THE PRODUCT SEMI-INVARIANTS OF THE MEAN AND A CENTRAL MOMENT IN SAMPLES

By Cecil C. Craig

The method developed by the author for calculating the semi-invariants and product semi-invariants of moments in samples from any infinite population[1] is not immediately applicable to the calculation of product semi-invariants of the mean and a central moment in such samples. In the present paper this method is adapted for this purpose so that the calculation of these product semi-invariants becomes routine. As it will be seen, the computing is a little heavier than in the case of central moments alone for results of equal weight. A table of results up to weight ten for the mean and the second, third and fourth central moments is given. The author plans to apply these to a further study of the sampling characteristics of the coefficient of variation and Fisher's t in samples from non-normal populations.

Let a random sample, $x_1, x_2, \cdots, x_N$ of N observations be drawn at random from an infinite population characterized by the semi-invariants, $\lambda_1, \lambda_2, \lambda_3, \cdots$. The sample mean is,

$$\bar{x} = \sum_{i=1}^{N} x_i/N,$$

and the n-th central moment of the sample is

$$m_n = \sum_{i=1}^{N} (x_i - \bar{x})^n/N.$$

Then the product semi-invariants of order kl of x and m_n, $S_{kl}(x, m_n)$, are defined by the formal identity in the parameters ϑ and ω:

$$(1)\qquad (S_{10}\vartheta + S_{01}\omega) + \frac{1}{2!}(S_{10}\vartheta + S_{01}\omega)^{(2)} + \frac{1}{3!}(S_{10}\vartheta + S_{01}\omega)^{(3)} + \cdots \equiv \log E(e^{\bar{x}\vartheta + m_n\omega}),$$

in which E denotes the mathematical expectation over the set of all such samples and

$$(S_{10}\vartheta + S_{01}\omega)^{(r)} = \sum_{j=1}^{r} \binom{r}{j} S_{j,r-j}(\bar{x}, m_n)\vartheta^j\omega^{r-j}.$$

---

[1] "An Application of Thiele's Semi-invariants to the Sampling Problem;" *Metron*, Vol. VII, part IV (1928), pp. 3–75.

If we denote $E(\bar{x}^k m_n^l)$ by M_{kl}, we have by definition the further formal identity in ϑ and ω:

$$E(e^{\bar{x}\vartheta + m_n\omega}) \equiv 1 + (M_{10}\vartheta + M_{01}\omega) + \frac{1}{2!}(M_{10}\vartheta + M_{01}\omega)^{(2)} + \cdots$$

in which $(M_{10}\vartheta + M_{01}\omega)^{(r)}$ is to be expanded in the same manner as $(S_{10}\vartheta + S_{01}\omega)^{(r)}$ above.

Let us write

$$\delta_i = x_i - \bar{x},$$

and then

$$E(e^{\bar{x}\vartheta + m_n\omega}) = E(e^{(\Sigma x_i)\vartheta/N + (\Sigma\delta_i^n)\omega/N}). \tag{2}$$

(Summations with respect to i and j always run from 1 to N.) Now we define a new set of product semi-invariants, $\lambda_{rst\cdots}$, of the sum Σx_i and the N δ_i's, by means of

$$(\lambda_{10}\vartheta + \Sigma\lambda_{0i}\omega_i) + \frac{1}{2!}(\lambda_{10}\vartheta + \Sigma\lambda_{0i}\omega_i)^{(2)} + \cdots \equiv \log E(e^{(\Sigma x_i)\vartheta + \Sigma\delta_i\omega_i}),$$

in which for example,

$$\left(\lambda_{10}\vartheta + \sum_{i=1}^{3}\lambda_{0i}\omega_i\right)^{(2)} = \lambda_{2000}\vartheta^2 + 2\lambda_{1100}\vartheta\omega_1 + 2\lambda_{1010}\vartheta\omega_2 + \cdots + \lambda_{0200}\omega_1^2 + \lambda_{0020}\omega_2^2 + \lambda_{0002}\omega_3^2.$$

We may set

$$\delta_i = \sum_{j=1}^{N} a_{ij}x_j \quad \text{with} \quad \begin{cases} a_{ij} = -\dfrac{1}{N}, & i \neq j \\ a_{ii} = \dfrac{N-1}{N} \end{cases}$$

Then

$$E(e^{(\Sigma x_i)\vartheta + \Sigma\delta_i\omega_i}) = E(e^{\sum_i x_i(\vartheta + \sum_j a_{ij}\omega_j)}) = E(e^{\alpha_1 x_1})\cdot E(e^{\alpha_2 x_2}) \cdots E(e^{\alpha_N x_N}),$$

in which

$$\alpha_i = \vartheta + \sum_j a_{ij}\omega_j.$$

It follows then that

$$(\lambda_{10}\vartheta + \Sigma\lambda_{0i}\omega_i) + \frac{1}{2!}(\lambda_{10}\vartheta + \Sigma\lambda_{0i}\omega_i)^{(2)} + \frac{1}{3!}(\lambda_{10}\vartheta + \Sigma\lambda_{0i}\omega_i)^{(3)} + \cdots \equiv \lambda_1\Sigma\alpha_i + \lambda_2\frac{\Sigma\alpha_i^2}{2!} + \lambda_3\frac{\Sigma\alpha_i^3}{3!} + \cdots,$$

from which

$$(\lambda_{10}\vartheta + \Sigma\lambda_{0i}\omega_i)^{(k+l)} = \lambda_{k+l}\sum_i (\vartheta + \sum_j a_{ij}\omega_j)^{k+l}.$$

From this

$$\lambda_{k00\cdots 0} = \lambda_{k0} = N\lambda_k\,,$$
$$\lambda_{k10\cdots 0} = \lambda_{k010\cdots 0} = \cdots = 0,$$

and generally,[2]

$$(3) \qquad \lambda_{kl_1l_2\cdots l_N} = \frac{\lambda_{k+l}}{N^l}[\Sigma(-1)^{l-l_i}(N-1)^{l_i}] \qquad (l_1 + l_2 + \cdots + l_N = l).$$

This is the first result to be used in calculating values of S_{kl}'s. Note that the value of $\lambda_{kl_1l_2\cdots l_N}$ is independent of the order in which a given set of l_i's occur.

Calculation of particular $\lambda_{kl_1l_2\cdots l_N}$'s in terms of N and the semi-invariants of the sampled population is both simple and rapid as one may see from a pair of examples:

$$\lambda_{22} = \lambda_{202} = \lambda_{2002} = \cdots$$

(suppressing superfluous zeros in the subscripts)

$$= \frac{\lambda_4}{N^2}[(N-1)^2 + (N-1)] = \frac{N-1}{N}\lambda_4\,.$$

Then, too,

$$\lambda_{k2} = \frac{N-1}{N}\lambda_{k+2}\,.$$

For a second example:

$$\lambda_{k+3} = \frac{\lambda_{k+7}}{N^7}[-(N-1)^4 + (N-1)^3 - (N-2)]$$
$$= -\frac{(N-2)(N^2-3N+3)}{N^6}\lambda_{k+7}\,.$$

Now the semi-invariants, S_{kl}, can be expressed directly in terms of the product moments, $\nu_{kl_1l_2\cdots l_N}$ of the sum Σk_i and the $N\delta$'s. These product moments are given by the appropriate moment generating function:

$$E(e^{(\Sigma x_i)\vartheta + \Sigma\delta_i\omega_i}) = 1 + (\nu_{10}\vartheta + \Sigma\nu_{0i}\omega_i) + \frac{1}{2!}(\nu_{10}\vartheta + \Sigma\nu_{0i}\omega_i)^{(2)} + \cdots.$$

[2] As written this result is valid if at least one of the l_i's is zero which is always the case if N, the size of the sample, is greater than l. (Cf. the author's paper cited above, p. 17.)

Then it is seen that,

$$E(e^{(\Sigma x_i)\vartheta+(\Sigma \delta_i^n)\omega}) = 1 + [\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega] + \frac{1}{2!}[\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega]^{(2)} + \cdots,$$

in which

$$[\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega]^{(2)}$$
$$= \nu_{20}\vartheta^2 + 2(\nu_{1n} + \nu_{10n} + \nu_{100n} + \cdots)\vartheta\omega + (\nu_{0,2n} + \nu_{00,2n} + \nu_{000,2n} + \cdots)\omega^2$$

etc. and by comparison with (1) and (2), we have

$$(S_{10}\vartheta + S_{01}\omega) + \frac{1}{2!}(S_{10}\vartheta + S_{01}\omega)^{(2)} + \cdots$$
$$\equiv \log\left\{1 + \frac{1}{N}[\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega] + \frac{1}{2!N^2}[\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega]^{(2)} + \cdots\right\}.$$

From this

$$(S_{10}\vartheta + S_{01}\omega)^{(k+l)}$$
$$\equiv \frac{1}{N^{k+l}}\Sigma\frac{(-1)^{p-1}(p-1)!(k+l)![\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega]^r\{[\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega]^{(2)}\}^s\cdots}{(1!)^r(2!)^s\cdots r!s!\cdots} \tag{4}$$

in which

$$r + s + t + \cdots = p,$$

the summation extending over all partitions $(1^r2^s3^t\cdots)$ of $k + l$. This, of course, is only the usual formula for semi-invariants in terms of moments appropriately modified. In particular,

$$(S_{10}\vartheta + S_{01}\omega)^{(2)} = \frac{1}{N^2}\{[\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega]^{(2)} - [\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega]^2\}.$$

If we write

$$[\nu_{10}\vartheta + (\Sigma\nu_{0,ni})\omega] = W$$

$$(S_{10}\vartheta + S_{01}\omega)^{(3)} = \frac{1}{N^3}(W^{(3)} - 3W^{(2)}W + 2W^3) \tag{5}$$

$$(S_{10}\vartheta + S_{01}\omega)^{(4)} = \frac{1}{N^4}[W^{(4)} - 4W^{(3)}W - 3(W^{(2)})^2 + 12W^{(2)}W^2 - 6W^4].$$

Now the $\nu_{kl_1l_2\cdots l_N}$'s can be replaced by their values in terms of the $\lambda_{kl_1l_2\cdots l_N}$'s, the details of which will be explained below, and it will be evident that any $\nu_{kl_1l_2\cdots l_N}$ is unaltered by a permutation of the l_i's in its subscript. Taking account of this, the formulae (5) may be written in the expanded forms:

$$S_{11}(\bar{x}, m_n) = \frac{1}{N}[\nu_{1n} - \nu_{10}\nu_{0n}]$$

$$S_{12}(\bar{x}, m_n) = \frac{1}{N^2}[\nu_{2n} - \nu_{20}\nu_{0n} - 2\nu_{1n}\nu_{10} + 2\nu_{10}^2\nu_{0n}]$$

$$S_{12}(\bar{x}, m_n) = \frac{1}{N^2}[\nu_{1,2n} + (N-1)\nu_{1nn} - \nu_{10}\nu_{0,2n} - (N-1)\nu_{10}\nu_{0nn} - 2N\nu_{1n}\nu_{0n} + 2N\nu_{10}\nu_{0n}^2].$$

But, with no loss in generality, the origin may be taken at the population mean so that $\lambda_1 = 0$. In this case it will be found that $\nu_{10} = 0$ and these formulae become:

$$S_{11}(\bar{x}, m_n) = \nu_{1n}/N$$

$$S_{21}(\bar{x}, m_n) = \frac{1}{N^2}[\nu_{2n} - \nu_{20}\nu_{0n}]$$

$$S_{12}(\bar{x}, m_n) = \frac{1}{N^2}[\nu_{1,2n} + (N-1)\nu_{1nn} - 2N\nu_{1n}\nu_{0n}]$$

$$S_{31}(\bar{x}, m_n) = \frac{1}{N^2}[\nu_{3n} - \nu_{30}\nu_{0n} - 3\nu_{1n}\nu_{20}] \tag{6}$$

$$S_{22}(\bar{x}, m_n) = \frac{1}{N^3}[\nu_{2,2n} + (N-1)\nu_{2nn} - 2N\nu_{2n}\nu_{0n} - \nu_{20}\nu_{0,2n} - (N-1)\nu_{20}\nu_{0nn} - 2N\nu_{1n}^2 + 2N\nu_{20}\nu_{0n}^2]$$

$$S_{13}(\bar{x}, m_n) = \frac{1}{N^3}[\nu_{1,3n} + 3(N-1)\nu_{1,2n,n} + (N-1)(N-2)\nu_{1nnn} - 3N\nu_{1,2n}\nu_{0n} - 3N(N-1)\nu_{1nn}\nu_{0n} - 3N\nu_{1n}\nu_{0,2n} - 3N(N-1)\nu_{1n}\nu_{0nn} + 6N^2\nu_{1n}\nu_{0n}^2].$$

These formulae are the second result used in the actual calculation of $S_{kl}(\bar{x}, m_n)$'s. One begins with them, putting in the particular value of n for the central moment in question. If for instance we wish to compute the product semi-invariants of the mean and variance in samples of N, we begin with the set of formulae:

$$S_{11}(\bar{x}, m_2) = \nu_{12}/N$$

$$S_{21}(\bar{x}, m_2) = \frac{1}{N^2}[\nu_{22} - \nu_{20}\nu_{02}] \tag{7}$$

$$S_{12}(\bar{x}, m_2) = \frac{1}{N^2}[\nu_{14} + (N-1)\nu_{122} - 2N\nu_{12}\nu_{02}],$$

etc.

The second step is to replace the product moments $\nu_{kl_1l_2\cdots l_N}$ which appear by their values in terms of the corresponding product semi-invariants. This process can perhaps be best explained by some examples.

Consider the complete calculation of $S_{12}(\bar{x}, m_2)$. From the expression for the fifth central moment in terms of semi-invariants:

$$\nu_5 = \lambda_5 + 10\lambda_3\lambda_2\,,$$

we can write the corresponding expression for product moments in terms of product semi-invariants

$$(\Sigma\nu_i\vartheta_i)^{(5)} \equiv (\Sigma\lambda_i\vartheta_i)^{(5)} + 10(\Sigma\lambda_i\vartheta_i)^{(3)}(\Sigma\lambda_i\vartheta_i)^{(2)}. \tag{8}$$

Then we get ν_{14} by comparing coefficients of $\dfrac{\vartheta_1\vartheta_2^4}{1!4!}$ and ν_{122} by comparing coefficients of $\dfrac{\vartheta_1\vartheta_2^2\vartheta_3^2}{2!2!}$ in this identity. For an index as low as 5, these coefficients are readily picked out by inspection; for larger indices the use of Hammond operators reduces this to a mechanical routine.[3] In this case we have

$$D_3D_2(14) = (12)(02) + (03)(11).$$

To the terms on the right the appropriate binomial coefficients must be applied giving

$$3(12)(02) + 2(03)(11).$$

The total of these coefficients is $5 = \dfrac{5!}{4!1!}$, a necessary check. Then multiplying these coefficients by $10/5$, we have

$$6\lambda_{12}\lambda_{02} + 4\lambda_{03}\lambda_{11}$$

for the required coefficients in the second term in (8). Thus

$$\nu_{14} = \lambda_{14} + (6\lambda_{12}\lambda_{02} + 4\lambda_{03}\lambda_{11}).$$

The two terms in parentheses arise from the same term in (8) and would both give rise to terms in $\lambda_3\lambda_2$ in the final result if λ_{11} were not identically zero from (3). In practice all terms in which λ_{k1} is a factor are crossed out as they appear. Next

$$D_3D_2(122) = 2(12)(02) + (111)(011) + 2(021)(11).$$

($\lambda_{002} = \lambda_{02}$; $\lambda_{012} = \lambda_{021}$.) With the binomial, or multinomial coefficients attached, the right member is rewritten

$$6(12)(02) + 12(111)(011) + 12(021)(11).$$

[3] Cf. the author, loc. cit., p. 24.

The total of these coefficients is $30 = \frac{5!}{2!2!1!}$. Then multiplying each coefficient by 10/30, we have

$$\nu_{122} = \lambda_{122} + (2\lambda_{12}\lambda_{02} + 4\lambda_{111}\lambda_{011} + 4\lambda_{012}\lambda_{11}).$$

Going on with the calculation of $S_{12}(\bar{x}, m_2)$:

$$\nu_{12} = \lambda_{12}, \qquad \nu_{02} = \lambda_{02},$$

and then we have:

$$S_{12}(\bar{x}, m_2) = \frac{1}{N^2}[\{\lambda_{14} + (N-1)\lambda_{122}\} + \{6\lambda_{12}\lambda_{02} + (N-1)(2\lambda_{12}\lambda_{02} + 4\lambda_{111}\lambda_{011}) - 2N\lambda_{12}\lambda_{02}\}].$$

The first set of terms within braces gives rise to terms in λ_5; the second to terms in $\lambda_3\lambda_2$. Next

$$\lambda_{14} = \frac{(N-1)(N^2 - 3N + 3)}{N^3}\lambda_5 \qquad \lambda_{111} = -\frac{\lambda_3}{N}$$

$$\lambda_{122} = \frac{2N-3}{N^3}\lambda_5 \qquad \lambda_{02} = \frac{N-1}{N}\lambda_2$$

$$\lambda_{03} = \frac{(N-1)(N-2)}{N^2}\lambda_3 \qquad \lambda_{011} = -\frac{\lambda_2}{N}.$$

$$\lambda_{021} = -\frac{(N-2)}{N^2}\lambda_3$$

This table of values will be of frequent use in further calculations of S_{kl}'s. Giving the values of both λ_{111} and λ_{011} here, was unnecessary duplication.

Now only the final reduction is to be carried out. We obtain

$$S_{12}(\bar{x}, m_2) = \frac{N-1}{N^4}[(N-1)\lambda_5 + 4N\lambda_3\lambda_2].$$

This result of order 3 and of weight 5 follows a quite mechanical procedure and is quite brief. The length of the algebraic computations required grows rapidly as the weight is increased but for weights no greater than 10 undue labor is not required. For greater weights only time and patience is required to get results if they are needed. It is to be noted that by this method one may calculate individual terms in the result without doing any of the work required for the remaining terms and that one may readily shorten the work by getting results to a desired degree of approximation with respect to powers of $1/N$.

There follows a table of the results so far calculated.

For $n = 2$:

$$S_{11} = \frac{N-1}{N} \lambda_3$$

$$S_{21} = \frac{N-1}{N^3} \lambda_4$$

$$S_{12} = \frac{N-1}{N^4} [(N-1)\lambda_5 + 4N\lambda_3\lambda_2]$$

$$S_{31} = \frac{N-1}{N^4} \lambda_5$$

$$S_{22} = \frac{N-1}{N^5} [(N-1)\lambda_6 + 4N(\lambda_4\lambda_2 + \lambda_3^2)]$$

$$S_{13} = \frac{N-1}{N^6} [(N-1)^2\lambda_7 + 12N(N-1)\lambda_5\lambda_2 + 4N(5N-7)\lambda_4\lambda_3 + 24N^2\lambda_3\lambda_2^2].$$

It is not difficult to see that in general

$$S_{k1}(\bar{x}, m_2) = \frac{N-1}{N^{k+1}} \lambda_{k+2}.$$

For $n = 3$:

$$S_{11} = \frac{(N-1)(N-2)}{N^3} \lambda_4$$

$$S_{21} = \frac{(N-1)(N-2)}{N^4} \lambda_5$$

$$S_{12} = \frac{(N-1)(N-2)}{N^6} [(N-1)(N-2)\lambda_7 + 9N(N-2)\lambda_5\lambda_2$$
$$+ 27N(N-2)\lambda_4\lambda_3 + 18N^2\lambda_3\lambda_2^2]$$

$$S_{31} = \frac{(N-1)(N-2)}{N^5} \lambda_6$$

$$S_{22} = \frac{(N-1)(N-2)}{N^7} [(N-1)(N-2)\lambda_8 + 9N(N-2)\lambda_6\lambda_2$$
$$+ 36N(N-2)\lambda_5\lambda_3 + 27N(N-2)\lambda_4^2 + 18N^2\lambda_4\lambda_2^2 + 36N^2\lambda_3^2\lambda_2]$$

$$S_{13} = \frac{(N-1)(N-2)}{N^{10}} [N(N-1)^2(N-2)^2\lambda_{10}$$
$$+ 9(N-1)(3N^4 - 12N^3 + 12N^2 - 5N + 5)\lambda_8\lambda_2$$
$$+ 27N(4N^4 - 21N^3 + 36N^2 - 20N + 3)\lambda_7\lambda_3$$
$$+ 27N^2(N-2)^2(7N-11)\lambda_6\lambda_4 + 54N^3(N-2)(4N-7)\lambda_6\lambda_2^2$$

$$+ 27N^2(N-2)^2(4N-7)\lambda_5^2 + 54N^3(N-2)(23N-50)\lambda_5\lambda_3\lambda_2$$
$$+ 162N^3(N-2)(5N-12)\lambda_4^2\lambda_2 + 54N^3(29N^2-126N+140)\lambda_4\lambda_3^2$$
$$+ 108N^4(5N-12)\lambda_4\lambda_2^3 + 324N^4(5N-12)\lambda_3^2\lambda_2^2].$$

For $n = 4$:

$$S_{11} = \frac{N-1}{N^4}[(N^2-3N+3)\lambda_5 + 6N(N-1)\lambda_3\lambda_2]$$

$$S_{21} = \frac{N-1}{N^5}[(N^2-3N+3)\lambda_6 + 6N(N-1)(\lambda_4\lambda_2+\lambda_3^2)]$$

$$\begin{aligned} S_{12} = \frac{N-1}{N^8}[&(N-1)(N^2-3N+3)^2\lambda_9 \\ &+ 4N(N^2-3N+3)(7N^2-18N+15)\lambda_7\lambda_2 \\ &+ 4N(N^2-3N+3)(19N^2-66N+63)\lambda_6\lambda_3 \\ &+ 4N(29N^4-195N^3+537N^2-639N+351)\lambda_5\lambda_4 \\ &+ 12N^2(17N^3-71N^2+117N-69)\lambda_5\lambda_2^2 \\ &+ 24N^2(35N^3-173N^2+309N-189)\lambda_4\lambda_3\lambda_2 \\ &+ 72N^2(N-2)^2(3N-5)\lambda_3^3 + 96N^3(4N^2-9N+6)\lambda_3\lambda_2^3] \end{aligned}$$

$$S_{31} = \frac{N-1}{N^6}[(N^2-3N+3)\lambda_7 + 6N(N-1)\lambda_5\lambda_2 + 18N(N-1)\lambda_4\lambda_3]$$

$$\begin{aligned} S_{22} = \frac{N-1}{N^9}[&(N-1)(N^2-3N+3)^2\lambda_{10} \\ &+ 4N(N^2-3N+3)(7N^2-18N+15)\lambda_8\lambda_2 \\ &+ 8N(N^2-3N+3)(13N^2-42N+39)\lambda_7\lambda_3 \\ &+ 12N(16N^4-106N^3+285N^2-360N+180)\lambda_6\lambda_4 \\ &+ 12N^2(17N^3-71N^2+117N-69)\lambda_6\lambda_2^2 \\ &+ 4N(29N^4-195N^3+537N^2-693N+351)\lambda_5^2 \\ &+ 48N^2(26N^3-125N^2+213N-129)\lambda_5\lambda_3\lambda_2 \\ &+ 24N^2(35N^3-173N^2+309N-189)\lambda_4^2\lambda_2 \\ &+ 24N^2(62N^3-326N^2+597N-369)\lambda_4\lambda_3^2 \\ &+ 96N^3(4N^2-9N+6)\lambda_4\lambda_2^3 + 288N^3(4N^2-9N+6)\lambda_3^2\lambda_2^2]. \end{aligned}$$

THE UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH.