of V which is V itself. Since ϕ_γ converges uniformly to zero on $G - W$, the second integral of (1.2) is in V if γ is large enough. Thus for sufficiently large γ, $T_{\phi_\gamma}(x) - x$ is in $2V$, and since V was an arbitrary admissible neighborhood, $T_{\phi_\gamma}(x)$ converges to x.

COROLLARY 1.2. *If A is a G-space and x is in A, $T_\chi(x) = 0$ for all characters χ implies $x = 0$.*

Proof. If $\{\phi_\gamma\}$ is an approximate identity, $T_{\phi_\gamma}(x)$ will be a finite linear combination of the $T_\chi(x)$. Since $T_{\phi_\gamma}(x)$ converges to x, it must be zero if all of the $T_\chi(x)$ are zero.

COROLLARY 1.3. A1 *plus* A2 *is equivalent to* A1 *plus* A4.

Proof. If A satisfies A1 plus A2, it is a G-space, and Theorem 1.1 can be applied. $T_{\phi_\gamma}(x)$ will be, if ϕ_γ is an element of an approximate identity, a finite linear combination of $T_\chi(x)$ and so is in A^0. Thus by Theorem 1.1, A^0 is dense, so A4 is satisfied. For the converse assume that A satisfies A1 and A4. The map $\sigma \rightarrow T_\sigma(x)$ is continuous for x in any A_χ since $T_\sigma(x) = \chi(\sigma)x$.

Thus this map will also be continuous for x in A^0. If y is any point in A and $\{x_\gamma\}$ a directed subset of A^0 that converges to y, the map $\sigma \to T_\sigma(y)$ is the uniform limit of the continuous maps $\sigma \to T_\sigma(x_\gamma)$ and thus is itself continuous. Therefore A must satisfy A4.

2. Convex sets in the formal topology

Let A be a G-space. Denote by $\hat{A}$ the set of all functions having as domain the character group $\hat{G}$ of G and as range A, and which besides satisfy $f(\chi) \in A_\chi$ for all χ in $\hat{G}$. $\hat{A}$ is a complete locally convex topological linear space under the topology of pointwise convergence on $\hat{G}$ (i.e., f_γ converges to f if $f_\gamma(\chi)$ converges to $f(\chi)$ in the topology of A for all χ in $\hat{G}$).

To each element x in A associate the function $\hat{x}$ in $\hat{A}$ defined by $\hat{x}(\chi) = T_\chi(x)$. The map $J:A \to \hat{A}$ defined by $J(x) = \hat{x}$ is a linear transformation that is continuous because the T_χ are continuous, and that is one-one because of Corollary 1.2. Thus A can be identified with the subspace $J(A)$ of $\hat{A}$. $J(A)$ inherits a topology as a subspace of $\hat{A}$, and this topology transferred to the isomorphic space A will be called the *formal topology* of A (in the G-spaces that occur in analysis, it is the topology of formal convergence of Fourier series or power series; see Lemma 2.4 and Corollary 2.5).

The formal topology can also be described as the weakest topology on A which agrees with the original topology on the A_χ and which is such that the projections $T_\chi:A \to A_\chi$ are continuous. Note that if x_γ is a directed subset of A, x_γ converges to x in the formal topology if and only if $T_\chi(x_\gamma)$ converges to $T_\chi(x)$ in the original topology for all χ. If the A_χ are finite-dimensional, as they are in the spaces that occur in analysis, the formal topology is weaker than the ordinary weak topology of A. Now that A has a second topology, its original topology will be called the strong topology.

It is a consequence of the theory of locally convex spaces that a closed convex subset K of A remains closed if the topology of A is weakened to the weak topology. Our main result is that if G-invariance is added the topology can be weakened to the formal topology and K remains closed.

THEOREM 2.1. *Let A be a G-space and K a closed, convex, G-invariant subset of A. Then K remains closed if the topology of A is weakened to the formal topology.*

Proof. Let x be any element in the closure of K in the formal topology. Then there is a directed subset $\{x_\delta\}$ of K with x_δ converging to x in the formal topology. Let $\{\phi_\gamma\}$ be an approximate identity on G. Since each T_{ϕ_γ} is a finite linear combination of the T_χ, $T_{\phi_\gamma}(A)$ is contained in the subspace spanned by a finite number of the A_χ, and on such a subspace the formal and strong topologies agree. Also since T_{ϕ_γ} is a finite linear combination of the T_χ, $T_{\phi_\gamma}(x_\delta)$ converges to $T_{\phi_\gamma}(x)$ in the formal topology, and thus in the strong topology since they agree on $T_{\phi_\gamma}(A)$. Recall that $T_{\phi_\gamma}(x_\delta)$ is defined by the vector valued integral

$$\int_G T_\sigma(x_\delta)\phi_\gamma(\sigma)\,d\sigma\,.$$

Since x_δ is in K which is G-invariant, $T_\sigma(x_\delta)$ will be in K for all σ in G. Thus $T_{\phi_\gamma}(x_\delta)$ will be in the convex closure of K which is K itself. Again using the fact that K is strongly closed, we have, since $T_{\phi_\gamma}(x_\delta)$ converges to $T_{\phi_\gamma}(x)$ in the strong topology, that $T_{\phi_\gamma}(x)$ is in K for each ϕ_γ in the approximate identity. Now Theorem 1.1 shows that $T_{\phi_\gamma}(x)$ converges to x in the strong topology, so x itself must be in K. This proves K to be closed in the formal topology.

Although it should be clear that the above could not hold without the G-invariance of K, it is instructive to look at an example. Let A be all continuous functions on G with $[T_\sigma(f)](\tau) = f(\sigma\tau)$. Let σ be some fixed point in G and K all functions in A that vanish at σ. Then K is closed and convex but becomes unclosed if the topology of A is weakened only as far as L_1 convergence.

If A is a vector space over the complex numbers, a semi-norm p on A is a nonnegative real valued function on A that satisfies $p(x + y) \leqq p(x) + p(y)$ and $p(\lambda x) = |\lambda| p(x)$. If A is a topological vector space, p is called lower semicontinuous if $p(x) \leqq \liminf p(x_\gamma)$ for any x in A and directed subset $\{x_\gamma\}$ of A converging to x. Lim inf for a directed set of numbers is defined as for sequences by

$$\liminf t_\alpha = \lim_\beta\,(\inf_{\alpha>\beta} t_\alpha)\,.$$

It is well known, and easy to check, that p is lower semicontinuous if and only if its unit sphere $S_p = \{x : p(x) \leqq 1\}$ is closed.

COROLLARY 2.2. *Let A be a G-space and p a G-invariant semi-norm that is lower semicontinuous in the strong topology of A. Then p remains lower semicontinuous if the topology is weakened to the formal topology.*

Proof. S_p is a G-invariant convex set that is strongly closed since p is lower semicontinuous in the strong topology. By Theorem 2.1, S_p is closed on the formal topology, so p is lower semicontinuous in the formal topology.

As a special case of Theorem 2.1 and Corollary 2.2, if A is a G-space which is a Banach space and $\|\cdot\|$ a G-invariant norm for A, the unit sphere is closed and $\|\cdot\|$ is lower semicontinuous in the formal topology.

Before giving several examples of concrete applications we shall show that in a class of G-spaces that occur in analysis, the formal topology is identical with that of termwise convergence of Fourier series. A G-space A will be called a *G-space of functions* if A is a subspace of $L_1(G)$ with topology stronger than that of $L_1(G)$ and G acts on A by translation, $[T_\sigma(f)](\tau) = f(\sigma\tau)$.

As examples of G-spaces of functions one has $C(G)$ and $L_p(G)$ for $1 \leqq p < \infty$.

In addition, if G is the circle group $\{z : |z| = 1\}$, one has the space of infinitely differentiable functions, all functions in $L_p(G)$, $1 \leqq p < \infty$, that are boundary values of functions analytic in the unit disk, and the space of all functions that can be extended to be entire in the plane.

If A is a G-space of functions, A_χ is the subspace of A generated by χ if χ is in A and is the zero element of A otherwise. For if f is an element of $L_1(G)$ that satisfies $T_\sigma(f) = \chi(\sigma)f$ for all σ in G, f must be a multiple of the character χ; and conversely, if f is a multiple of χ, $T_\sigma(f) = \chi(\sigma)f$ for all σ in G. Thus the subspace A^0 of A, defined earlier as the direct sum of the A_χ, is seen to be the subspace of linear combinations of characters, and so because of Corollary 1.3 linear combinations of characters are dense in A.

If A is a G-space of functions, $T_\chi(f)$ being in A_χ will be χ multiplied by some constant. We proceed to identify this constant.

LEMMA 2.3. *If A is a G-space of functions,* $T_\chi(f) = \alpha_\chi(f)\chi$ for each f in A *and each character χ of G, where*

$$\alpha_\chi(f) = \int_G f(\sigma)\chi^{-1}(\sigma)\, d\sigma$$

is the Fourier coefficient of f with respect to χ.

Proof. $T_\chi(f)$ is the A valued integral

$$\int_G T_\sigma(f)\chi^{-1}(\sigma)\, d\sigma\,. \tag{2.1}$$

Since A is a subspace of $L_1(G)$ and the topology of $L_1(G)$ is weaker than that of A, (2.1) can be considered to be an $L_1(G)$ valued integral. (2.1) as an $L_1(G)$ valued integral is well known to be the convolution of f with χ which is $\alpha_\chi(f)\chi$.

If A is a G-space of functions, f_γ converging to f in the formal topology, which is equivalent to $T_\chi(f_\gamma)$ converging to $T_\chi(f)$ in the strong topology for all χ, means the same as $\alpha_\chi(f_\gamma)$ converging to $\alpha_\chi(f)$ for each χ because of the preceding lemma. But $\alpha_\chi(f_\gamma)$ converging to $\alpha_\chi(f)$ for all χ is equivalent to the Fourier series of f_γ converging termwise to the Fourier series of f. Thus we have proven

LEMMA 2.4. *If A is a G-space of functions, the formal topology is identical with that of termwise convergence of Fourier series.*

We shall illustrate applications of the preceding to two concrete examples.

COROLLARY 2.5. *Let A be the space of entire functions. Let D be a disk in the plane having the origin as center, and for every f in A let*

$$p(f) = \operatorname*{Sup}_{z \in D} |f(z)|\,.$$

If f_γ is a directed subset of A with the power series of f_γ converging termwise to the power series of some function f in A, then $p(f) \leqq \liminf p(f_\gamma)$.

Proof. Let G be the circle group $\{w : |w| = 1\}$ in the plane. It is easy to check that A is a G-space if it is supplied with the topology of uniform convergence on compact subsets of the plane and the operation of G is defined by $[T_w(f)](z) = f(wz)$. p is a G-invariant semi-norm continuous in the topology of A, so the corollary will follow from Corollary 2.2 when it is established that the formal topology on A is identical with that of termwise convergence of power series. For each f in A, denote by $\bar{f}$ its restriction to the circle G. Let $\bar{A}$ be the space of all functions on G thus obtained. $\bar{A}$ is, if supplied with the topology of the naturally isomorphic space A, a G-space of functions, and so by Lemma 2.4 the formal topology and that of termwise convergence of Fourier series are identical. But the natural mapping $f \to \bar{f}$ preserves the formal topologies, and since the Fourier series of $\bar{f}$ is essentially the same as the power series of f, our result is established.

COROLLARY 2.6. *Let A be the space of all functions on the circle group G having m continuous derivatives ($m = 1, 2, \cdots, \infty$). Let λ_i ($i = 0, 1, \cdots, m$) be positive real numbers and K the subset of A consisting of those f with $\| f^{(i)} \| \leqq \lambda_i$, where $f^{(i)}$ is the i^{th} derivative of f and*

$$\| h \| = \sup_{\sigma \in G} | h(\sigma)|$$

for any continuous function on G. Then K is closed in the topology of termwise convergence of Fourier series.

Proof. It is known that A is a G-space of functions if supplied with the topology determined by the semi-norms p_i ($i = 0, \cdots, m$) defined by $p_i(f) = \| f^{(i)} \|$. K is a closed convex G-invariant set, so the result follows from Theorem 2.1 and Lemma 2.4.

Note that as a consequence K must be closed in all topologies stronger than that of termwise convergence of Fourier series, for example those of uniform convergence, L_p convergence, or convergence in the sense of distributions.

3. Continuity of linear transformations

In this section we establish a result concerning the equivalence of several types of continuity for linear transformations between two G-spaces that commute with the operation of G. The closed graph theorem is needed so it will be necessary to assume that the G-spaces involved are metrizable (and thus F-spaces in the sense of [1]). It will furthermore be necessary to assume that the G-spaces are of *finite type*, that is, that the subspaces A_χ are all finite-dimensional. This includes the case of G-spaces of functions defined in the previous section.

If A is a G-space, A_s will denote A supplied with its strong topology, and A_f will be A supplied with the formal topology.

Our result is as follows:

THEOREM 3.1. *Let A and B be two metrizable G-spaces of finite type and $L:A \to B$ a linear transformation that commutes with the action of G (i.e., $LT_\sigma = T_\sigma L$ for all σ in G). Then the following are equivalent:*

1. *$L:A_s \to B_s$ is continuous.*
2. *$L:A_s \to B_f$ is continuous.*
3. *$L:A_f \to B_f$ is continuous.*
4. *$LT_\chi = T_\chi L$ for all χ in $\hat{G}$.*

Proof. 1 implies 4: If x is in A, $LT_\chi(x) = L(\int T_\sigma(x)\chi^{-1}(\sigma)d\sigma) = \int L(T_\sigma(x))\chi^{-1}(\sigma)d\sigma = \int T\sigma(L(x))\chi^{-1}(\sigma)d\sigma = T_\chi L(x)$, by using the fact that a continuous linear transformation commutes with integration. 4 implies 3: It suffices to show that if $\{x_\gamma\}$ is a directed subset of A_f that converges to zero, $L(x_\gamma)$ converges to zero in B_f. If x_γ converges to 0 in A_f, $T_\chi(x_\gamma)$ converges to 0 in A_f for all χ in $\hat{G}$. Since $T_\chi(A) = A_\chi$ is finite-dimensional, L restricted to $T_\chi(A)$ is automatically continuous, and so $L(T_\chi(x_\gamma))$ converges to 0 in B_f for all χ in $\hat{G}$. But since $LT_\chi = T_\chi L$, $T_\chi(L(x_\gamma))$ converges to 0 in B_f for all χ in $\hat{G}$, which is equivalent to $L(x_\gamma)$ converging to 0 in B_f. 3 implies 2: The proof is trivial. 2 implies 4: Since $L:A_s \to B_f$ is continuous and $T_\chi:B_f \to B_f$ is also, the composite $T_\chi L:A_s \to B_f$ is continuous. Similarly $LT_\chi:A_s \to B_f$ is continuous. If χ_0 is a character of G, $L(x)$ is in B_{χ_0} if x is in A_{χ_0} since $T_\sigma L = LT_\sigma$ for all σ in G. Thus if x is in A_{χ_0} and $\chi \neq \chi_0$, the formula

$$LT_\chi(x) = T_\chi L(x) \tag{3.1}$$

is valid since both sides are zero. If $\chi = \chi_0$, (3.1) is valid since both sides will be $L(x)$. Since both sides of (3.1) are linear in x, it will be valid for any x in A^0. Finally it is true for all x in A, since A^0 is dense in A_s and we have shown that $T_\chi L$ and LT_χ are continuous on A_s. 4 implies 1: Since A_s and B_s are metrizable and complete, to prove $L:A_s \to B_s$ continuous it suffices by the closed graph theorem (see [1], p. 37) to prove that the graph of L as a subset of $A_s \times B_s$ is closed. Let $\{(x_\gamma, L(x_\gamma))\}$ be a directed subset of the graph that converges to a point (x, y) in $A_s \times B_s$. To prove the graph closed it suffices to show that $y = L(x)$. Since $L(x_\gamma)$ converges to y in B_s, $LT_\chi(x_\gamma) = T_\chi L(x_\gamma)$ converges to $T_\chi(y)$ in B_s. Also since x_γ converges to x in A_s, $T_\chi(x_\gamma)$ converges to $T_\chi(x)$ in A_s. L is continuous on the finite-dimensional subspace $T_\chi(A_s)$, so $LT_\chi(x_\gamma)$ converges to $LT_\chi(x)$ in B_s. We have shown that the directed set $\{LT_\chi(x_\gamma)\}$ has as limit both $T_\chi L(x) = LT_\chi(x)$ and $T_\chi(y)$. Thus $T_\chi L(x) = T_\chi(y)$ for all characters χ, so by Corollary 1.2, $y = L(x)$, and the proof is complete.

4. Duality

In this section some results concerning duality for G-spaces are obtained. For simplicity it will be assumed that the spaces under consideration are Banach spaces.

If A is a G-space that is a Banach space, G-invariant norms for A exist, and we shall take one fixed one $\|\cdot\|$ as part of the structure of A. Let A^* be the dual space of A, $\langle\cdot,\cdot\rangle$ the pairing between them. For each σ in G, we shall denote by T_σ^* the adjoint $(T_\sigma)^*$ of T_σ, and for each complex valued continuous function ϕ on G, we shall denote by T_ϕ^* the adjoint $(T_\phi)^*$ of T_ϕ.

G acts as a group of operators on A^* since $T_{\sigma\tau}^* = T_\sigma^* T_\tau^*$, and if A^* is supplied with the norm $\|\cdot\|^*$ dual to $\|\cdot\|$, defined by

$$\| F \|^* = \operatorname*{Sup}_{\|x\| \leqq 1} | \langle F, x\rangle |,$$

the T_σ^* are isometries. Nevertheless A^* may fail to be a G-space since there may be F in A^* for which the mapping $\sigma \to T_\sigma^*(F)$ is discontinuous.

There is however a relatively large closed G-invariant subspace $A^\#$ of A^* that will be a G-space under the action of the T_σ^*. We simply take $A^\#$ to be the subspace of A^* consisting of those F in A^* for which the map $\sigma \to T_\sigma^*(F)$ is continuous. $A^\#$ is G-invariant and so will be a G-space if it is closed in A^*. But $A^\#$ must be closed, for if $\{F_i\}$ is a sequence in $A^\#$ converging to a point F in A^*, the map $\sigma \to T_\sigma^*(F)$ is the uniform limit of the continuous maps $\sigma \to T_\sigma^*(F_i)$ and thus is itself continuous, so F is in $A^\#$. $A^\#$ supplied with the norm $\|\cdot\|^*$ and the action of the T_σ^* will be called the *G-dual* of the G-space A.

It is possible to characterize the subspace $A^\#$ of A^* in a different manner. Denote by A_χ^* the closed subspace of A^* that consists of all F that satisfy $T_\sigma^*(F) = \chi(\sigma)F$ for all σ in G. It is easy to check that the adjoint T_χ^* of the projection T_χ is a projection of A^* onto the subspace A_χ^*.

LEMMA 4.1. *$A^\#$ is the closed linear subspace of A^* that is generated by the A_χ^*. $A^\#$ is dense in A^* in the weak* topology.*

Proof. If F is in A_χ^*, $T_\sigma^*(F) = \chi(\sigma)F$ so the map $\sigma \to T_\sigma^*(F)$ is continuous and F is in $A^\#$. Thus A_χ^* is actually the same as $A_\chi^\#$ and the first assertion of the lemma follows from the fact that in a G-space $A^\#$, the subspace $(A^\#)^0$ spanned by the $A_\chi^\#$ must be dense. If $\{\phi_\gamma\}$ is an approximate identity for G, $T_{\phi_\gamma}^*(A^*) \subset A^\#$ for each ϕ_γ since $T_{\phi_\gamma}^*$ is a finite linear combination of the T_χ^*. Because of Theorem 1.1, $T_{\phi_\gamma}(x)$ converges to x for each x in A and thus $T_{\phi_\gamma}^*(F)$ converges weak* to F for each F in A^*. Therefore each F in A^* is a weak* limit point of $A^\#$ and $A^\#$ is weak* dense.

Since $A^\#$ is a G-space, it will have a G-dual $(A^\#)^\#$ of its own. The main result of this section is the fact that if A is of finite type, A and $(A^\#)^\#$ are canonically isomorphic. (For example if A is $L_1(G)$, A^* is $L_\infty(G)$, $A^\#$ is $C(G)$, $(A^\#)^*$ is the space of measures on G, and $(A^\#)^\#$ is $L_1(G)$.)

The canonical isomorphism is set up as follows: For each x in A, the map $F \to \langle F, x\rangle$ of $A^\#$ into the complex numbers is a continuous linear functional. It will be denoted by ϕ_x. The map $I:A \to (A^\#)^*$ defined by $I(x) = \phi_x$ is clearly linear.

THEOREM 4.2. *If A is of finite type, A and $(A^\#)^\#$ are canonically isomorphic. In particular the map I defined above is an isomorphism, an isometry, commutes with the operation of G, and $I(A) = (A^\#)^\#$.*

Proof. I is one-one. For if $I(x) = \phi_x = 0$, $\langle F, x\rangle = 0$ for all F in $A^\#$, and x must be zero since $A^\#$ is weak* dense in A^*. Thus since I is linear, it is an isomorphism. It is known that in any Banach space the canonical mapping from A to $(A^*)^*$ is an isometry. Equivalently,

$$\| x \| = \operatorname*{Sup}_{\|F\|^* \leqq 1, F \in A^*} | \langle F, x\rangle |. \tag{4.1}$$

I will be proven to be an isometry when we have shown that

$$\| x \| = \operatorname*{Sup}_{\|F\|^* \leqq 1, F \in A^\#} | \langle F, x\rangle |. \tag{4.2}$$

Let $\{\phi_\gamma\}$ be an approximate identity on G. For any F in A^*,

$$\lim_\gamma \langle T^*_{\phi_\gamma}(F), x\rangle = \lim_\gamma \langle F, T_{\phi_\gamma}(x)\rangle = \langle F, x\rangle. \tag{4.3}$$

If x is in the unit sphere of A, $T_\sigma(x)$ is also for all σ in G, and so

$$T_{\phi_\gamma}(x) = \int_G T_\sigma(x)\phi_\gamma(\sigma)\, d\sigma$$

will be in the convex closure of the sphere which is the sphere itself. As a consequence, each $T^*_{\phi_\gamma}$ takes the unit sphere of A^* into itself. Each $T^*_{\phi_\gamma}$, being a finite linear combination of the T^*_χ, takes A^* into $A^\#$, and thus by the preceding comment, will take the unit sphere of A^* into the unit sphere of $A^\#$. Therefore (4.2) is a consequence of (4.1) and (4.3), and I is an isometry. Since it is clear from the definition of I that it commutes with the operation of G, it remains only to prove that $I(A) = (A^\#)^\#$. For the first time we shall use the fact that A is of finite type. To prove $I(A) = (A^\#)^\#$, it suffices to prove that $I(A_\chi) = (A^\#)^*_\chi$. For by Lemma 4.1, $(A^\#)^\#$ is the closed subspace of $(A^\#)^*$ generated by the $(A^\#)^*_\chi$, the closed subspace of A generated by the A_χ is A itself, and I is an isometry. $I(A)$ is dense in $(A^\#)^*$ in the weak* topology that $(A^\#)^*$ receives as the dual of $A^\#$ because of the well known fact that the canonical image of A is dense in $(A^*)^*$ in the weak* topology that $(A^*)^*$ has as the dual of A^*. Now take any θ in $(A^\#)^*_\chi$ and directed subset $\{x_\gamma\}$ of A so that $I(x_\gamma)$ converges to θ in the weak* topology. Let $T^{\#*}_\chi:(A^\#)^* \to (A^\#)^*$ be the adjoint of $T^\#_\chi:A^\# \to A^\#$ which is the restriction to $A^{\#}$ of the adjoint T_χ^* of T_χ. $IT_\chi = T_\chi^{\#*}I$, $T_\chi^{\#*}(\theta) = \theta$, and $T_\chi^{\#*}$ is continuous in the weak* topology of $(A^{\#})^*$. Thus $IT_\chi(x_\gamma) = T_\chi^{\#*}I(x_\gamma)$ converges to $T_\chi^{\#*}(\theta) = \theta$ in the weak* topology, and so $I(A_\chi)$ is weak* dense in $(A^{\#})_\chi^*$. But A_χ is finite-dimensional since A is of finite type, so $I(A_\chi)$ must be equal to $(A^{\#})_\chi^*$ which was all that was needed to conclude that $I(A) = (A^{\#})^{\#}$.

COROLLARY 4.3. *If A is of finite type, $A^{\#} = A^*$, and $(A^{\#})^{\#} = (A^{\#})^*$, then A must be reflexive.*

Proof. The map $I:A \to (A^{\#})^{\#}$ is simply the canonical map of A into $(A^*)^*$ in this case, and A is reflexive when this map is onto.

The converse is easily established.

THEOREM 4.4. *Let A be reflexive. Then $A^{\#} = A^*$ and $(A^{\#})^{\#} = (A^{\#})^*$.*

Proof. We prove only $A^{\#} = A^*$. Then the second assertion will follow since the dual A^* is reflexive if A is. By Lemma 4.1, $A^{\#}$ is dense in A^* in the weak* topology. But since A is reflexive, the weak and weak* topologies agree on A^*, so $A^{\#}$ is weakly dense in A^*. It is known that a weakly dense subspace of a Banach space must be strongly dense, and since $A^{\#}$ is closed in the strong topology, $A^{\#} = A^*$.

In the light of Corollary 4.3 and Theorem 4.4 the question arises whether $A^{\#} = A^*$ and finite type are sufficient to insure the reflexivity of A. It is interesting that the answer is in the negative. For an example take A to be the space of all distributions on the circle group G having Fourier coefficients that are 0 at infinity with G acting by translation. Then A^* can be identified with the space of functions on G having absolutely convergent Fourier series and $A^{\#} = A^*$, but there is no reflexivity.

Most of the results of this section can be obtained without added difficulty if A is no longer a Banach space. Let us point out however that to obtain a result analogous to Theorem 4.2 it seems necessary to make some further assumptions concerning A, for example A being a t-space in the sense of [1]. Also in the hypothesis of Theorem 4.2, finite type can be weakened to the A_χ being reflexive, the proof becoming slightly more complicated.

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THE UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN
STANFORD UNIVERSITY
STANFORD, CALIFORNIA