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CHARACTERIZATIONS OF ESTIMABILITY IN THE GENERAL LINEAR MODEL[1]

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In the general linear model $\mathcal{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$, the vector $\mathbf{A}\boldsymbol{\beta}$ is *estimable* whenever there is a matrix $\mathbf{B}$ so that $\mathcal{E}(\mathbf{By}) = \mathbf{A}\boldsymbol{\beta}$. Several characterizations of estimability are presented along with short easy proofs. The characterizations involve rank equalities, generalized inverses, Schur complements and partitioned matrices.

1. Introduction.

Consider the general linear model

$$\mathcal{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}, \tag{1}$$

where $\mathbf{X}$ is a given $n \times p$ matrix, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unrestricted unknown parameters, and $\mathbf{y}$ is an $n \times 1$ vector of observable random variables. We wish to estimate the $s \times 1$ vector

$$\mathbf{A}\boldsymbol{\beta}, \tag{2}$$

where $\mathbf{A}$ is a given $s \times p$ matrix. When there exists an $s \times n$ matrix $\mathbf{B}$ so that

$$\mathcal{E}(\mathbf{By}) = \mathbf{A}\boldsymbol{\beta}, \tag{3}$$

then $\mathbf{A}\boldsymbol{\beta}$ is said to be *estimable* (Bose, 1944; Scheffé, 1959, page 13). It follows at once, using (1), that (3) holds if and only if

$$\mathbf{A} = \mathbf{BX}, \tag{4}$$

or equivalently,

$$r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}\end{pmatrix} = r(\mathbf{X}), \tag{5}$$

where $r(\cdot)$ denotes rank.

Roy and Roy (1959) showed that the null hypothesis $\mathbf{A}\boldsymbol{\beta} = \mathbf{0}$ is *completely testable* whenever

$$r(\mathbf{XT}) = r(\mathbf{X}) - r(\mathbf{A}), \tag{6}$$

where the matrix $\mathbf{T}$ spans the null space of $\mathbf{A}$. Milliken (1971) showed that (4) and (6) are equivalent; see also Alalouf (1975, page 50) and Baksalary and Kala (1976).

Other characterizations of (4) have been obtained. If $\mathbf{X}^-$ is any generalized inverse or *g-inverse* of $\mathbf{X}$ satisfying

$$\mathbf{XX}^-\mathbf{X} = \mathbf{X}, \tag{7}$$

then (4) holds if and only if

$$\mathbf{AX}^-\mathbf{X} = \mathbf{A}. \tag{8}$$

This was proved by Searle (1965) with $\mathbf{X}^- = (\mathbf{X}'\mathbf{X})^-\mathbf{X}'$, which is a g-inverse also

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satisfying

$$\mathbf{X}^{-}\mathbf{X}\mathbf{X}^{-} = \mathbf{X}^{-} \tag{9}$$

and

$$(\mathbf{X}\mathbf{X}^{-})' = \mathbf{X}\mathbf{X}^{-}. \tag{10}$$

Following Rao and Mitra (1971, page 16) we say that $\mathbf{X}^{-}$ is a *reflexive* g-inverse if both (7) and (9) are satisfied, and then set $\mathbf{X}^{-} = \mathbf{X}_r^{-}$, and say that $\mathbf{X}^{-}$ is a *least squares* g-inverse if both (7) and (10) are satisfied and then set $\mathbf{X}^{-} = \mathbf{X}_l^{-}$. It is easy to see that a g-inverse $\mathbf{X}^{-}$ is reflexive if and only if $r(\mathbf{X}^{-}) = r(\mathbf{X})$. Moreover, $\hat{\boldsymbol{\beta}} = \mathbf{K}\mathbf{y}$ is a solution of the normal equations $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$ if and only if $\mathbf{K}$ is a least-squares g-inverse of $\mathbf{X}$, cf. Rao and Mitra (1971, page 48). (Golub and Styan (1973, page 266) asserted that such a g-inverse must also be reflexive, i.e., of the form $\mathbf{X}^{-} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$, but this is not necessary.)

It has also been shown, e.g., by Scheffé (1959, page 14), that if $\mathbf{A}\boldsymbol{\beta}$ is estimable then $\mathbf{A}\hat{\boldsymbol{\beta}}$ is invariant to the choice of solution $\hat{\boldsymbol{\beta}}$ to the normal equations. Equivalently, $\mathbf{A}\mathbf{X}_l^{-}$ is invariant to the choice of least-squares g-inverse $\mathbf{X}_l^{-}$. We show that this condition also implies estimability. Mitra (1972) showed that $r(\mathbf{A}\mathbf{X}^{-})$ invariant for every g-inverse $\mathbf{X}^{-}$ is equivalent to (4).

Milliken (1971) claimed that estimability is generally difficult to check, and recommended using (6) with

$$r(\mathbf{X}\mathbf{T}) = \operatorname{tr} \mathbf{X}\mathbf{T}(\mathbf{X}\mathbf{T})^{+}, \tag{11}$$

where $(\mathbf{X}\mathbf{T})^{+}$ is the *Moore-Penrose* g-inverse of $\mathbf{X}\mathbf{T}$, i.e., that reflexive least-squares g-inverse $(\mathbf{X}\mathbf{T})^{-}$ for which $(\mathbf{X}\mathbf{T})^{-}\mathbf{X}\mathbf{T}$ is symmetric. We prefer using (5) or (8) after an orthogonal reduction of $\mathbf{X}$ to triangular form, as recommended by Golub and Styan (1973, page 269).

Our purpose in this paper is to collect together the various characterizations of estimability and as far as possible to supply short easy proofs. We believe that some of the characterizations and almost all of the proofs are new. We will assume only that

$$r(\mathbf{X}) < \min(n, p). \tag{12}$$

2. Results. The characterizations of estimability fall easily into those that involve $\mathbf{X}$ and those that involve $\mathbf{X}'\mathbf{X}$.

THEOREM 1. *Characterizations of estimability based on* $\mathbf{X}$. *The vector* $\mathbf{A}\boldsymbol{\beta}$ *is estimable when* $\mathcal{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$, *if and only if any one of the following seven conditions holds.*

(1.1) $\mathbf{A} = \mathbf{B}\mathbf{X}$ *for some matrix* $\mathbf{B}$,
(1.2) $r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}\end{pmatrix} = r(\mathbf{X})$,
(1.3) $r\{\mathbf{X}(\mathbf{I} - \mathbf{A}^{-}\mathbf{A})\} = r(\mathbf{X}) - r(\mathbf{A})$ *for some g-inverse* $\mathbf{A}^{-}$,
(1.4) $\mathbf{A}\mathbf{X}^{-}\mathbf{X} = \mathbf{A}$ *for some g-inverse* $\mathbf{X}^{-}$,
(1.5) $\mathbf{A}\mathbf{X}_l^{-}$ *is invariant for every least-squares g-inverse* $\mathbf{X}_l^{-}$,
(1.6) $r(\mathbf{A}\mathbf{X}_l^{-})$ *is invariant for every least-squares g-inverse* $\mathbf{X}_l^{-}$,
(1.7) $r(\mathbf{A}\mathbf{X}_l^{-}) = r(\mathbf{A})$ *for every least-squares g-inverse* $\mathbf{X}_l^{-}$.

If any one of these seven conditions holds then (1.3), (1.4), (1.6) *and* (1.7) *hold for all g-inverses* $\mathbf{A}^-$ *and* $\mathbf{X}^-$.

Conditions (1.5), (1.6) and (1.7) appear to be new characterizations of estimability. They point to the interesting fact that the weaker condition (1.6) implies the stronger condition (1.5). Characterizations involving least-squares g-inverses are especially interesting since these are the g-inverses that provide estimators and test statistics. Moreover, computationally stable procedures, such as those suggested by Golub and Styan (1973), provide a least-squares g-inverse in addition to estimators and statistics.

From a computational point of view forming the matrix $\mathbf{X}'\mathbf{X}$ and solving the normal equations $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$ may lead to difficulties, cf. e.g., Stewart (1973, page 225). In some design situations, however, the matrix $\mathbf{X}'\mathbf{X}$ may be known explicitly, cf. e.g., Searle (1971, Chapter 7), and it might then be easier to check estimability using the square $p \times p$ matrix $\mathbf{X}'\mathbf{X}$ than the (usually larger) $n \times p$ $\mathbf{X}$.

THEOREM 2. *Characterizations of estimability based on* $\mathbf{X}'\mathbf{X}$. *The vector* $\mathbf{A}\boldsymbol{\beta}$ *is estimable when* $\mathcal{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$, *if and only if any one of the following ten conditions holds*:

(2.1) $\mathbf{A}\hat{\boldsymbol{\beta}}$ *is invariant for every* $\hat{\boldsymbol{\beta}}$ *satisfying* $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$,

(2.2) $r\begin{pmatrix} \mathbf{X}'\mathbf{X} \\ \mathbf{A} \end{pmatrix} = r(\mathbf{X}'\mathbf{X})$,

(2.3) $r\{\mathbf{X}'\mathbf{X}(\mathbf{I} - \mathbf{A}^-\mathbf{A})\} = r(\mathbf{X}'\mathbf{X}) - r(\mathbf{A})$ *for some g-inverse* $\mathbf{A}^-$,

(2.4) $\mathbf{A}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'\mathbf{X} = \mathbf{A}$ *for some g-inverse* $(\mathbf{X}'\mathbf{X})^-$,

(2.5) $\mathbf{A}(\mathbf{X}'\mathbf{X})^-\mathbf{A}'$ *is invariant for every g-inverse* $(\mathbf{X}'\mathbf{X})^-$,

(2.6) $r\{\mathbf{A}(\mathbf{X}'\mathbf{X})^-\mathbf{A}'\}$ *is invariant for every g-inverse* $(\mathbf{X}'\mathbf{X})^-$,

(2.7) $r\{\mathbf{A}(\mathbf{X}'\mathbf{X})^-\mathbf{A}'\} = r(\mathbf{A})$ *for every g-inverse* $(\mathbf{X}'\mathbf{X})^-$,

(2.8) $r\begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{A}' \\ \mathbf{A} & \mathbf{0} \end{pmatrix} = r(\mathbf{X}'\mathbf{X}) + r\{\mathbf{A}(\mathbf{X}'\mathbf{X})^-\mathbf{A}'\}$ *for some g-inverse* $(\mathbf{X}'\mathbf{X})^-$,

(2.9) $$\begin{pmatrix} (\mathbf{X}'\mathbf{X})^- + (\mathbf{X}'\mathbf{X})^-\mathbf{A}'\mathbf{S}^-\mathbf{A}(\mathbf{X}'\mathbf{X})^-, & -(\mathbf{X}'\mathbf{X})^-\mathbf{A}'\mathbf{S}^- \\ -\mathbf{S}^-\mathbf{A}(\mathbf{X}'\mathbf{X})^-, & \mathbf{S}^- \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{A}' \\ \mathbf{A} & \mathbf{0} \end{pmatrix}^-$$

for some g-inverses $(\mathbf{X}'\mathbf{X})^-$ *and* $\mathbf{S}^-$, *where* $\mathbf{S} = -\mathbf{A}(\mathbf{X}'\mathbf{X})^-\mathbf{A}'$,

(2.10) $$\begin{pmatrix} (\mathbf{X}'\mathbf{X})^+ + (\mathbf{X}'\mathbf{X})^+\mathbf{A}'\mathbf{S}^+\mathbf{A}(\mathbf{X}'\mathbf{X})^+, & -(\mathbf{X}'\mathbf{X})^+\mathbf{A}'\mathbf{S}^+ \\ -\mathbf{S}^+\mathbf{A}(\mathbf{X}'\mathbf{X})^+, & \mathbf{S}^+ \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{A}' \\ \mathbf{A} & \mathbf{0} \end{pmatrix}^+,$$

where $\mathbf{S} = -\mathbf{A}(\mathbf{X}'\mathbf{X})^+\mathbf{A}'$.

If any one of these ten conditions holds then (2.3), (2.4), (2.8) *and* (2.9) *hold for all g-inverses* $\mathbf{A}^-$, $(\mathbf{X}'\mathbf{X})^-$ *and* $\mathbf{S}^-$. *Moreover the g-inverses* $(\mathbf{X}'\mathbf{X})^-$ *and* $\mathbf{S}^-$ *in* (2.9) *may then all be chosen differently.*

Conditions (2.5), (2.6) and (2.7) are known to be necessary for estimability. It does not seem to be known, however, that these conditions are also sufficient. Conditions (2.8), (2.9) and (2.10) appear to be entirely new.

The matrices $\mathbf{S} = -\mathbf{A}(\mathbf{X}'\mathbf{X})^-\mathbf{A}'$ and $-\mathbf{A}(\mathbf{X}'\mathbf{X})^+\mathbf{A}'$ in (2.8), (2.9) and (2.10) are

Schur complements of $\mathbf{X'X}$ in the partitioned matrix

$$\begin{pmatrix} \mathbf{X'X} & \mathbf{A'} \\ \mathbf{A} & \mathbf{0} \end{pmatrix};$$

a partitioned matrix of this type has been extensively considered by Rao (1973, pages 294–298) and is called "the fundamental bordered matrix of linear estimation" by Hall and Meyer (1975). Thus (2.8) represents rank additivity on the Schur complement, while (2.9) and (2.10) show then that the usual formula for the inverse of a partitioned matrix may be extended using g-inverses. As shown by Rao (1973, pages 294–298), the fundamental bordered matrix of linear estimation plays a central unifying role in his treatment of the general linear model. It is, therefore, not unlikely that the matrix

$$\begin{pmatrix} \mathbf{X'X} & \mathbf{A'} \\ \mathbf{A} & \mathbf{0} \end{pmatrix}$$

will prove useful in this context also.

3. Proofs. To establish Theorem 1 we will use the following lemma.

LEMMA 1 (Marsaglia and Styan, 1974a, page 274). *For conformable matrices* $\mathbf{E}$ *and* $\mathbf{F}$ *and for any choices of their g-inverses* $\mathbf{E}^-$ *and* $\mathbf{F}^-$

$$r\begin{pmatrix} \mathbf{E} \\ \mathbf{F} \end{pmatrix} = r(\mathbf{E}) + r\{\mathbf{F}(\mathbf{I} - \mathbf{E}^-\mathbf{E})\} = r\{\mathbf{E}(\mathbf{I} - \mathbf{F}^-\mathbf{F})\} + r(\mathbf{F}), \tag{13}$$

$$r(\mathbf{E}, \mathbf{F}) = r(\mathbf{E}) + r\{(\mathbf{I} - \mathbf{E}\mathbf{E}^-)\mathbf{F}\} = r\{(\mathbf{I} - \mathbf{F}\mathbf{F}^-)\mathbf{E}\} + r(\mathbf{F}). \tag{14}$$

PROOF. Since the matrix

$$\mathbf{G} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{F}\mathbf{E}^- & \mathbf{I} \end{pmatrix}$$

is nonsingular, we have that

$$r\begin{pmatrix} \mathbf{E} \\ \mathbf{F} \end{pmatrix} = r\left\{\mathbf{G}\begin{pmatrix} \mathbf{E} \\ \mathbf{F} \end{pmatrix}\right\} = r\begin{pmatrix} \mathbf{E} \\ \mathbf{F}(\mathbf{I} - \mathbf{E}^-\mathbf{E}) \end{pmatrix} = r(\mathbf{E}) + r\{\mathbf{F}(\mathbf{I} - \mathbf{E}^-\mathbf{E})\},$$

as the row spaces of $\mathbf{E}$ and $\mathbf{F}(\mathbf{I} - \mathbf{E}^-\mathbf{E})$ are virtually disjoint. The other equality in (13) follows similarly, while (13) $\Rightarrow$ (14) by transposition. □

PROOF OF THEOREM 1. From (13) we have

$$r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} = r(\mathbf{X}) + r\{\mathbf{A}(\mathbf{I} - \mathbf{X}^-\mathbf{X})\} = r\{\mathbf{X}(\mathbf{I} - \mathbf{A}^-\mathbf{A})\} + r(\mathbf{A}), \tag{15}$$

and so (1.2) $\Leftrightarrow$ (1.4) $\Leftrightarrow$ (1.3) follow at once; putting $\mathbf{B} = \mathbf{AX}^-$ gives (1.4) $\Leftrightarrow$ (1.1). Moreover (1.1) $\Rightarrow$ (1.5) because the symmetric projector $\mathbf{XX}_l^-$ is unique. Clearly (1.5) $\Rightarrow$ (1.6). We now prove that (1.6) $\Rightarrow$ (1.7) $\Rightarrow$ (1.1). Let $\mathbf{X}$ have rank r and singular value decomposition

$$\mathbf{X} = \mathbf{P}\begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{Q}', \tag{16}$$

where $\mathbf{D}$ is $r \times r$ diagonal positive definite, $\mathbf{P}$ is $n \times n$ orthogonal and $\mathbf{Q}$ is $p \times p$

orthogonal. Then, cf. Marsaglia and Styan (1974a, page 273),

$$\mathbf{X}_l^- = \mathbf{Q}\begin{pmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{G} & \mathbf{H} \end{pmatrix}\mathbf{P}' \tag{17}$$

is the general form of a least-squares g-inverse, where $\mathbf{G}$ is $(p-r) \times r$ and $\mathbf{H}$ is $(p-r) \times (n-r)$. From (12) we see that both $\mathbf{G}$ and $\mathbf{H}$ are always present in (17). Substituting (17) into (1.6) yields

$$r(\mathbf{A}\mathbf{X}_l^-) = r(\mathbf{A}_1 + \mathbf{A}_2\mathbf{G}, \mathbf{A}_2\mathbf{H}) = q, \tag{18}$$

say, for every $\mathbf{G}$ and $\mathbf{H}$, where

$$\mathbf{A}\mathbf{Q} = (\mathbf{A}_1, \mathbf{A}_2), \tag{19}$$

with $\mathbf{A}_1$ $s \times r$. It follows then that with $\mathbf{G} = \mathbf{0}$,

$$q = r(\mathbf{A}_1) = r(\mathbf{A}_1, \mathbf{A}_2\mathbf{H}) = r(\mathbf{A}_1) + r\{(\mathbf{I} - \mathbf{A}_1\mathbf{A}_1^-)\mathbf{A}_2\mathbf{H}\} \tag{20}$$

for every $\mathbf{H}$, using (14). Thus $(\mathbf{I} - \mathbf{A}_1\mathbf{A}_1^-)\mathbf{A}_2\mathbf{H} = \mathbf{0}$ for every $\mathbf{H}$ and hence $(\mathbf{I} - \mathbf{A}_1\mathbf{A}_1^-)\mathbf{A}_2 = \mathbf{0}$, so that using (14) again,

$$q = r(\mathbf{A}_1) = r(\mathbf{A}_1) + r\{(\mathbf{I} - \mathbf{A}_1\mathbf{A}_1^-)\mathbf{A}_2\} = r(\mathbf{A}), \tag{21}$$

and so (1.6) $\Rightarrow$ (1.7). Now choose $\mathbf{G} = -\mathbf{A}_2^-\mathbf{A}_1$ and $\mathbf{H} = \mathbf{0}$ in (18). Then

$$q = r\{(\mathbf{I} - \mathbf{A}_2\mathbf{A}_2^-)\mathbf{A}_1\} = r(\mathbf{A}) - r(\mathbf{A}_2) \tag{22}$$

using (14), and so $\mathbf{A}_2 = \mathbf{0}$. Thus

$$\mathbf{A} = (\mathbf{A}_1, \mathbf{0})\mathbf{Q}' = (\mathbf{A}_1\mathbf{D}^{-1}, \mathbf{0})\mathbf{P}'\mathbf{P}\begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{Q}', \tag{23}$$

and (1.7) $\Rightarrow$ (1.1), with $\mathbf{B} = (\mathbf{A}_1\mathbf{D}^{-1}, \mathbf{0})\mathbf{P}'$.

If any one of the conditions (1.1) through (1.7) holds then (1.3) and (1.4) hold for all g-inverses $\mathbf{A}^-$ and $\mathbf{X}^-$ by Lemma 1; with $\mathbf{A} = \mathbf{B}\mathbf{X}$ from (1.1) we see that $r(\mathbf{A}\mathbf{X}^-) = r(\mathbf{B}\mathbf{X}\mathbf{X}^-) = r(\mathbf{B}\mathbf{X}) = r(\mathbf{A})$, since $r(\mathbf{X}\mathbf{X}^-) = r(\mathbf{X})$, and so (1.6) and (1.7) hold for every g-inverse $\mathbf{X}^-$. □

To prove Theorem 2 we will use Lemma 1 above and the following three lemmas.

LEMMA 2 (Rao and Mitra, 1971, page 48). *Let* $\mathbf{K}$ *be a* $p \times n$ *matrix. Then* $\mathbf{K}$ *is a least-squares g-inverse of* $\mathbf{X}$ *if and only if* $\mathbf{X}'\mathbf{X}\mathbf{K} = \mathbf{X}'$.

LEMMA 3 (Rao, 1973, page 296). *Let* $\mathbf{X}$ *and* $\mathbf{A}$ *each have p columns. Then*

$$r\begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{A}' \\ \mathbf{A} & \mathbf{0} \end{pmatrix} = r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} + r(\mathbf{A}). \tag{24}$$

LEMMA 4 (Marsaglia and Styan, 1974b). *Let* $\mathbf{E}^{\sim}$ *be a particular g-inverse of* $\mathbf{E}$ *and let* $\mathbf{S} = \mathbf{H} - \mathbf{G}\mathbf{E}^{\sim}\mathbf{F}$. *Then for a particular g-inverse* $\mathbf{S}^-$,

$$\begin{pmatrix} \mathbf{E}^{\sim} + \mathbf{E}^{\sim}\mathbf{F}\mathbf{S}^-\mathbf{G}\mathbf{E}^{\sim} & -\mathbf{E}^{\sim}\mathbf{F}\mathbf{S}^- \\ -\mathbf{S}^-\mathbf{G}\mathbf{E}^{\sim} & \mathbf{S}^- \end{pmatrix} = \begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{pmatrix}^- \tag{25}$$

if and only if

$$r\begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{pmatrix} = r(\mathbf{E}) + r(\mathbf{H} - \mathbf{G}\mathbf{E}^{\sim}\mathbf{F}), \tag{26}$$

and then (26) *holds for every g-inverse* $\mathbf{S}^-$. *Moreover if* $\mathbf{S} = \mathbf{H} - \mathbf{G}\mathbf{E}^+\mathbf{F}$ *then*

$$\begin{pmatrix} \mathbf{E}^+ + \mathbf{E}^+\mathbf{F}\mathbf{S}^+\mathbf{G}\mathbf{E}^+ & -\mathbf{E}^+\mathbf{F}\mathbf{S}^+ \\ -\mathbf{S}^+\mathbf{G}\mathbf{E}^+ & \mathbf{S}^+ \end{pmatrix} = \begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{pmatrix}^+ \tag{27}$$

if and only if

$$r\begin{pmatrix} \mathbf{E} \\ \mathbf{G} \end{pmatrix} = r(\mathbf{E}, \mathbf{F}) = r(\mathbf{E}) \quad \textit{and} \quad r\begin{pmatrix} \mathbf{F} \\ \mathbf{H} \end{pmatrix} = r(\mathbf{G}, \mathbf{H}) = r(\mathbf{S}). \tag{28}$$

While both Lemma 2 and Lemma 3 are easy to prove, we have not been able to find a short and easy proof of Lemma 4.

Proof of Theorem 2. We note that $\hat{\boldsymbol{\beta}} = \mathbf{K}\mathbf{y}$ solves $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y} \Leftrightarrow \mathbf{X}'\mathbf{X}\mathbf{K} = \mathbf{X}'$ and so (2.1) $\Leftrightarrow$ (1.5) by Lemma 2. From (13) we have

$$r\begin{pmatrix} \mathbf{X}'\mathbf{X} \\ \mathbf{A} \end{pmatrix} = r(\mathbf{X}'\mathbf{X}) + r\{\mathbf{A}[\mathbf{I} - (\mathbf{X}'\mathbf{X})^- \mathbf{X}'\mathbf{X}]\} = r\{\mathbf{X}'\mathbf{X}(\mathbf{I} - \mathbf{A}^-\mathbf{A})\} + r(\mathbf{A}), \tag{29}$$

and thus (2.2) $\Leftrightarrow$ (2.4) $\Leftrightarrow$ (2.3). Putting $\mathbf{A}(\mathbf{X}'\mathbf{X})^-\mathbf{X}' = \mathbf{B}$ in (2.4) implies (1.1), while (1.1) $\Rightarrow$ (2.4) since $\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'\mathbf{X} = \mathbf{X}$. Also (1.1) $\Rightarrow$ (2.5) as $\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ is invariant, and (2.5) clearly implies (2.6). Choosing $(\mathbf{X}'\mathbf{X})^-$ positive definite gives (2.6) $\Rightarrow$ (2.7). To show that (2.7) $\Rightarrow$ (1.1) we use (16) to write

$$\mathbf{A}(\mathbf{X}'\mathbf{X})^- \mathbf{A}' = \mathbf{A}\mathbf{Q}\begin{pmatrix} \mathbf{D}^{-2} & \mathbf{L} \\ \mathbf{M} & \mathbf{N} \end{pmatrix}\mathbf{Q}'\mathbf{A}'. \tag{30}$$

Now let $\mathbf{M} = -\mathbf{A}_2^-\mathbf{A}_1\mathbf{D}^{-2} = \mathbf{L}'$ and $\mathbf{N} = \mathbf{A}_2^-\mathbf{A}_1\mathbf{D}^{-2}(\mathbf{A}_2^-\mathbf{A}_1)'$, where $\mathbf{A}\mathbf{Q} = (\mathbf{A}_1, \mathbf{A}_2)$, cf. (19). Then

$$\begin{aligned} r(\mathbf{A}) = r\{\mathbf{A}(\mathbf{X}'\mathbf{X})^- \mathbf{A}'\} &= r\left\{(\mathbf{A}_1, \mathbf{A}_2)\begin{pmatrix} \mathbf{I} \\ -\mathbf{A}_2^-\mathbf{A}_1 \end{pmatrix}\mathbf{D}^{-2}\right\} \\ &= r\{(\mathbf{I} - \mathbf{A}_2\mathbf{A}_2^-)\mathbf{A}_1\} = r(\mathbf{A}) - r(\mathbf{A}_2), \end{aligned} \tag{31}$$

using Lemma 1. Thus $\mathbf{A}_2 = \mathbf{0}$, and using (23) we see that (2.7) $\Rightarrow$ (1.1). Now suppose (2.8) holds. Then using Lemma 3 we have that

$$\begin{aligned} r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} + r(\mathbf{A}) &= r(\mathbf{X}'\mathbf{X}) + r\{\mathbf{A}(\mathbf{X}'\mathbf{X})^- \mathbf{A}'\} \\ &\leqslant r(\mathbf{X}) + r(\mathbf{A}) \leqslant r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} + r(\mathbf{A}), \end{aligned} \tag{32}$$

and so (2.8) $\Rightarrow$ (1.2). To see that estimability implies (2.8) we note from (2.7) and (1.2) that

$$r(\mathbf{X}'\mathbf{X}) + r\{\mathbf{A}(\mathbf{X}'\mathbf{X})^- \mathbf{A}'\} = r(\mathbf{X}) + r(\mathbf{A}) = r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} + r(\mathbf{A}). \tag{33}$$

Then (2.8) follows at once using Lemma 3. The first part of Lemma 4 proves that (2.8) $\Leftrightarrow$ (2.9), while the second part of Lemma 4 shows that (2.10) holds if and only if, from (28),

$$r\begin{pmatrix} \mathbf{X}'\mathbf{X} \\ \mathbf{A} \end{pmatrix} = r(\mathbf{X}'\mathbf{X}) \quad \text{and} \quad r(\mathbf{A}) = r\{\mathbf{A}(\mathbf{X}'\mathbf{X})^+ \mathbf{A}'\}, \tag{34}$$

cf. (2.2) and (2.7).

It follows from Lemma 1 that if (2.2) holds then (2.3) and (2.4) hold for all g-inverses $\mathbf{A}^-$ and $(\mathbf{X}'\mathbf{X})^-$. Moreover, (2.5) shows that (2.8) holds for every g-inverse $(\mathbf{X}'\mathbf{X})^-$, while Lemma 4 then implies that (2.9) is valid for all g-inverses $(\mathbf{X}'\mathbf{X})^-$ and $\mathbf{S}^- = \{-\mathbf{A}(\mathbf{X}'\mathbf{X})^-\mathbf{A}'\}^-$; to see that these g-inverses may all be chosen differently, multiply out

(35)

$$\begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{A}' \\ \mathbf{A} & \mathbf{0} \end{pmatrix} \begin{pmatrix} (\mathbf{X}'\mathbf{X})_1^- + (\mathbf{X}'\mathbf{X})_2^-\mathbf{A}'\mathbf{S}_1^-\mathbf{A}(\mathbf{X}'\mathbf{X})_3^- & -(\mathbf{X}'\mathbf{X})_4^-\mathbf{A}'\mathbf{S}_2^- \\ -\mathbf{S}_3^-\mathbf{A}(\mathbf{X}'\mathbf{X})_5^- & \mathbf{S}_4^- \end{pmatrix} \begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{A}' \\ \mathbf{A} & \mathbf{0} \end{pmatrix}$$

using (2.4) and (2.5). Also needed are the invariance of $\mathbf{A}'\mathbf{S}^-\mathbf{A}$ and of $\mathbf{A}'\mathbf{S}^-\mathbf{S}$ which follow from (2.7). □

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