A NOTE ON COMBINED INTERBLOCK AND INTRABLOCK ESTIMATION IN INCOMPLETE BLOCK DESIGNS[1]

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1. Introduction. In some experiments it is necessary to use incomplete block designs in order to keep the block size small. One example is the balanced incomplete block design [1], in which every variety occurs in r blocks and every pair of varieties occurs in λ blocks. Thus it is possible to estimate a variety difference $v_i - v_j$ in the λ blocks containing both varieties, but estimates of $v_i - v_j$ from any other blocks will be confounded with blocks. The former estimates (free from block effects) are the intrablock estimates denoted by $\hat{v}_i$ (or $\hat{v}_i - \hat{v}_j$ for the differences) and are obtained by minimizing

$$\sum(y_{ij} - \mu - v_i - b_j)^2,$$

where y_{ij} is the observation corresponding to variety i in block j and is an $N(\mu + v_i + b_j, \sigma^2)$ variate.

If the block effects b_j are random $N(0, \sigma_b^2)$ variates and are small, it is possible to extract information about variety differences from blocks which do not contain both varieties. This gives rise to interblock recovery of information and the interblock estimates v_i'' discussed in [9]. The interblock estimates are obtained by minimizing

$$\sum_j \{ \sum_i (y_{ij} - \mu - v_i) \}^2,$$

where the y_{ij} are $N(\mu + v_i, \sigma_b^2 + \sigma^2)$ variates. When the recovery of interblock information was first discussed [9], the interblock and intrablock estimates were combined to form the "best combined estimate" v_i', the linear combination of $\hat{v}_i$ and v_i'' having minimum variance; that is,

$$v_i' = \frac{\hat{v}_i(\text{var } v_i'') + v_i''(\text{var } \hat{v}_i)}{\text{var } \hat{v}_i + \text{var } v_i''}. \tag{1.1}$$

The variance of v_i' defined in this way is

$$(\text{var } v_i'')(\text{var } \hat{v}_i)/(\text{var } v_i'' + \text{var } \hat{v}_i).$$

However, it can be shown, [4], [8], that the best combined linear estimates (that is, estimates which are linear functions of the observations, are functions of intrablock and interblock information, and have minimum variance) can be found by minimizing

$$W \sum_{i,j} \left(y_{ij} - \frac{B_{j-v_i}}{k} + \frac{\sum_{(j)} v_i}{k} \right)^2 + \frac{W'}{k} \sum_j \{ \sum_i (y_{ij} - \mu - v_i \}^2, \tag{1.2}$$

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where $W = 1/\sigma^2$, $W' = 1/(k\sigma_b^2 + \sigma^2)$, and $\sum(j)v_i$ is the sum of the variety effects of varieties contained in block b_j. (Setting $W' = 0$ will give the intrablock equations and setting $W = 0$ will give the interblock equations.) Let the combined estimates found by minimizing (1.2) be denoted by v_i^*. There have thus been presented two possible methods of finding the best combined estimate of v_i; we shall denote them by Method 1 (yielding v_1') and Method 2 (yielding v_1^*). Method 1 was the one first used [9] and is used in [5] and [6], all in connection with the balanced incomplete block design.

It is the purpose of this paper to show that Methods 1 and 2 are not the same ($v_i' \neq v_i^*$) in general; therefore Method 1 does not of itself yield the best combined estimate. The conditions for $v_i' = v_i^*$ and $(v_i - v_j)' = v_i^* - v_j^*$ will be derived, and the resulting designs will be balanced or special cases of partially balanced incomplete block designs [3], such as group-divisible designs [2].

2. Formation of the estimates. Suppose that there are v varieties occurring in b blocks of k distinct varieties each, so that each variety occurs r times and varieties v_i and v_j occur together λ_{ij} times. The intrablock estimates $\hat{v}_i$ are obtained by minimizing

$$\sum\nolimits_{i,j} (y_{ij} - \mu - v_i - b_i)^2$$

with respect to μ, v_i, and b_j, subject to the restrictions $\sum v_i = \sum b_j = 0$. After eliminating the b_j and μ, the resulting equations are

$$c_i = kV_i - T_i = r(k-1)\hat{v}_i - \sum_{\mu \neq i} \lambda_{i\mu}\hat{v}_\mu,$$

where V_i is the sum of the observations containing variety v_i, and T_i is the sum of the block totals (B_j) of all blocks containing variety v_i. Since the v_i sum to zero, v_v can be replaced by $-(v_1 + v_2 + \cdots + v_{v-1})$, the resulting equation being

$$c_i = [r(k-1) + \lambda_{iv}]\hat{v}_i - \sum_{\mu \neq i}^{v-1} (\lambda_{i\mu} - \lambda_{iv})\hat{v}_\mu,$$

where $c_i = kV_i - T_i$ as before. This is a set of $(v-1)$ equations for $\hat{v}_1$, $\hat{v}_2$, $\cdots$, $\hat{v}_{v-1}$, and they can be written in the matric form

$$\begin{aligned} C &= A\hat{V}, \qquad \text{where } A = (a_{ij}), \\ a_{ij} &= -(\lambda_{ij} - \lambda_{iv}), \qquad\qquad (2.1) \\ a_{ii} &= r(k-1) + \lambda_{iv}. \end{aligned}$$

Similarly, the interblock estimates v_i'' are obtained by minimizing

$$\sum_j (B_j - k\mu - \sum_{(j)} v_i)^2, \qquad (\textstyle\sum v_i = 0),$$

where B_j is the sum of the observations for block b_j and $\sum_{(j)}v_i$ is the sum of the variety effects of varieties contained in block b_j. The resulting equations are

$$\tilde{c}_i = rv_i'' \sum_{\mu \neq i} \lambda_{i\mu} v_\mu'',$$

where $\tilde{c}_i = T_i - rk\hat{\mu}$, $rv\hat{\mu} = \sum y_{ij}$. Since $\sum v_i'' = 0$, we get, as before,

$$\tilde{c}_i = (r - \lambda_{iv})v_i'' + \sum_{\mu \neq i}^{v-1} (\lambda_{i\mu} - \lambda_{iv})v_\mu'',$$

This is a set of $(v - 1)$ equations for $v_1'', v_2'', \cdots, v_{v-1}''$, and they can be written in matric form:

$$\begin{cases} \tilde{C} = \tilde{A}V'', & \text{where } \tilde{A} = (\tilde{a}_{ij}), \\ \tilde{a}_{ij} = (\lambda_{ij} - \lambda_{iv}), \\ \tilde{a}_{ii} = r - \lambda_{iv}. \end{cases} \tag{2.2}$$

Using Method 2 the combined estimates v_1^* are found by minimizing

$$W \sum_{i,j} \left(y_{ij} - \frac{B_j}{k} - v_i + \frac{\sum_{(j)} v_i}{k} \right)^2 + \frac{W'}{k} \sum_j (B_j - k\mu - \sum_{(j)} v_i)^2.$$

The resulting equations for $v_1^*, v_2^*, \cdots, v_{v-1}^*$ can be shown to be

$$W(C - AV^*) + W'(\tilde{C} - \tilde{A}V^*) = 0. \tag{2.3}$$

Using Method 1, the best linear combination of the estimates $\hat{V}$ and V'' is

$$V' = (K_i)\hat{V} + (1 - K_i)V'',$$

where (K_i) is the diagonal matrix

$$\left(\frac{\text{var } v_i''}{\text{var } v_i'' + \text{var } \hat{v}_i} \right).$$

3. Condition for the equivalence of Methods 1 and 2 for all variety estimates. If Methods 1 and 2 both produce the same combined estimate then

$$V^* = V' = (K_i)\hat{V} + (1 - K_i)V''.$$

Substituting this expression for V^* into (2.3), we get

$$W\{C - A(K_i)\hat{V} - A(1 - K_i)V''\} + W'\{\tilde{C} - \tilde{A}(K_i)\hat{V} - \tilde{A}(1 - K_i)V''\} = 0.$$

Noting that $C = A\hat{V}$ and $\tilde{C} = \tilde{A}V''$, this becomes

$$W\{AV - A(K_i)\hat{V} - A(1 - K_i)V''\} + W'\{\tilde{A}V'' - \tilde{A}(K_i)\hat{V} - \tilde{A}(1 - K_i)V''\} = 0;$$

that is,

$$[WA(1 - K_i) - W'\tilde{A}(K_i)]\hat{V} - [WA(1 - K_i) - W'\tilde{A}(K_i)]V'' = 0.$$

Because the intrablock estimates are statistically independent of the interblock estimates, there cannot be a linear relation connecting the $\hat{v}$ and the v''. Hence

$$[WA(1 - K_i) - W'\tilde{A}(K_i)]\hat{V} = 0.$$

But as $v_1, v_2, \cdots, v_{v-1}$ are linearly independent, there cannot be a linear relation involving $\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_{v-1}$. Therefore

$$WA(1 - K_i) = W'\tilde{A}(K_i).$$

Since W and W' are scalar constants, multiplying the matrices together and equating the corresponding entries gives

$$Wa_{ij}(1 - K_j) = W'\tilde{a}_{ij}K_j,$$

where K_j and $(1 - K_j)$ are entries from the corresponding diagonal matrices; consequently, $a_{ij} = m_j\tilde{a}_{ij}$, where m_j is a constant depending only on j. Substituting the appropriate expressions from (2.1) and (2.2) for a_{ij} and $\tilde{a}_{ij}$ gives

$$-(\lambda_{ij} - \lambda_{iv}) = m_j(\lambda_{ij} - \lambda_{iv});$$

that is,

$$\lambda_{ij} = \lambda_{iv} \text{ (all } i) \text{ or } m_j = -1.$$

However, we also have $a_{jj} = m_j\tilde{a}_{jj}$, that is,

$$r(k - 1) + \lambda_{jv} = m_j(r - \lambda_{jv}),$$

and therefore $m_j = -1$ is impossible. Hence

$$\lambda_{ij} = \lambda_{iv} \qquad \text{for all } i \text{ and } j. \tag{3.1}$$

But

$$\sum_{\substack{\mu=1\\ \mu\neq i}}^{v} \lambda_{i\mu} = r(k - 1); \qquad \sum_{\substack{\mu=1\\ \mu\neq i}}^{v-1} \lambda_{i\mu} = r(k - 1) - \lambda_{iv}$$

$$= \sum_{\substack{\mu=1\\ \mu\neq i}}^{v-1} \lambda_{iv} \qquad \text{because of (3.1)}$$

$$= (v - 2)\lambda_{iv}.$$

Thus $\lambda_{iv} = \lambda = r(k - 1)/(v - 1) =$ constant for all i. Therefore all

$$\lambda_{ij} = r(k - 1)/(v - 1) = \lambda,$$

and the design is completely balanced. Consequently, only for balanced incomplete blocks designs is $v_i' = v_i^*$ for all varieties.

4. Estimates of variety differences.

THEOREM. *In an incomplete block design $(v, b, r, k, \lambda_{ij})$ a necessary and sufficient condition that there exist a subset of varieties $v_1, v_2, \cdots, v_a$, such that $(v_i - v_j)' = (v_i^* - v_j^*)$ for $i, j = 1, 2, \cdots, a$, is that all pairs v_i, v_j occur together a constant number λ_m of times, and that any other variety v_μ occur a constant number λ_μ of times with $v_1, v_2, \cdots, v_a$.*

PROOF. First, equations analogous to those of Section 2 must be derived for the intrablock and interblock estimates of variety differences. Thus

$$c_i = r(k - 1)\hat{v}_i - \sum\lambda_{i\mu}\hat{v}_\mu;$$

and as $\sum \lambda_{i\mu} = r(k - 1)$, we have

$$0 = r(k-1)\hat{v}_a - \sum \lambda_{i\mu}\hat{v}_a .$$

Subtracting,

$$\begin{aligned} c_i &= r(k-1)(\hat{v}_i - \hat{v}_a) - \sum_{\mu \neq i} \lambda_{i\mu}(\hat{v}_\mu - \hat{v}_a) \\ &= r(k-1)\hat{\delta}_i - \sum_{\mu \neq i} \lambda_{i\mu}\hat{\delta}_\mu , \end{aligned}$$

where $\hat{v}_i - \hat{v}_a = \hat{\delta}_i$. Therefore

$$\begin{aligned} c_1 - c_i &= r(k-1)(\hat{\delta}_1 - \hat{\delta}_i) - \sum_{\mu \neq 1} \lambda_{1\mu}\hat{\delta}_\mu + \sum_{\mu \neq i} \lambda_{i\mu}\hat{\delta}_\mu \\ &= [r(k-1) + \lambda_{1i}](\hat{\delta}_1 - \hat{\delta}_i) - \sum_{\mu \neq 1, i} (\lambda_{1i} - \lambda_{i\mu})\hat{\delta}_\mu . \end{aligned}$$

This is a set of $v - 1$ equations for $\hat{\delta}_1, \cdots, \hat{\delta}_{a-1}, \hat{\delta}_{a+1}, \cdots \hat{\delta}_v$; they can be written in the partitioned matric form

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} \hat{\Delta}_1 \\ \hat{\Delta}_2 \end{pmatrix}, \tag{4.1}$$

where D_1 is the column vector $(c_1 - c_i)$, $i = 2, 3, \cdots, a$, and D_2 is the column vector $(c_1 - c_i)$, $i = a + 1, a + 2, \cdots, v$; $\hat{\Delta}_1$ and $\hat{\Delta}_2$ are the corresponding column vectors of estimates; also,

$$\begin{aligned} A_1 = (a^1_{ij}), \quad & \text{where } a^1_{ii} = -[r(k-1) + \lambda_{1i}], \\ & a^1_{1i} = [r(k-1) + \lambda_{1i}], \\ & a^1_{i\mu} = -(\lambda_{1\mu} - \lambda_{i\mu}), \qquad \mu, i = 2, 3, \cdots, a, \end{aligned}$$

(note that $\delta_a \equiv 0$);

$$\begin{aligned} A_2 = (a^2_{ij}), \quad & \text{where } a^2_{i\mu} = -(\lambda_{1\mu} - \lambda_{i\mu}), \\ & i = 2, 3, \cdots, a, \ \mu = a + 1, a + 2, \cdots, v \\ A_3 = (a^3_{ij}), \quad & \text{where } a^3_{1i} = [r(k-1) + \lambda_{i1}], \\ & a^3_{i\mu} = -(\lambda_{1\mu} - \lambda_{i\mu}), \\ & i = a + 1, a + 2, \cdots, v, \ \mu = 2, 3, \cdots, a. \\ A_4 = (a^4_{ij}), \quad & \text{where } a^4_{ii} = -[r(k-1) + \lambda_{i1}], \\ & a^4_{i\mu} = -(\lambda_{1\mu} - \lambda_{i\mu}), \\ & i, \mu = a + 1, \cdots, v. \end{aligned}$$

The corresponding equations for the interblock estimates can be formed:

$$\begin{aligned} \tilde{c}_i &= rv''_i + \sum_{\mu \neq i} \lambda_{i\mu} v''_\mu , \\ rkv''_a &= rv''_a + \sum_{\mu \neq i} \lambda_{i\mu} v''_a , \end{aligned}$$

since $\Sigma\lambda_{i\mu} = r(k - 1)$. Therefore,

$$\tilde{c}_i - rkv''_a = r\delta''_i + \sum_{\mu \neq i} \lambda_{i\mu}\delta''_\mu,$$

and

$$\tilde{c}_1 - \tilde{c}_i = (r - \lambda_{1i})(\delta''_1 - \delta''_i) + \sum_{\mu \neq i,1} (\lambda_{1\mu} - \lambda_{i\mu})\delta''_\mu.$$

Writing this in matric form,

$$\begin{pmatrix} \tilde{D}_1 \\ \tilde{D}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{pmatrix} \begin{pmatrix} \Delta''_1 \\ \Delta''_2 \end{pmatrix}. \tag{4.2}$$

where the $\tilde{A}$ matrices are formed from the corresponding A matrices by replacing $r(k - 1)$ by r and λ_{ij} by $-\lambda_{ij}$. Thus, for example,

$$\tilde{A}_1 = (\tilde{a}^1_{ij}), \qquad \text{where } \tilde{a}^1_{ii} = -(r - \lambda_{1i})$$
$$\tilde{a}^1_{i1} = (r - \lambda_{1i})$$
$$\tilde{a}^1_{i\mu} = (\lambda_{1\mu} - \lambda_{i\mu}), \qquad i, \mu = 2, 3, \cdots, a;$$

also, $A_2 = -\tilde{A}_2$.

Using Method (2), the combined estimates δ^*_i satisfy the equations (analogous to (2.3)):

$$W\left\{\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} - \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} \Delta^*_1 \\ \Delta^*_2 \end{pmatrix}\right\} + W'\left\{\begin{pmatrix} \tilde{D}_1 \\ \tilde{D}_2 \end{pmatrix} - \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{pmatrix}\begin{pmatrix} \Delta^*_1 \\ \Delta^*_2 \end{pmatrix}\right\} = 0. \tag{4.3}$$

The theorem requires that $(v_1 - v_i)' = (v^*_1 - v^*_i)$ for $i \leq a$; that is,

$$\Delta^*_1 = (K_i)\hat{\Delta}_1 + (1 - K_i)\Delta''_1,$$

where (K_i) is the diagonal matrix

$$\left(\frac{(\hat{v}_1 - \hat{v}_i)\operatorname{var}(v''_1 - v''_i) + (v''_1 - v''_i)\operatorname{var}(\hat{v}_1 - \hat{v}_i)}{\operatorname{var}(\hat{v}_1 - \hat{v}_i) + \operatorname{var}(v''_1 - v''_i)}\right).$$

This expression can be substituted into (4.3) and the matrices multiplied together. Noting from (4.1) and (4.2) that

$$D_1 = A_1\hat{\Delta}_1 + A_2\hat{\Delta}_2, \qquad D_2 = A_3\hat{\Delta}_1 + A_4\hat{\Delta}_2,$$

and

$$\tilde{D}_1 = \tilde{A}_1\Delta''_1 + \tilde{A}_2\Delta''_2, \qquad \tilde{D}_2 = \tilde{A}_3\Delta''_1 + \tilde{A}_4\Delta''_2,$$

and rearranging the terms as in Section 3, we get

$$\begin{aligned} &\{WA_1(1 - K_i) - W'\tilde{A}_1(K_i)\}(\hat{\Delta}_1 - \Delta''_1) + WA_2\hat{\Delta}_2 + W'\tilde{A}_2\Delta''_2 \\ &\qquad - (WA_2 + W'\tilde{A}_2)\Delta^*_2 = 0, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\{WA_3(1 - K_i) - W'\tilde{A}_3(K_i)\}(\hat{\Delta}_1 - \Delta''_1) + WA_4\hat{\Delta}_2 + W'\tilde{A}_4\Delta''_2 \\ &\qquad - (WA_4 + W'\tilde{A}_4)\Delta^*_2 = 0. \end{aligned} \tag{4.5}$$

These equations must hold for arbitrary W and W'. Because $A_2 = -\tilde{A}_2$, letting $W = W'$ eliminates the term Δ_2^* from (4.4). Hence $A_2 = \tilde{A}_2 = 0$, for otherwise there would be a linear relation connecting $\hat{\Delta}_2$ and Δ_2''. Equation (4.4) now has the form of those equations considered in Section 3; thus

$$a_{ij}^1 = m_j \tilde{a}_{ij}^1 .$$

Combining these results gives

$$\lambda_{1\mu} = \lambda_{i\mu}, \qquad i = 1, 2, \cdots, a, \mu = a+1, a+2, \cdots, v$$

and

$$\lambda_{1\mu} = \lambda_{i\mu}, \qquad i, \mu = 2, 3, \cdots, a.$$

However, the choice of variety 1 in forming the equations involving $c_1 - c_i$ was arbitrary, any variety v_j in the set $v_1, v_2, \cdots, v_a$ being possible. Consequently, any two varieties v_i and v_j from this set occur a constant number λ_m of times together, and any other variety v_μ occurs a constant number λ_μ of times with each of the varieties $v_1, v_2, \cdots, v_a$.

SPECIAL CASES.

THEOREM. *In a partially balanced incomplete block design, a necessary and sufficient condition for $(v_i - v_j)' = v_i^* - v_j^*$ for any two mth associates v_i and v_j is $p_{ij}^m = 0$; that is, the matrix P_m is the diagonal matrix (p_{ii}^m).*

PROOF. Let v_1 and v_2 be mth associates; hence they occur λ_m times together. The ith associates of v_1 occur λ_i times with v_1 and therefore by the preceding theorem must also occur λ_i times with v_2, and so are ith associates of v_2. Thus the number of ith associates common to v_1 and v_2 is the total number of ith associates of v_1 (or v_2); that is,

$$p_{ii}^m = n_i .$$

Since

$$\sum_{j=1}^{m} p_{ij}^m = n_1(i \neq m), \qquad \sum_{j=1}^{m} p_{mj}^m = n_m - 1,$$

we have $p_{ij}^m = 0$ for $i \neq j$.

Conversely, if $p_{ij}^m = 0$, then the number of ith associates common to both v_1 and v_2 (where v_1 and v_2 are mth associates) is n_i, so that any ith associate of v_1 is also an ith associate of v_2. This means that any v_i not an mth associate of v_1 or v_2 occurs λ_i times with v_1 and v_2 and with all other varieties in the mth associate class. Hence by the preceding theorem the difference between any two mth associates can be estimated either by Method (1) or (2).

COROLLARY. *If there are only two associate classes, the resulting design is group divisible.*

PROOF. $p_{ij}^2 = 0(i \neq j)$. Therefore,

$$P_2 = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 - 1 \end{pmatrix},$$

which is sufficient to ensure group divisibility [2].

For the group-divisible design, using the usual notation [5], if v_i and v_j are second associates, var $(v_i'' - v_j'') = 2kB_{22}''/W'\Delta'' = 2k/W'A_{12}''$, since $p_{12}^2 = 0$, and var $(\hat{v}_i - \hat{v}_j) = 2k/WA_{12}$. The variance of the combined estimate as found by Method (1) is, consequently,

$$\frac{(2k/W'A_{12}'')(2k/WA_{12})}{2k/W'A_{12}'' + 2k/WA_{12}} = \frac{2k}{WA_{12} + W'A_{12}''} = \frac{2k}{A_{12}'} = \frac{2kB_{22}'}{\Delta'},$$

which is the variance of the combined estimate as found by Method (2). It can easily be verified that this is not true for first-associate variety differences.

Consider the case of four associate classes, where $p_{ij}^4 = 0$ $(i \neq j)$. Using the notation of [7], and noting that in this case $A_{24} = A_{34} = A_{44} = 0$ ([7], p. 134), the variance of the combined estimate found by Method (2) is

$$\text{Var } (v_i^* - v_j^*) = 2k/A_{14}' = 2k/(WA_{14} + W'A_{14}''),$$

where v_i and v_j are 4th associates, and $A_{14}'' = r - \lambda_4$. Thus

$$\begin{aligned} 2k/A_{14}' &= 2k/(WA_{14} + W'A_{14}'') \\ &= \frac{(2k/WA_{14})(2k/W'A_{14}'')}{2k/NA_{14} + 2k/W'A_{14}''}, \end{aligned}$$

which is the variance of the combined estimate as found by Method (1), since var $(\hat{v}_i - \hat{v}_j) = 2k/WA_{14}$ ([7], pp. 129–130), and by analogy var $(v_i'' - v_j'') = 2k/W'A_{14}''$.

5. Estimates of subsets of varieties. Section 4 showed that it is possible sometimes to obtain combined estimates of certain subsets of variety differences by either Method (1) or Method (2). Obviously, if Methods (1) and (2) give the same results for all variety differences, the design is completely balanced. This introduces the question as to whether there exists a design for which it is possible to obtain the best combined estimate either by Method (1) or Method (2) for a subset of varieties and not for the remaining varieties.

THEOREM. *If, in an incomplete block design* $(v, b, r, k, \lambda_{ij})$, $v_i' = v_i^*$ *for* $1, 2, \cdots, a$, *then all varieties occur the same number of times with varieties* $v_1, v_2, \cdots, v_a$.

PROOF. Equation (2.3) can be written, using the methods of Section 4,

$$W\left\{\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}\right\} + W'\left\{\begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} - \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{pmatrix}\begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}\right\} = 0,$$

where

$$C_1 = A_1\hat{V}_1 + A_2\hat{V}_2, \qquad C_2 = A_3\hat{V}_1 + A_4\hat{V}_2,$$

$$\tilde{C}_1 = \tilde{A}_1V_1'' + \tilde{A}_2V_2'', \qquad \tilde{C}_2 = \tilde{A}_3V_1'' + \tilde{A}_4V_2'',$$

and $V_1^* = V_1' = (K_i)\hat{V}_1 + (1 - K_i)V_1''$. Substituting these values back into the equation, we get

$$\begin{aligned}[WA_1(1 - K_i) - W'\tilde{A}_1(K_i)]\hat{V}_1 - [WA_1(1 - K_i) - W'\tilde{A}_1(K_i)]V_1'' \\ + WA_2\hat{V}_2 + W'\tilde{A}_2V_2'' - (WA_2 + W'\tilde{A}_2)V_2^* = 0,\end{aligned} \tag{5.1}$$

$$\begin{aligned}[WA_3(1 - K_i) - W'\tilde{A}_3(K_i)]\hat{V}_1 - [WA_3(1 - K_i) - W'\tilde{A}_3(K_i)]V_1'' \\ + WA_4\hat{V}_2 + W'\tilde{A}_4V_2'' - (WA_4 + W'\tilde{A}_4)V_2^* = 0.\end{aligned} \tag{5.2}$$

As in Section 4, these equations must hold for all W and W', and hence must be true for $W = W'$. When this is so, the term in V_2^* disappears in (5.1), since $A_2 = -\tilde{A}_2$. But $\hat{V}_1$, $\hat{V}_2$, V_1'', and V_2'' cannot be linearly related, and so $A_2 = \tilde{A}_2 = 0$; because these matrices do not contain W or W', they must therefore always be 0. Then

$$WA_1(1 - K_i) = W'\tilde{A}_1(K_i),$$

and the same arguments as those used in Sections 3 and 4 can be applied. Thus

$$\lambda_{ij} = \lambda_{iv}, \qquad i = 1, 2, \cdots, a, j = a + 1, a + 2, \cdots, v,$$

and

$$\lambda_{ij} = \lambda_{iv}, \qquad i = 1, 2, \cdots, a, j = 1, 2, \cdots, a.$$

This means that any variety v_μ occurs a constant number of times with v_1, v_2, $\cdots$, v_a, since we can show (as at the end of Section 3) that λ_{iv} = constant for $i \leqq a$.

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