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A CONSTRUCTIVE RENEWAL THEOREM

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Let P be a distribution on R with a positive, finite mean. A constructive, unified version of the renewal theorem is proved. A routine method is provided with which one can compute, in principle at least, the point x_0 where the renewal measure Q settles down. As a corollary, it is shown that x_0 depends continuously on P.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with a common distribution P which has a positive, finite mean μ. As usual, define the convolution $P_1 * P_2$ of two distributions P_1 and P_2 by $\int f(x)\,dP_1 * P_2(x) = \int\int f(x+y)\,dP_1(x)\,dP_2(y)$. Define the renewal measure Q on R by

$$\int f(x)\,dQ(x) = \sum_{j=1}^{\infty} \mathscr{E}(f(X_1 + \cdots + X_j)) = \sum_{j=1}^{\infty} \int f(x)\,dP^{*j}(x)$$

where f is any continuous function on R with compact support. $Q[x, x+h]$ is the expected number of partial sums $X_1 + \cdots + X_j$ which fall in the interval $[x, x+h]$. The classical renewal theorems assert that, far from the origin, Q behaves like a discrete or continuous uniform measure, ([2], [3], [4]).

To illustrate, consider a distribution P supported by $[0, \infty)$. In case it is not a lattice measure, i.e., in case it is not supported by $\{jL : j = 0, 1, \cdots\}$ for any $L > 0$, the well-known Blackwell's theorem asserts that for any $h \geqq 0$, $Q[x, x+h] \to h/\mu$ as $x \to \infty$. The proof given by Smith [4] is constructive, which means that given $\varepsilon > 0$, we can follow the steps of the proof and compute a sufficiently large x_0 so that $|Q[x, x+h] - h/\mu| < \varepsilon$ whenever $x > x_0$ and $0 \leqq h \leqq 1$. One has to, of course, interpret the hypothesis constructively. For example, the assumption that P is nonlattice is taken to mean that there is a way to compute a positive lower bound for $|(1 - \phi(t))/t|$ on every finite t-interval. Here $\phi(t)$ stands for the characteristic function of the distribution P. Proofs for Blackwell's theorem based on compactness arguments have also been given. Though elegant, they have little computational consequence.

There is also an analogous theorem in case P is a lattice measure. This was given a proof, also constructive, by Erdös, Feller, and Pollard in [2]. Later authors (see [3]) removed the restriction that P be supported by $[0, \infty)$.

To find the "sufficiently large x_0 beyond which Q settles down," one would therefore first determine whether P is lattice. If it is nonlattice, one follows Smith's proof and calculates x_0. If it is lattice, one follows Erdös–Feller–Pollard's proof to find x_0.

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However, deciding whether a given measure is of the lattice type is not a finitely performable operation. It is not possible to write down a program for a (human or mechanical) computer such that the input is a sequence of digits representing a measure P, and the output, after finitely many steps, is 0 or 1 indicating the type of P. Furthermore, it is not possible constructively to determine the period of a lattice measure P (the largest L such that $\{jL : j = 0, 1, \cdots\}$ supports P). And the calculations of x_0, if one follows Erdös–Feller–Pollard's proof, depends in an essential way on L.

To illustrate the point, consider the distribution P^r which assigns measures $\frac{1}{2}$ to each of the points r and 1. If r is taken to be Euler's constant $\gamma = \lim (1 + \frac{1}{2} + \cdots + 1/n - \log n) = .57721\cdots$, the classical theorems cannot tell us when Q^γ settles down, because it is not known whether P^γ is lattice or nonlattice. (It is not known whether γ is rational or irrational.)

One may argue that this counterexample is not interesting, and that the distributions encountered in practice can always, somehow, be verified to be of one type or the other. But there is another way to look at the issue raised. The measure P is often only a close approximation of some "true" measure P'. This happens, for example, when we replace P' by P which assigns mass $P'[jL, (j+1)L)$ to jL. Now there is the question whether x_0 (where Q settles down) has anything to do with x_0' (where the "true" renewal measure Q' settles down). It seems highly intuitive that the answer is yes. However, it is not at all obvious how to prove this from the classical theorems.

We will now prove a renewal theorem which does not refer to whether P is lattice or to its period if lattice. From this, the two classical versions will be deduced. We will also show that x_0, when defined properly, depends continuously on P relative to the metric $\lambda(P', P) = \int |F'(x) - F(x)|\, dx$ where $F(x) = P(-\infty, x]$.

2. Formulation of the theorem. By changing the scale if necessary, we will assume that the given distribution P has mean μ strictly between 3 and 4. This will simplify presentation without loss of generality.

Define a function $\hat{F}(x)$ by $\hat{F}(x) = 1 - F(x)$ if $x > 0$, and $\hat{F}(x) = -F(x)$ if $x < 0$. Let $\hat{P}$ be the measure with density $\hat{F}$. Note that $\int_{-\infty}^{\infty} |\hat{F}(x)|\, dx = \int_{-\infty}^{\infty} |x|\, dF(x)$ exists by assumption. So there is a continuous, positive, nonincreasing function b defined on $(0, \infty)$ such that

$$\int_{|x|>b(a)} |\hat{F}(x)|\, dx < a \qquad \text{for all } a > 0\,. \tag{2.1}$$

Since the number $b(1)$ will enter many times, we will write B for $b(1)$. Note that $3 < \mu = \int_{-\infty}^{\infty} \hat{F}(x)\, dx \leqq 1 + \int_{-B}^{B} |\hat{F}(x)|\, dx \leqq 1 + 2B$. Hence $B > 1$.

The Fourier transform $\hat{\phi}(t) = \int e^{itx}\hat{F}(x)\, dx$ of $\hat{P}$ is related to the characteristic function $\phi(t) = \int e^{itx}\, dP(x)$ of P by $\hat{\phi}(t) = (\phi(t) - 1)/it$. Classically the period L of P is given by $2\pi/\tau$ where $\tau \equiv \inf\{t \geqq 0 : \hat{\phi}(t) = 0\}$. Our difficulty, as implied in the introduction, is that the calculation of τ is not possible from the most general data.

We therefore use a substitute for τ. For arbitrary positive real numbers n

and c, we say that the real number $T \in [0, n]$ is an (n, c)-*frequency* for P if

$$c < m \equiv \min \{|\hat{\phi}(t)| : t = 0 \text{ or } 0 \leqq t \leqq T - n^{-1}\} , \tag{2.2}$$

and

$$n^5 |\hat{\phi}(T)| < m^2 \qquad \text{in case} \quad T < n . \tag{2.3}$$

(Note that $m \leqq |\hat{\phi}(0)| = \mu < 4$.) Thus, instead of the first zero τ of $\hat{\phi}$, we use T where $|\hat{\phi}(T)|$ is very small compared to $|\hat{\phi}(t)|$ for $t \in [0, T - n^{-1}]$. Note that if $n' \leqq n$, and T is an (n, c)-frequency, then $T \wedge n'$ is an (n', c)-frequency. Given any distribution P and any $n > 0$, we have a routine method to find an (n, c)-frequency T for some c. [To find one such T, we may assume $n > 1$. Choose positive real numbers c_k ($k = 0, 1, \cdots, N$ where N is some integer $> n^2$) such that $c_0 < 3$, such that $c_k < n^{-5} c_{k-1}^2$, and such that the sets $A_k \equiv \{t \in [0, n] : |\hat{\phi}(t)| \leqq c_k \text{ or } t = n\}$ are totally bounded. (Theorem 8 on page 101 of [1] shows that this can be done constructively.) Then $t_k \equiv \inf A_k$ exists. Partition the t_k's into two sets A and B so that $t_k - t_{k-1} < n^{-1}$ if $t_k \in A$, and so that $t_k - t_{k-1} > n/N$ if $t_k \in B$. Let $T \equiv \min \{t : t \in A \text{ or } t = n\}$ and let c be any positive number with $c < c_N$. Then clearly $T \leqq t_N$ and so (2.2) is satisfied. Moreover, if $T < n$, then $T = t_k$ for some $t_k \in A$ and so whenever $0 \leqq t \leqq T - n^{-1} = t_k - n^{-1} < t_{k-1}$ we have $|\hat{\phi}(t)|^2 \geqq c_{k-1}^2 > n^5 c_k = n^5 |\hat{\phi}(t_k)| = n^5 |\hat{\phi}(T)|$. Thus condition (2.3) is also satisfied.] If T is an (n, c)-frequency, we will call $L \equiv 2\pi/T$ an (n, c)-*period*. We will need the fact that

$$T \geqq B^{-\frac{3}{2}} \qquad \text{if} \quad T \quad \text{is an } (n, c)\text{-frequency with} \quad n \geqq 2 . \tag{2.4}$$

Suppose $n \geqq 2$ and $T < B^{-\frac{3}{2}}$. We will deduce a contradiction. First note that $1 - \cos Tx \leqq T^2 x^2 / 2 \leqq 2^{-1} B^{-1}$ for all $x \in [-B, B]$. Hence

$$\begin{aligned} |\mathrm{Re}\, \hat{\phi}(T)| &= |\textstyle\int \cos Tx \hat{F}(x)\, dx| \\ &\geqq |\textstyle\int \hat{F}(x)\, dx| - |\textstyle\int_{|x| \leqq B} 2^{-1} B^{-1}\, dx| - \textstyle\int_{|x| > B} |\hat{F}(x)|\, dx \\ &> \mu - 1 - 1 > 1 . \end{aligned}$$

On the other hand we have $n > 1 \geqq B^{-\frac{3}{2}} > T$. Hence (2.3) implies $|\hat{\phi}(T)| < n^{-5} m^2 \leqq 2^{-5} 4^2 < 1$, a contradiction.

We can now state the theorem.

THEOREM. *Let P be a distribution on R with positive, finite mean, satisfying* (2.1) *for some b. Let $\varepsilon > 0$ be given. Then there exists $n = n(b, \varepsilon) > 0$ with the following properties. If $L > 0$ is any (n, c)-period for P, we can find $x_0 = x_0(\varepsilon, b, n, c) > 0$ such that*

$$|\textstyle\int G(x - y)\, dQ(y) - \mu^{-1} \int G(y)\, dy| < \varepsilon \qquad x \geqq x_0 \tag{2.5}$$

$$|\textstyle\int G(x - y)\, dQ(y)| < \varepsilon \qquad x \leqq -x_0 \tag{2.5$'$}$$

where $G(y)$ is any function of the form $1 \wedge L^{-1}(a + L - |y|)_+$ for some $a \in [0, 1]$. Moreover, x_0 depends continuously on ε, n, c (relative to ordinary convergence in $(0, \infty)$), and on b (relative to uniform convergence on compact intervals). x_0 is independent of L. □

Since we can always find some (n, c)-period for a given n, the theorem is always applicable.

Consider a distribution P, satisfying (2.1) for some b. Let $\varepsilon > 0$ be given, let $n = n(\varepsilon, b)$ be as given by the theorem, and let L be some (n, c)-period of P. Suppose P_j is a sequence of distributions converging to P with respect to the metric $\lambda(P_j, P) = \int |F_j(x) - F(x)|\, dx$. Then clearly $|\hat{\phi}_j(t) - \hat{\phi}(t)| \leqq \lambda(P_j, P) \to 0$ uniformly in t. Hence (2.2), (2.3) are satisfied by $\hat{\phi}_j$ if j is large enough. Consequently L is also an (n, c)-period for P_j if j is large enough. We now prove that P_j satisfies (2.1) for some b_j which converges to b uniformly on compact subintervals of $(0, \infty)$. There is no loss of generality in assuming $b(a) \uparrow \infty$ as $a \downarrow 0$. But

$$\int_{|x|>b(a)} |\hat{F}_j(x)|\, dx < a + \lambda_j \qquad a > 0$$

if we write $\lambda_j = \lambda(P_j, P)$. Take any b_j' such that (2.1) is satisfied by b_j' and P_j, and define $b_j(a) \equiv b((a - \lambda_j)_+) \wedge (b_j'(a) \vee b(a))$. Then, since $b_j(a) \geqq b((a - \lambda_j)_+) \wedge b_j'(a)$, condition (2.1) is satisfied by b_j and P_j also. Because the function b is continuous, the expression b_j tends to b uniformly on compact subintervals of $(0, \infty)$, as desired. Now, since x_0 depends continuously on b, we have $x_0(\varepsilon, b_j, n, c) \to x_0(\varepsilon, b, n, c)$ when $P_j \to P$, as promised in the introduction.

It is easy to deduce the classical versions of the renewal theorem from the one presented here. For example, let P be a lattice measure with the known period $L > 0$. Then for any n, the number L is an (n, c)-period of P for some $c > 0$. If we let $G(y) = L^{-1}(L - |y|)_+$, then the integral $\int G(y)\, dy$ is equal to L, while $\int G(kL - y)\, dQ(y)$ is equal to Q_{kL}, the mass assigned to kL by Q, $(k = 0, \pm 1, \cdots)$. So the theorem reads

$$\begin{aligned} |Q_{kL} - \mu^{-1}L| &< \varepsilon \qquad \text{if} \quad k \geqq L^{-1}x_0\,, \\ |Q_{kL}| &< \varepsilon \qquad \text{if} \quad k \leqq -L^{-1}x_0\,, \end{aligned}$$

This is the lattice version.

Likewise, if P is a nonlattice measure, every $n > 0$ is an (n, c)-frequency of P for some $c > 0$. Let $\varepsilon > 0$ be given, and consider the indicator χ of some interval $[-a, a]$ with $0 < a \leqq 1$. Then for n large enough, the (n, c)-period $L \equiv 2\pi/n$ is so small that the two functions defined by

$$G'(y) \equiv 1 \wedge L^{-1}(a - |y|)_+\,, \qquad \text{and} \qquad G''(y) = 1 \wedge L^{-1}(L + a - |y|)_+$$

are such that $0 < \mu^{-1} \int G''(y)\, dy - \mu^{-1} \int G'(y)\, dy < \varepsilon$. Applying the theorem to G' and G'', and using the fact that $G' \leqq \chi \leqq G''$, we deduce

$$\begin{aligned} \left|\int \chi(x - y)\, dQ(y) - \mu^{-1} \int \chi(y)\, dy\right| &< 5\varepsilon \qquad \text{if} \quad x \geqq x_0\,, \\ \left|\int \chi(x - y)\, dQ(y)\right| &< \varepsilon \qquad \text{if} \quad x \leqq -x_0\,. \end{aligned}$$

This is the nonlattice version.

Before proceeding with the proof of the theorem, we remark that while the classical theorems permit the case $\mu = \infty$, we have not attempted to give a method to calculate x_0 in this case or when we do not know if $\mu < \infty$ or $\mu = \infty$. The reason is that, evidently, we do not have the function b in this case, and some different data will be needed to compute x_0. The presentation would therefore be substantially different.

3. Proof of the theorem. The proof will be broken into several elementary lemmas, whose proofs are presented in the next section. Throughout the proof, "const" stands for an absolute constant, not necessarily the same in different places.

Let P, b, ε be as given in the hypothesis of the theorem. Let $L > 0$ be an (n, c)-period of P for some n and c. Let G be a function of the form $G(y) = 1 \wedge L^{-1}(a + L - |y|)_+$ for some $a \in [0, 1]$.

Let K be any integer such that $KL \geqq n^{-\frac{1}{6}} \geqq (K - 2)L$. Write D for KL and define $G_0(y) \equiv 1 \wedge D^{-1}(a + D - |y|)_+$. We prefer to work with G_0 because we have the convenient lower bound $D \geqq n^{-\frac{1}{6}}$. In view of (2.4), we see that whenever $n \geqq 2$, we have also $D = KL \leqq n^{-\frac{1}{6}} + 2L \leqq 1 + 2L = 1 + 4\pi/T < 2^4 B^{\frac{3}{2}}$.

Define ϕ, $\hat{F}$, $\hat{P}$, $\hat{\phi}$ as in last section. Let g_0 denote the Fourier transform of G_0

$$g_0(t) \equiv \textstyle\int_{-\infty}^{\infty} e^{itx} G_0(x)\, dx = 2D^{-1}t^{-2}(\cos at - \cos (a + D)t)\,.$$

The key step in the proof is to introduce

$$g_n(t) \equiv (g_0(t) - 2n^{-1}B^3)_+ + (g_0(t) + 2n^{-1}B^3)_-\,.$$

($x_+ \equiv x \vee 0$ and $x_- \equiv x \wedge 0$). Since g_n is bounded in absolute value by $|g_0|$, it is integrable and has an inverse Fourier transform

$$G_n(x) \equiv (2\pi)^{-1} \textstyle\int_{-\infty}^{\infty} e^{-itx} g_n(t)\, dt\,.$$

The next lemma shows that G_n is integrable and converges in a strong sense to G_0 as $n \to \infty$.

LEMMA 1. $|G_0(x) - G_n(x)| \leqq \text{const}\,\{n^{-\frac{5}{12}} B^{\frac{3}{2}} \wedge n^{\frac{1}{12}} B^{\frac{21}{4}} x^{-2}\}$ *if* $n \geqq 2$.

The function G_n has the desirable property that its Fourier transform g_n vanishes whenever $\hat{\phi}$ is close to 0.

LEMMA 2. $|\hat{\phi}(t)| \geqq \text{const}\, n^{-3}c^2$ *if* $|g_n(t)| > 0$ *and if* $n \geqq 2^6 B^{\frac{3}{2}}$.

The function $g_n/\hat{\phi}$ is therefore well defined if we equate it to 0 whenever $|\hat{\phi}|$ is smaller than const $n^{-3}c^2$.

LEMMA 3. *If* $n \geqq 2^6 B^{\frac{3}{2}}$, *then* $g_n/\hat{\phi}$ *is the Fourier transform of some integrable function* $H(x)$ *such that* $\int_{|x|>r(\alpha)} |H(x)| < \alpha$ *for all* $\alpha > 0$, *where* $r(\alpha) = r(\alpha, b, n, c)$ *depends continuously on its arguments.*

LEMMA 4. $\hat{P} * Q = m_+ - \hat{P}$ *where* m_+ *is the Lebesgue measure on* $[0, \infty)$.

PROOF OF THE THEOREM. Suppose $n \geqq 2^6 B^{\frac{3}{2}}$. Lemma 3 implies

$$G_n(v) = \textstyle\int H(v - z)\, d\hat{P}(z)\,.$$

In particular, since $\int_{-\infty}^{\infty} d\hat{P}(z) = \int_{-\infty}^{\infty} \hat{F}(z)\, dz = \mu$, integrating the above displayed equality yields

$$\int G_n(v)\,dv = \mu \int H(v)\,dv \,. \tag{3.1}$$

We compute, using Lemma 4,

$$\begin{aligned} \int G_n(x-y)\,dQ(y) &= \int\int H(x-y-z)\,d\hat{P}(z)\,dQ(y) \\ &= \int H(x-u)\,d\hat{P} * Q(u) \\ &= \int_0^\infty H(x-u)\,du - \int H(x-u)\,d\hat{P}(u) \\ &= \int_{-\infty}^x H(v)\,dv - G_n(x) \,. \end{aligned} \tag{3.2}$$

In view of (3.1), the last equality can also be written as

$$\int G_n(x-y)\,dQ(y) = \mu^{-1} \int G_n(v)\,dv - \int_x^\infty H(v)\,dv - G_n(x) \,. \tag{3.3}$$

From Lemma 1, we have

$$\left|\int G_0(v)\,dv - \int G_n(v)\,dv\right| \leqq \text{const}\; n^{-\frac{1}{6}} B^{\frac{27}{8}} \,. \tag{3.4}$$

Next note that $3 < \mu \leqq 1 + B \cdot P[1, \infty) + 1$ and so $P[1, \infty) > B^{-1}$. Hence ([3], pages 359–360) the measure Q assigns mass no greater than const B to any unit interval. Therefore, using Lemma 1 again,

$$\left|\int G_0(x-y)\,dQ(y) - \int G_n(x-y)\,dQ(y)\right| \leqq \text{const}\; n^{-\frac{1}{6}} B^{\frac{35}{8}} \,. \tag{3.5}$$

Next, since $n \geqq 2^6 B^{\frac{3}{2}} > 2$, we have $D < 2^4 B^{\frac{3}{2}}$, and so $G_0(x)$ vanishes if $|x| > 2^5 B^{\frac{3}{2}} (> D + 1)$. Lemma 1 then implies

$$|G_n(x)| < \text{const}\; n^{-\frac{5}{12}} B^{\frac{3}{2}} \qquad \text{if} \quad |x| > n^{\frac{1}{4}} B^{\frac{15}{8}} \vee 2^5 B^{\frac{3}{2}} \,. \tag{3.6}$$

Finally, if $|x| > r(\varepsilon/8)$ where r is as given in Lemma 3, then

$$\int_{|v|>|x|} |H(v)|\,dv < \varepsilon/8 \,. \tag{3.7}$$

Combining the formulas (3.3) through (3.7), we see that

$$\begin{aligned} &\left|\int G_0(x-y)\,dQ(y) - \mu^{-1} \int G_0(y)\,dy\right| \\ &\qquad < (\varepsilon/8) + \text{const}\; n^{-\frac{1}{6}} B^{\frac{35}{8}} + \text{const}\; n^{-\frac{5}{12}} B^{\frac{3}{2}} < \varepsilon/4 \end{aligned} \tag{3.8}$$

whenever

$$\begin{aligned} &n \geqq n_1 \equiv \text{const}\; 2^6 B^{\frac{3}{2}} \vee \text{const}\; \varepsilon^{-6} B^{27} \vee \text{const}\; \varepsilon^{-\frac{12}{5}} B^{\frac{18}{5}} \,, \\ &x \geqq x_1 \equiv r(\varepsilon/8) \vee n^{\frac{1}{4}} B^{\frac{15}{8}} \vee 2^5 B^{\frac{3}{2}} \,. \end{aligned} \tag{3.9}$$

A fortiori, inequality (2.5) is valid when $L > n^{-\frac{1}{6}}$ (for we can pick $K = 1$ and have $G = G_0$) and when

$$\begin{aligned} &n \geqq n_0 \equiv \text{const}\; \varepsilon^{-6} B^{27} \,, \quad (\text{const is chosen so large that } n_0 \geqq n_1) \,, \\ &x \geqq x_0 \equiv x_1 + 1 \,. \end{aligned} \tag{3.10}$$

In case $L < 2n^{-\frac{1}{6}}$ we have $D = KL \leqq 2L + n^{-\frac{1}{6}} < 5n^{-\frac{1}{6}}$. Define $\bar{G}(y) \equiv D^{-1}(D - |y|)_+$. Then (3.8) is satisfied by $\bar{G}$ if $n \geqq n_1$ and $x \geqq x_1$. But $0 \leqq G_0(y) - G(y) \leqq \bar{G}(y + a) + \bar{G}(y - a)$. Hence, whenever $n \geqq n_0$ and $x \geqq x_0$, we have

$$
\begin{aligned}
0 &\leqq \int G_0(x-y)\,dQ(y) - \int G(x-y)\,dQ(y) \\
&\leqq \int \bar{G}(x-y-a)\,dQ(y) + \int \bar{G}(x-y+a)\,dQ(y) \\
&\leqq 2\mu^{-1} \int \bar{G}(y)\,dy + 2\varepsilon/4 \\
&\leqq 10n^{-\frac{1}{6}} + \varepsilon/2 .
\end{aligned}
\tag{3.11}
$$

Likewise

$$
0 \leqq \mu^{-1} \int G_0(y)\,dy - \mu^{-1} \int G(y)\,dy \leqq 10n^{-\frac{1}{6}} . \tag{3.12}
$$

Combining (3.8), (3.11) and (3.12), we see that

$$
\left|\int G(x-y)\,dQ(y) - \mu^{-1} \int G(y)\,dy\right| \leqq 3\varepsilon/4 + 20n^{-\frac{1}{6}} \leqq \varepsilon
$$

whenever $n \geqq n_0$ and $x \geqq x_0$. This is inequality (2.5). The proof of (2.5)′ is a repetition of the steps after (3.7), but making use of (3.2) instead of (3.3).

4. Proof of the lemmas.

PROOF OF LEMMA 1. We may assume that there exist numbers $0 < t_1 < t_2 < \cdots < t_{2k} = \infty$ such that

$$
\begin{aligned}
|g_0(t)| < 2n^{-1}B^3 \qquad &\text{if} \quad t \in (-t_{2k}, -t_{2k-1}) \cup \cdots \cup (-t_2, -t_1) \\
&\qquad \cup (t_1, t_2) \cup \cdots \cup (t_{2k-1}, t_{2k}) , \\
|g_0(t)| > 2n^{-1}B^3 \qquad &\text{if} \quad t \in (-t_{2k-1}, -t_{2k-2}) \cup \cdots \cup (-t_1, t_1) \cup \cdots \\
&\qquad \cup (t_{2k-2}, t_{2k-1}) .
\end{aligned}
\tag{4.1}
$$

Integration by parts yields

$$
\begin{aligned}
|G_0(x) - G_n(x)| &= \left|(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itx}(g_0(t) - g_n(t))\,dt\right| \\
&= \left|(2\pi i x)^{-1} \int_{-\infty}^{\infty} e^{-itx}(g_0'(t) - g_n'(t))\,dt\right| .
\end{aligned}
$$

But $g_n' = g_0'$ or 0 according as $|g_0| > 2n^{-1}B^3$ or $< 2n^{-1}B^3$. Hence, using (4.1), we have

$$
\begin{aligned}
&\left|(2\pi i x)^{-1} \int_0^{\infty} e^{-itx}(g_0'(t) - g_n'(t))\,dt\right| \\
&\quad = \left|(2\pi x)^{-1} \textstyle\sum_{j=1}^{k} \int_{t_{2j-1}}^{t_{2j}} e^{-itx} g_0'(t)\,dt\right| \\
&\quad \leqq \left|(2\pi x^2)^{-1} \textstyle\sum_{j=1}^{k} \{e^{-itx} g_0'(t)|_{t_{2j-1}}^{t_{2j}}\}\right| + (2\pi x^2)^{-1} \textstyle\sum_{j=1}^{k} \int_{t_{2j-1}}^{t_{2j}} |g_0''(t)|\,dt \\
&\quad \leqq (2\pi x^2)^{-1} \textstyle\sum_{j=1}^{2k} |g_0'(t_j)| + (2\pi x^2)^{-1} \int_0^{\infty} |g_0''(t)|\,dt .
\end{aligned}
\tag{4.2}
$$

Now, because $n \geqq 2$ by hypothesis, we have $D < 2^4 B^{\frac{3}{2}}$. Hence straightforward differentiation yields the estimates

$$
|g_0'(t)| \leqq \text{const}\, B^{\frac{3}{2}} D^{-1} t^{-2} , \qquad \text{and} \qquad |g_0''(t)| \leqq \text{const}\, B^3 D^{-1} t^{-2} \qquad \text{if} \quad |t| \geqq 1 .
$$

To estimate $|g_0'(t)|$ and $|g_0''(t)|$ when $|t| \leqq 1$, notice that $g_0(t)$ can be written as $2D^{-1} \int_a^{a+D} t^{-1} \sin tu\,du$. Hence

$$
\begin{aligned}
|g_0'(t)| &= \left|2D^{-1} \int_a^{a+D} t^{-2}(tu \cos tu - \sin tu)\,du\right| \\
&\leqq D^{-1} \int_a^{a+D} u^2\,du \\
&\leqq (a + D)^2 \\
&\leqq \text{const}\, B^3 .
\end{aligned}
$$

Here we used the elementary inequality $|\theta\cos\theta - \sin\theta| \leqq \theta^2/2$. A similar reasoning shows $|g_0''(t)| \leqq \text{const}\, B^{\frac{9}{2}}$ for all t. Now write $A \equiv 2^2 D^{-\frac{1}{2}} B^{\frac{3}{4}}$. Then $A > 1$, and we can use the bounds obtained to estimate the second summand in the last expression of (4.2):

$$\begin{aligned}(2\pi x^2)^{-1} \textstyle\int_0^\infty |g_0''(t)|\, dt &\leqq (2\pi x^2)^{-1} \text{const}\, \{\textstyle\int_0^A B^{\frac{9}{2}}\, dt + \int_A^\infty B^3 D^{-1} t^{-2}\, dt\} \\ &\leqq \text{const}\, B^{2\frac{1}{4}} D^{-\frac{1}{2}} x^{-2} \\ &= \text{const}\, B^{2\frac{1}{4}} n^{\frac{1}{12}} x^{-2}\,.\end{aligned}$$

We will show that the first summand in the last expression of (4.2) has the same bound. First we prove that for any interval $[\alpha, \beta]$

(i) $\sum_{t_j \in [\alpha,\beta]} |g_0'(t_j)| \leqq \text{const}\, B^{\frac{3}{2}} D^{-1} \alpha^{-2} + \text{const}\, (\beta - \alpha) B^3 D^{-1} \alpha^{-2}$ if $\alpha \geqq 1$,

(ii) $\sum_{t_j \in [\alpha,\beta]} |g_0'(t_j)| \leqq \text{const}\, B^3 + \text{const}\, (\beta - \alpha) B^{\frac{9}{2}}$ for all $\alpha \geqq 0$.

It suffices to give the proof for those t_j's in $[\alpha, \beta]$ for which $g_0(t_j) = +2n^{-1}B^3$. Denote these t_j's by $s_1 < \cdots < s_q$. Rolle's theorem implies

$$\begin{aligned}\textstyle\sum_{i=1}^q |g_0'(s_i)| &\leqq |g_0'(s_1)| + \textstyle\sum_{i=2}^q (s_i - s_{i-1}) \max_{[\alpha,\beta]} |g''| \\ &\leqq |g_0'(s_1)| + (\beta - \alpha) \max_{[\alpha,\beta]} |g''|\,.\end{aligned}$$

Combining this with the bounds obtained for $|g_0'|$ and $|g_0''|$, we have inequalities (i) and (ii). Now we can apply (ii) to the interval $[0, A + 1]$, and apply (i) to the intervals $[A + 1, A + 2]$, $[A + 2, A + 3]$, $\cdots$ to obtain

$$\begin{aligned}\textstyle\sum |g_0'(t_j)| &\leqq \text{const}\, \{B^3 + (A + 1) B^{\frac{9}{2}}\} \\ &\qquad + \text{const}\, \{B^{\frac{3}{2}} D^{-1} + B^3 D^{-1}\}\{(A + 1)^{-2} + (A + 2)^{-2} + \cdots\} \\ &\leqq \text{const}\, A B^{\frac{9}{2}} + \text{const}\, B^3 D^{-1} A^{-1} \\ &\leqq \text{const}\, B^{2\frac{1}{4}} n^{\frac{1}{12}}\,.\end{aligned}$$

So we see that $|G_0(x) - G_n(x)|$, along with the expressions in (4.2), is bounded by const $B^{2\frac{1}{4}} n^{\frac{1}{12}} x^{-2}$.

At the same time, if we write $A \equiv n^{\frac{1}{2}} D^{-\frac{1}{2}} B^{-\frac{3}{2}}$, then

$$\begin{aligned}|G_0(x) - G_n(x)| &\leqq (2\pi)^{-1} \textstyle\int_{-\infty}^\infty |g_0(t) - g_n(t)|\, dt \\ &\leqq \text{const}\, \{\textstyle\int_0^A 2n^{-1} B^3\, dt + \int_A^\infty 2D^{-1} t^{-2}\, dt\} \\ &\leqq \text{const}\, n^{-\frac{1}{2}} D^{-\frac{1}{2}} B^{\frac{3}{2}} \\ &\leqq \text{const}\, n^{-\frac{5}{12}} B^{\frac{3}{2}}\,.\end{aligned}$$

The lemma is proved.

PROOF OF LEMMA 2. Suppose $n \geqq 2^6 B^{\frac{3}{2}}$, $|g_n(t)| > 0$, and $|\hat{\phi}(t)| < 2^{-6} n^{-3} c^2$. We will deduce a contradiction. Since $g_n(-t) = g_n(t)$ and $|\hat{\phi}(-t)| = |\hat{\phi}(t)|$, we may assume $t \geqq 0$. From the definition of g_n, we have $|g_0(t)| > 2n^{-1}B^3$. But $|g_0(t)| \leqq 2Dt/Dt^2 = 2t^{-1}$ as evident from the definition of g_0. Hence $t < nB^{-3} < n$. Since $|\hat{\phi}(t)| < 2^{-6} n^{-3} c^2 < 2^{-6} \cdot 4^2 < 1$, we also have $t \geqq B^{-\frac{3}{2}}$. (See the proof of (2.4).) Next, let j be any integer. Observe that since $D = KL = 2\pi K/T$, we have $|g_0(t)| = 2D^{-1} t^{-2} |\cos at - \cos(at + Dt - DjT)| \leqq 2D^{-1} t^{-2} D |t - jT| \leqq 2B^3 |t - jT|$. Hence

$$|t - jT| \geqq 2^{-1} B^{-3} |g_0(t)| > n^{-1} \qquad j \text{ any integer.}$$

In particular, since $T > t + n^{-1}$ would imply (because of (2.2)) that $|\hat{\phi}(t)| > c > 2^{-6}n^{-3}c^2$, we must have $T < t - n^{-1} < t < n$. Hence $|\hat{\phi}(T)| < n^{-5}m^2$ according to (2.3). Now choose the integer j such that $jT < t < (j+1)T$. Then $t - jT < T - n^{-1}$. But, using the elementary inequality $(1 - \cos(\alpha + \beta))^{\frac{1}{2}} \leqq (1 - \cos\alpha)^{\frac{1}{2}} + (1 - \cos\beta)^{\frac{1}{2}}$, we have

$$\begin{aligned} \int 1 - \cos(t - jT)u \, dP(u) &\leqq \int ((1 - \cos tu)^{\frac{1}{2}} + j(1 - \cos Tu)^{\frac{1}{2}})^2 \, dP(u) \\ &\leqq \{(\int 1 - \cos tu \, dP(u))^{\frac{1}{2}} + j(\int 1 - \cos Tu \, dP(u))^{\frac{1}{2}}\}^2 \\ &\leqq \{(t|\hat{\phi}(t)|)^{\frac{1}{2}} + j(T|\hat{\phi}(T)|)^{\frac{1}{2}}\}^2 \\ &\leqq \{(n \cdot 2^{-6}n^{-3}c^2)^{\frac{1}{2}} + j(Tn^{-5}m^2)^{\frac{1}{2}}\}^2 \\ &\leqq \{2^{-3}n^{-1}c + (jT)T^{-\frac{1}{2}}n^{-\frac{5}{2}}m\}^2 \\ &\leqq \{2^{-3}n^{-1}m + (n)B^{\frac{3}{4}}n^{-\frac{5}{2}}m\}^2 \\ &\leqq \{2^{-3}n^{-1}m + 2^{-3}n^{-1}m\}^2 \qquad n > 2^6B^{\frac{3}{2}} \\ &= 2^{-4}n^{-2}m^2 \,. \end{aligned}$$

Hence, since $|t - jT| > n^{-1}$,

$$|\mathrm{Im}\,\hat{\phi}(t - jT)| = |(t - jT)^{-1} \int 1 - \cos(t - jT)u \, dP(u)| \leqq 2^{-4}n^{-1}m^2 < 2^{-1}m \,.$$

Likewise

$$\begin{aligned} |\int \sin(t - jT)u \, dP(u)| &\leqq 2^{\frac{1}{2}} \int (1 - \cos(t - jT)u)^{\frac{1}{2}} \, dP(u) \\ &\leqq 2^{\frac{1}{2}}\{\int 1 - \cos(t - jT)u \, dP(u)\}^{\frac{1}{2}} \\ &\leqq 2^{\frac{1}{2}}2^{-2}n^{-1}m \,. \end{aligned}$$

So

$$|\mathrm{Re}\,\hat{\phi}(t - jT)| = |(t - jT)^{-1} \int \sin(t - jT)u \, dP(u)| \leqq 2^{\frac{1}{2}}2^{-2}m < 2^{-1}m \,.$$

Combining, we have $|\hat{\phi}(t - jT)| < m \leqq \min\{|\hat{\phi}(s)| : 0 \leqq s \leqq T - n^{-1}\}$, contradicting the fact that $0 \leqq t - jT \leqq T - n^{-1}$.

Proof of Lemma 3. The proof is a modification of the proof of Lévy–Wiener's theorem for Fourier transforms of (complex valued) finite lattice measures on R, as given in [6], page 245.

Suppose $n \geqq 2^6B^{\frac{3}{2}}$ as in the hypothesis. Let $\eta \equiv \text{const}\, n^{-3}c^2$ so that $|\hat{\phi}(t)| \geqq 7\eta$ whenever $|g_n(t)| > 0$, as guaranteed by Lemma 2. Write $X \equiv b(\eta)$ and define $S(t) \equiv \int_{-X}^{X} e^{itx}\hat{F}(x)\,dx$. Let $0 < t_1 < \cdots < t_{2k} = \infty$ be numbers as in (4.1). Since t_{2k-1} will enter many times, we will write y for t_{2k-1}. Thus $y^2 = 2D^{-1}|\cos ay - \cos(a + D)y| \cdot 2nB^{-3} \leqq \text{const}\, nD^{-1} \leqq \text{const}\, n^{\frac{7}{8}}$. Because $|S(t) - \hat{\phi}(t)| < \eta$ by the definition of X, we see that

$$(4.3) \qquad |S(t)| > 6\eta \qquad \text{on} \quad (-t_{2k-1}, -t_{2k-2}) \cup \cdots \cup (-t_1, t_1) \cup \cdots \cup (t_{2k-2}, t_{2k-1}) \,.$$

From Cauchy's formula, we have

$$(4.4) \qquad g_n(t)/\hat{\phi}(t) = (2\pi)^{-1} \int_0^{2\pi} g_n(t)(S(t) + 2\eta e^{ip} - \hat{\phi}(t))^{-1}(S(t) + 2\eta e^{ip})^{-1}2\eta e^{ip} \, dp$$

where the integrand is taken to be 0 if $g_n(t) = 0$. The lemma would be proved if we could show that each of the three factors in the integrand is the Fourier transform of an integrable function on R. Only the third factor causes trouble. Standing alone, it is not even defined for all t. Fortunately (4.4) is still valid if we replace $S(t)$ by any modification $\bar{S}(t)$ which coincides with $S(t)$ when $g_n(t) \neq 0$.

For the construction of the modification $\bar{S}$, we note that $|S'(t)| \leqq X \int |\hat{F}(x)|\, dx \leqq X(2B+1) \leqq 3XB$, and similarly that $|S''(t)| \leqq 3X^2B$. With this observation, we can, by elementary means, construct a continuously differentiable function $\bar{S}$ on R, twice differentiable except at finitely many points, such that

$$
\begin{aligned}
&\text{(i)} && \bar{S} = S \quad \text{on} \quad (-t_{2k-1}, -t_{2k-2}) \cup \cdots \cup (t_{2k-2}, t_{2k-1}), \\
&\text{(ii)} && |\bar{S}| \geqq 4\eta \quad \text{on} \quad R, \\
&\text{(iii)} && |\bar{S}'| \leqq \text{const}\, XB, \quad \text{and} \quad |\bar{S}''| \leqq \text{const}\, X^2B^2\eta^{-1} \quad \text{on} \quad [-y, y], \\
(4.5) \quad &\text{(iv)} && \bar{S}(t) = S(y) + S'(y)(t-y) \\
& && \qquad + 6X^2B^2\eta^{-1}(t-y)^2 S(y)|S(y)|^{-1} \quad \text{if} \quad t \geqq y; \\
& && \bar{S}(t) = S(-y) + S'(-y)(t+y) \\
& && \qquad + 6X^2B^2\eta^{-1}(t+y)^2 S(-y)|S(-y)|^{-1} \quad \text{if} \quad t \leqq -y.
\end{aligned}
$$

As remarked earlier, equality (4.4) becomes

$$(4.6) \qquad g_n(t)/\hat{\phi}(t) = (2\pi)^{-1} \int_0^{2\pi} g_n(t) 2\eta e^{ip} (S(t) + 2\eta e^{ip} - \hat{\phi}(t))^{-1} (\bar{S}(t) + 2\eta e^{ip})^{-1}\, dp.$$

Consider the last factor in the integrand. From (4.3) and (4.5) we deduce that for $|t| \geqq y$,

$$
\begin{aligned}
(4.7) \qquad |\bar{S}(t)| &\geqq 6\eta + 6X^2B^2\eta^{-1}(|t| - y)^2 - 3XB(|t| - y) \\
&\geqq 4\eta + 4X^2B^2\eta^{-1}(|t| - y)^2.
\end{aligned}
$$

Hence $(\bar{S}(t) + 2\eta e^{ip})^{-1}$ is bounded in modulus by $(2\eta + 4X^2B^2\eta^{-1}(|t| - y)^2)^{-1}$ for $|t| \geqq y$. For $|t| \leqq y$ it is bounded by $(2\eta)^{-1}$ because of (4.5)(ii). Hence it is integrable. Its inverse Fourier transform

$$(4.8) \qquad G_p(x) \equiv (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itx} (\bar{S}(t) + 2\eta e^{ip})^{-1}\, dt$$

is bounded in modulus by

$$
\begin{aligned}
(4.9) \qquad & (2\pi)^{-1} 2\{y(2\eta)^{-1} + (4(2)^{\frac{1}{2}} XB)^{-1}\pi\} \\
&\qquad \leqq \text{const}\, \{y\eta^{-1} + (XB)^{-1}\} \leqq \text{const}\, n^{\frac{7}{12}}\eta^{-1}
\end{aligned}
$$

(For the last simplification we used the fact that $y \leqq \text{const}\, n^{\frac{7}{12}}$ and that $XB > 1$.) Moreover, since $\bar{S}'(t)$ is continuous on R and $\bar{S}''(t)$ exists except at finitely many points, we can integrate by parts twice in (4.8) and obtain

$$G_p(x) = -(2\pi x^2)^{-1} \int_{-\infty}^{\infty} e^{-itx} \{(\bar{S}(t) + 2\eta e^{ip})^{-1}\}''\, dt.$$

The last integrand is equal in modulus to

$$|2(\bar{S}(t) + 2\eta e^{ip})^{-3} \bar{S}'^2(t) - (\bar{S}(t) + 2\eta e^{ip})^{-2} \bar{S}''(t)|.$$

For $|t| \leqq y$, this function is bounded by const $X^2B^2\eta^{-3}$, thanks to (4.5)(ii)(iii). For $|t| \geqq y$, it is bounded, in view of (4.7), by

$$\text{const}\, \{(2\eta + 4X^2B^2\eta^{-1}(|t| - y)^2)^{-3} X^2B^2 + (2\eta + 4X^2B^2\eta^{-1}(|t| - y)^2)^{-2} X^2B^2\eta^{-1}\}.$$

Straightforward integration with respect to t yields

$$|G_p(x)| \leqq \text{const } x^{-2}\{yX^2B^2\eta^{-3} + XB\eta^{-2} + XB\eta^{-2}\} \leqq \text{const } X^2B^2\eta^{-3}n^{\frac{7}{12}}x^{-2} .$$

Combining this bound for $|G_p(x)|$ and the one in (4.9), we obtain via direct integration

$$\text{(4.10)} \qquad \begin{aligned} &\textstyle\int_{-\infty}^{\infty} |G_p(x)|\, dx \leqq \text{const } XB\eta^{-2}n^{\frac{7}{12}} , \\ &\textstyle\int_{|x|>\lambda_3(\alpha)} |G_p(x)|\, dx \leqq \alpha \qquad \text{for all } \alpha > 0 \end{aligned}$$

where $\lambda_3(\alpha) \equiv \text{const } X^2B^2\eta^{-3}n^{\frac{7}{12}}\alpha^{-1}$.

Let M be a positive real number and let λ be a positive decreasing function on $(0, \infty)$. For convenience we will say loosely that a function $f(t)$ has rate λ and total mass at most M if $f(t)$ is the Fourier transform of some (complex) integrable function $J(x)$ such that $\int_{-\infty}^{\infty} |J(x)|\, dx \leqq M$, and $\int_{|x|>\lambda(\alpha)} |J(x)|\, dx \leqq \alpha$ for all $\alpha > 0$. Thus (4.10) says $(\bar{S}(t) + 2\eta e^{ip})^{-1}$ has rate λ_3 and total mass at most $M_3 \equiv \text{const } XB\eta^{-2}n^{\frac{7}{12}}$. In general, if $a_1, \cdots, a_N$ are positive reals with $a_1 + \cdots + a_N \leqq 1$ and if $f_1(t), \cdots, f_N(t)$ have rates $\lambda_1, \cdots, \lambda_N$ and total masses at most $M_1, \cdots, M_N$ respectively, then

$$\text{(4.11)} \qquad f_1 + \cdots + f_N \qquad \text{has rate} \quad \lambda(\alpha) \equiv \lambda_1(a_1\alpha) \vee \cdots \vee \lambda_N(a_N\alpha) ,$$

$$\text{(4.12)} \qquad f_1 \cdots f_N \qquad \text{has rate} \quad \lambda(\alpha) \equiv N \textstyle\bigvee_{j=1}^{N} \lambda_j(\alpha(NM_1 \cdots \hat{M}_j \cdots M_N)^{-1}) ,$$

where the hat ^ signifies omission. For illustration we prove the assertion for the (ordinary) product $f_1 \cdots f_N$. Thus let $J_1, \cdots, J_N$ be integrable functions with Fourier transforms $f_1, \cdots, f_N$ respectively. Then $f_1 \cdots f_N$ is the Fourier transform of $J_1 * \cdots * J_N$, and

$$\begin{aligned} &\textstyle\int_{|x|>\lambda(\alpha)} |J_1 * \cdots * J_N(x)|\, dx \\ &\quad = \textstyle\int \cdots \int_{|x_1+\cdots+x_N|>\lambda(\alpha)} |J_1(x_1) \cdots J_N(x_N)|\, dx_1 \cdots dx_N \\ &\quad \leqq \textstyle\sum_{j=1}^{N} \int \cdots \int_{|x_j|>\lambda(\alpha)/N} |J_1(x_1) \cdots J_N(x_N)|\, dx_1 \cdots dx_N \\ &\quad \leqq \textstyle\sum_{j=1}^{N} M_1 \cdots \hat{M}_j \cdots M_N(\alpha(NM_1 \cdots \hat{M}_j \cdots M_N)^{-1}) \\ &\quad = \alpha . \end{aligned}$$

The first inequality holds because $|x_1 + \cdots + x_N| > \lambda(\alpha)$ implies $|x_j| > \lambda(\alpha)/N$ for some j. The second follows from the definition of λ and from Fubini's theorem.

Now we can look at the second factor in the integrand of (4.6). It can be written as

$$\text{(4.13)} \qquad \textstyle\sum_{j=0}^{\infty} (-2\eta e^{ip})^{-j}(S(t) - \hat{\phi}(t))^j .$$

But $(-2\eta e^{ip})^{-1}(S(t) - \hat{\phi}(t))$ is the Fourier transform of $(2\eta e^{ip})^{-1}\hat{F}(x)\chi_{|x|>X}$. Hence it has rate $\lambda(\alpha) \equiv b(2\eta\alpha)$ and, by definition of X, total mass at most $(\frac{1}{2})$. From (4.12), then, the jth summand in (4.13) has rate $jb(2^j j^{-1}\eta\alpha)$. Now let $\alpha > 0$ be given. Let N be an integer with $-\log_2 \alpha + 1 < N < -\log_2 \alpha + 3$. Using (4.11) with $a_j \equiv 3^{-1}2^{-j}j$, we see that the function $\sum_{j=0}^{N} (-2\eta e^{ip})^{-j}(S(t) - \hat{\phi}(t))^j$ is the

Fourier transform of an integrable function whose modulus has integral on $\{x: |x| > Nb(\eta\alpha/6)\}$ bounded by $\alpha/2$. We already know that $\sum_{N+1}^{\infty}(-2\eta e^{ip})^{-j}(S(t) - \hat{\phi}(t))^j$ has total mass $\leqq 2^{-N} < \alpha/2$. Combining, we see that the expression (4.13) is the Fourier transform of an integrable function whose modulus has integral over $\{x: |x| > (-\log_2 \alpha + 3)b(\eta\alpha/6)\}$ bounded by α. In other words, the second factor in the integrand of (4.6) has rate

$$\lambda_2(\alpha) \equiv \text{const}\,(\log_2 \alpha^{-1})b(\eta\alpha/6)\,, \tag{4.14}$$

and total mass at most $M_2 \equiv 2$.

Finally, the first factor $g_n(t)$ in the integrand of (4.6) is the Fourier transform of the function $G_n(x)$. From Lemma 1, $g_n(t)$ has rate $\lambda_1(\alpha) \equiv \text{const}\, B^{2\frac{1}{4}} n^{\frac{1}{12}} \alpha^{-1} \vee B^{\frac{3}{2}}$ and total mass at most $M_1 \equiv \text{const}\, B^{2\frac{7}{8}} n^{-\frac{1}{6}}$.

Summing up and using (4.12), we see that the integrand of (4.6) has rate

$$r(\alpha) \equiv 3\lambda_1(\alpha(3M_2M_3)^{-1}) \vee 3\lambda_2(\alpha(3M_1M_3)^{-1}) \vee 3\lambda_3(\alpha(3M_1M_2)^{-1})\,.$$

It is easy to check that the last expression is a continuous function of α, b, n, and c. Moreover, since r is independent of the integration variable p in (4.6), the integral $g_n(t)/\hat{\phi}(t)$ in (4.6) also has rate $r(\alpha)$. (This is in analogy of Lemma 5.5 (b) in [6], page 246.) This proves our lemma.

Proof of Lemma 4. Recall that $\hat{P}$ has density $a(x) - F(x)$ where $a(x) = 0$ if $x < 0$, and $a(x) = 1$ if $x > 0$. Thus $\hat{P} * Q$ is a measure on the real line with density $\sum_{j=1}^{\infty} \int [a(x-y) - F(x-y)]\, dF^{*j}(y) = F(x)$, which is also the density of $m_+ - \hat{P}$.

Remark. We have carried out a computation to find x_0 for the distribution P^γ in the introduction. By refining various steps of the proof, we find that $x_0 = 350$ will work for $\varepsilon = 0.1$. One can, of course, then obtain a smaller x_0 (indeed the smallest x_0) by examining Q^γ from 0 to 350.

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