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ROTATION AND SCALE SPACE RANDOM FIELDS AND THE GAUSSIAN KINEMATIC FORMULA

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We provide a new approach, along with extensions, to results in two important papers of Worsley, Siegmund and coworkers closely tied to the statistical analysis of fMRI (functional magnetic resonance imaging) brain data. These papers studied approximations for the exceedence probabilities of scale and rotation space random fields, the latter playing an important role in the statistical analysis of fMRI data. The techniques used there came either from the Euler characteristic heuristic or via tube formulae, and to a large extent were carefully attuned to the specific examples of the paper.

This paper treats the same problem, but via calculations based on the so-called Gaussian kinematic formula. This allows for extensions of the Worsley–Siegmund results to a wide class of non-Gaussian cases. In addition, it allows one to obtain results for rotation space random fields in any dimension via reasonably straightforward Riemannian geometric calculations. Previously only the two-dimensional case could be covered, and then only via computer algebra.

By adopting this more structured approach to this particular problem, a solution path for other, related problems becomes clearer.

1. Introduction. In this paper we provide a new approach, along with extensions, to results in two important papers closely tied to the statistical analysis of fMRI (functional magnetic resonance imaging) brain data. The first, by David Siegmund and the late Keith Worsley [10], treated the problem of testing for a spatial signal with unknown location and scale perturbed by a Gaussian random field on $\mathbb{R}^N$. While their motivation came from fMRI, their results are also relevant to areas such as astronomy and genetics, or indeed to many applications in which one searches a large space of noisy data for a relatively small number of signals which manifest as "bumps" in the random noise.

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In a second paper, together with Shafie and Sigal [9], they treated an extended version of the same basic problem, allowing the asymmetric bumps to also have a direction associated to them, in addition to scale changes that were direction dependent. This added an additional "rotation parameter" to the model and complicated it to the extent that they were unable to treat anything beyond the two-dimensional ($N = 2$) case. Thus, for example, while their results could be used to examine cortical surface images, they could not be applied to three-dimensional fMRI analysis.

For etymological reasons, the first of these cases was called the *scale space* case and the second the *rotation space* case. In both cases, the mathematics of these papers concentrated on approximating the exceedence probabilities

$$\mathbb{P}\Big\{\sup_{t\in M} f(t) \geq u\Big\} \tag{1.1}$$

for large u, a smooth, Gaussian, real-valued, random field f, and some parameter space M. For reasons that will become clearer soon, in the N-dimensional scale space case, the effective dimension of M was $N + 1$, while in the rotation space case it jumped to $N + N(N + 1)/2$ (or 5, for the only case they could handle, when $N = 2$).

The aims of the current paper are twofold. First, we shall generalize the results of the rotation space to any dimension, but by a very different method. Rather than taking a "from first principles" approach, as was the case in [9], we shall compute everything from a basic result known as the *Gaussian kinematic formula* [GKF, cf. (2.14) below], using techniques of Riemannian geometry. In fact, we shall derive the rotation space results from a more general setting, which is of interest in itself. This is quite different from the analysis in [9], in which calculations were so involved that they could only be handled by symbolic calculus using computers and, then, as already mentioned, only in two dimensions. According to the "no free lunch" principle, there is a price to pay here, in that the specific $N = 2$ approximations of [9] have a slightly higher level of precision.

Second, we shall recover the results of the scale space case, again using the GKF. We shall do this in two ways. First, we shall obtain it as an immediate corollary to the "more general setting" mentioned above. This is quick, but once again does not give the best results. However, we shall also derive the "best" results from a more detailed calculation, something which seems too difficult to do for rotation space fields.

At first this might seem as if we are attempting to kill the proverbial ant with a sledge hammer, but it turns out that there are at least three advantages to this approach. First of all, the ant in question is a rather large one, and any way of dealing with it requires considerable effort. The basic calculations in both [10] and here take about 10 pages, and the essential difference is whether one prefers to work with Riemannian topology or integral geometry, hard calculus and occasionally symbolic (machine) calculus. This may be more a question of mathematical taste than anything else. The second advantage of using the GKF as the basic starting point, which is significant and no longer merely an issue of taste, is that while the results of [9] and [10] apply, as stated, only to Gaussian fields, the GKF allows immediate extension to a much wider class of examples. Section 2.5 explains this.

Finally, both of these cases are instructive in terms of how to use the GKF in more general settings, and our hope is that presenting them in detail will provide examples which will make the GKF easier to use in the future for other, perhaps quite different, examples.

We shall say no more in the way of motivation for this paper. Discussions and examples can be found in both [9] and [10] as well as the papers cited there, and the many more recent papers citing [9] and [10]. In particular, it is explained there why (1.1) is an important quantity in statistical hypothesis testing. Rather, we shall, in the following section, immediately turn to formal definitions of scale and rotation random fields, a description of the GKF and its relation to [9] and [10], and a statement of the main results of the paper in Section 2.6. Fuller results for scale space fields, which go considerably beyond those of Section 2.6, are given at the end of the paper, in Section 8.

Understanding or using the results of the paper does not require a deeper knowledge of differential geometry beyond that which we describe along the way. This is not true of the proofs, and in Section 4 we point to what else is needed. The remainder of the paper is taken up with proofs.

2. Random fields, the GKF and our main results. The common element behind all the random fields in this paper is Gaussian white noise.

2.1. *Gaussian white noise.* The simplest way to define Gaussian white noise W on $\mathbb{R}^N$ is as a random field indexed by the Borel sets $\mathcal{B}^N$, finitely additive, taking independent values on disjoint sets, and such that, for all $A \in \mathcal{B}^N$ with finite Lebesgue measure $\lambda_N(A) \equiv |A|$, $W(A) \sim N(0, |A|)$. It is standard fare to define integrals $\int_{\mathbb{R}^N} \varphi(t) W(dt)$, for *deterministic* $\varphi \in L_2(\mathbb{R}^N)$, which are mean zero, Gaussian random variable with covariances

$$\mathbb{E}\left\{\int_{\mathbb{R}^N} \varphi(t) W(dt) \int_{\mathbb{R}^N} \psi(t) W(dt)\right\} = \int_{\mathbb{R}^N} \varphi(t)\psi(t)\lambda_N(dt). \tag{2.1}$$

2.2. *Scale space fields.* The Gaussian scale space random field is obtained by smoothing white noise with a spatial filter over a range of filter widths or scales. Formally, for a "filter" $h \in L_2(\mathbb{R}^N)$ it is defined as

$$f(\sigma, t) = \sigma^{-N/2} \int_{\mathbb{R}^N} h\left(\frac{t-u}{\sigma}\right) dW(u). \tag{2.2}$$

Typically t ranges over a nice subset T of $\mathbb{R}^N$, and we take $\sigma \in [\sigma_l, \sigma_u]$, for finite $0 < \sigma_l < \sigma_u$.

The parameter space therefore includes both scale and location parameters [10, 16]. From a statistical point of view, the scale space field is a continuous wavelet transform of white noise that is designed to be powerful at detecting a localized signal of unknown spatial scale and location.

This concludes the definition, and we now turn to some additional assumptions, as well as setting up some notation and further definitions that we shall require to state the main results and to carry out the proofs.

For convenience, and without loss of generality, we shall assume that the filter h is normalized so that $\int_{\mathbb{R}^N} h^2(u)\,du = 1$, which will ensure that the variance of f is also one. We shall also assume that h is twice continuously differentiable on $\mathbb{R}^N$ and that all partial derivatives are in $L_2(\mathbb{R}^N)$.

With some loss of generality, that, in practice, is not generally too restrictive, we shall assume that there exists a $\gamma > 0$ such that

$$\int_{\mathbb{R}^N} \nabla h(u)\bigl(\nabla h(u)\bigr)'\,du = \gamma I_{N\times N}, \tag{2.3}$$

where we write I or $I_{N\times N}$ to denote the $N \times N$ identity matrix. (Note: We take all our vectors to be column vectors.) Equation (2.3) will follow, for example, if the filter is spherically symmetric. Two common examples are given by the standard Gaussian kernel $h(t) = \pi^{-N/4} e^{-|t|^2/2}$ and the Marr or "Mexican hat" wavelet

$$h(t) = \left[\frac{4N}{(N+2)\pi^{N/2}}\right]^{1/2}\left(1 - \frac{|t|^2}{N}\right)e^{-|t|^2/2}.$$

It is easy to check that in the Gaussian case (2.3) holds with $\gamma = 1/2$ and in the Marr wavelet case with $\gamma = (N+4)/(2N)$. Furthermore, the covariance function of the scale space field is easily calculated, via (2.1), to be

$$C\bigl((\sigma_1, t_1), (\sigma_2, t_2)\bigr) = \frac{1}{(\sigma_1\sigma_2)^{N/2}} \int_{\mathbb{R}^N} h\left(\frac{t_1 - t_2 - v}{\sigma_1}\right) h\left(\frac{-v}{\sigma_2}\right) dv.$$

Note that, for fixed σ, it follows from the previous line that f is stationary in t. However, f is definitely not stationary as a process in the pair (σ, t).

To apply the GKF and to state our results, we shall need information about the derivatives of f, which exist due to our assumptions on h. Either from first principles, by differentiating the covariance function or by using (5.5.5) of [3], we find that all first-order partial derivatives of f with respect to the space variables are uncorrelated with the first-order derivative in the scale variable. This lack of correlation, which of course is equivalent to independence since all fields are Gaussian, is *crucial* to the proofs, and without it, it is unlikely that we could complete the detailed calculations that we shall carry out later. The same techniques also give

$$\kappa \triangleq \sigma^2 \cdot \operatorname{Var}\left(\frac{\partial f(\sigma, t)}{\partial \sigma}\right) = \int_{\mathbb{R}^N} \bigl[\langle u, \nabla h(u)\rangle + N h(u)/2\bigr]^2\,du, \tag{2.4}$$

$$\Lambda_\sigma \triangleq \operatorname{Var}\bigl(\nabla_t f(\sigma, t)\bigr) = \sigma^2 \int_{\mathbb{R}^N} \nabla h(u)\bigl(\nabla h(u)\bigr)'\,du, \tag{2.5}$$

where $\nabla_t f$ denotes the gradient of f with respect to the elements of t only. In view of (2.3), we have $\Lambda_\sigma = \sigma^2 \gamma I_{N\times N}$. We should note that it is not obvious at this point that the quantity $\sigma^2 \operatorname{Var}(\partial f(\sigma,t)/\partial\sigma)$ should be independent of σ. However, this is in fact a consequence of calculations in Section 6. For the Gaussian and Marr wavelet kernels the values of κ are $1/2$ and $N/2$, respectively.

Excluding Section 8, from now on we shall assume that h is spherically symmetric, so that we can define $k:\mathbb{R}_+ \to \mathbb{R}$ by

$$h(x) = k\big(\|x\|^2\big). \tag{2.6}$$

In this case, γ, κ and other constants which will appear throughout can be expressed in terms of k. In particular, retaining consistency with (2.3) and (2.4) and introducing three new terms, we have

$$\begin{aligned} \gamma &= 4\int_{\mathbb{R}^N} \dot{k}^2(u'u)u_1^2\,du, &\qquad \kappa_{2,2} &= \int_{\mathbb{R}^N} \dot{k}^2(u'u)u_1^2u_2^2\,du, \\ \kappa_4 &= \int_{\mathbb{R}^N} \dot{k}^2(u'u)u_1^4\,du, &\qquad \rho &= \int_{\mathbb{R}^N} k(u'u)\dot{k}(u'u)u_1^2\,du \end{aligned} \tag{2.7}$$

and also define

$$\widetilde{C}^2_{k,N} \triangleq \frac{\kappa}{4} = N^2\left(\frac{1}{16} + \frac{\rho}{2} + \kappa_{2,2}\right) + 2N\kappa_{2,2}. \tag{2.8}$$

Note for later use that

$$\kappa_4 - 3\kappa_{2,2} = \int_{\mathbb{R}^N} \dot{k}^2(u'u)\big(u_1^4 - 3u_1^2u_2^2\big)\,du = 0. \tag{2.9}$$

This can be easily proved for a general isotropic kernel [instead of $\dot{k}^2(u'u)$ above] by using Fubini's theorem to restrict to the two-dimensional case and then expressing the integral using spherical coordinates.

2.3. *Rotation space fields.* As already described, rotation space random fields are based on the same underlying white noise model as scale space fields, but also allow for rotations and scale changes that are direction dependent. More precisely, write

$$\mathbb{S} = \mathbb{S}\big(\sigma_2^{-2}, \sigma_1^{-2}\big) \triangleq \big\{S \in \mathrm{Sym}_N^+ : \sigma_2^{-2} I \le S \le \sigma_1^{-2} I\big\},$$

where $\mathrm{Sym}_N^+ \subset \mathrm{Sym}_N$ is the set of $N\times N$ positive definite symmetric matrices, $0 < \sigma_1 < \sigma_2 < \infty$, $K \triangleq N(N+1)/2$, $D \triangleq N + K$ and for two matrices A and B we write $A \le B$ if, and only if, $B - A$ is nonnegative definite. Then the rotation space random field, defined over the D-dimensional space $M \triangleq T \times \mathbb{S}$, is given by

$$f(t,S) = |S|^{1/4} \int_{\mathbb{R}^N} h\big(S^{1/2}(t-u)\big)\,dW(u). \tag{2.10}$$

Again, we shall assume that $h(t)$ is spherically symmetric, twice continuously differentiable on $\mathbb{R}^N$, that all partial derivatives are in $L_2(\mathbb{R}^N)$ and normalized so that $\int_{\mathbb{R}^N} h^2(u)\,du = 1$. Note that the eigenvalues and eigenvectors of S determine the scalings and the corresponding "principal directions" for the rotation space field.

The constant analogous to $\widetilde{C}_{k,N}$ in rotation space is

$$C_{k,N} \triangleq (2\pi)^{D/2-N} \prod_{j=1}^{N} \frac{\Gamma(1/2)}{\Gamma(j/2)} c_{k,N}, \tag{2.11}$$

where

$$c_{k,N}^2 \triangleq (2\kappa_{2,2})^K \left(1 + N(2\kappa_{2,2})^{-1}\left(\frac{1}{16} + \frac{\rho}{2} + \kappa_{2,2}\right)\right). \tag{2.12}$$

Reassuringly, when $N = 1$, in which case scale space is equivalent to rotation space, it is easy to check that $C_k = \widetilde{C}_k$. Another constant which will appear later is

$$D_{k,N}^2 \triangleq 2\kappa_{2,2} \frac{\widetilde{C}_{k,N}^2/N}{\widetilde{C}_{k,N-1}^2/(N-1)}. \tag{2.13}$$

As for scale space fields, we shall once again need information about variances and covariances of the first-order derivatives of rotation space fields. This time, however, the formulae are somewhat more complicated, and, since one of the parameters of $f(t, S)$ is a matrix, we shall have to explain what we mean by differentiation with respect to this parameter. Details will be given below in Section 5.1. At this point we note only that, as before, derivatives with respect to t and S are independent, and, again, this is crucial to our ability to carry out detailed computations. Furthermore, although once again f is neither stationary nor isotropic, it does have zero mean and constant variance.

2.4. *Regularity conditions*. Before we state the GKF, we need to say something about the regularity conditions needed for it to hold. These are of three kinds, relating to the smoothness of the scale and rotation space fields, the structure of T (and so of M) and the smoothness of a transformation F that we shall meet only in the next subsection. Full details are given in Chapters 11 and 12 of [3].

We start with T, which we take to be a compact domain in $\mathbb{R}^N$ with a C^2 boundary (although [3] would allow us to assume less). For the remainder of this paper we require no convexity conditions on T, although in order to apply the Euler characteristic heuristic of Section 3 it is necessary to assume that T is locally convex in the sense of Definition 8.2.1 of [3].

The second set of conditions is required to ensure that f is, with probability one, C^2, as a function of each of the space and scale or rotation variables, and also nondegenerate in a certain (Morse theoretic) sense. Sufficient conditions under which this holds are given in Sections 11.3 and 12.1 of [3]. It is easy to check that these conditions are satisfied for the Gaussian, Marr and other smooth kernels.

2.5. *The Gaussian kinematic formula.* We shall state the GKF in more generality than needed for this paper, since to do so requires little more than an additional sentence or two. Suppose M is a C^2, Whitney stratified manifold satisfying mild side conditions and D a similarly nice stratified submanifold of $\mathbb{R}^k$. For the exact definitions of "mild" and "nice" see [3], but, for the purposes of this paper, it suffices that M is either a scale or rotation space, with T satisfying the conditions of the previous subsection. Also, if $F:\mathbb{R}^k \to \mathbb{R}$ is C^2, then it will suffice that $D = F^{-1}([u,\infty))$ for some u.

Let $f = (f^1,\dots,f^k): M \to \mathbb{R}^k$ be a vector valued random process, the components of which are independent, identically distributed, real-valued, C^2, centered, unit variance, Gaussian processes, satisfying the conditions of the previous subsection. The Gaussian kinematic formula (due originally to Taylor in [12, 13] and extended in [3, 14]) states that, for certain additive set functionals $\mathcal{L}_0,\dots,\mathcal{L}_{\dim M}$,

$$\mathbb{E}\{\mathcal{L}_i(M\cap f^{-1}(D))\} = \sum_{j=0}^{\dim M - i} \begin{bmatrix} i+j \\ j \end{bmatrix} (2\pi)^{-j/2}\mathcal{L}_{i+j}(M)\mathcal{M}_j^k(D). \tag{2.14}$$

There is a lot to explain here. The "flag" coefficients $\left[{n \atop j}\right] = \binom{n}{j}\omega_n/\omega_{n-j}\omega_j$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. The $\mathcal{M}_j^k(D)$, known as the Gaussian Minkowski functionals of D, are determined via the tube expansion

$$\mathbb{P}\{\xi \in \{y\in\mathbb{R}^k : \mathrm{dist}(y,D)\le\rho\}\} = \sum_{j=0}^{\infty}\frac{\rho^j}{j!}\mathcal{M}_j^k(D), \tag{2.15}$$

where $\xi \sim N(0, I_{k\times k})$ and dist is the usual Euclidean distance. In the case $D = F^{-1}([u,\infty))$ it typically involves no more than multivariate calculus to compute the $\mathcal{M}_j^k(D)$, and many examples are given in [3] and [4]. We shall give one example in a moment.

The most important elements of (2.14) are the $\mathcal{L}_j$, known as Lipschitz–Killing curvatures (LKCs). The first of these is rather simple, as it is always the Euler characteristic χ, that is, $\mathcal{L}_0(A) \equiv \chi(A)$ for all nice A. Thus, (2.14), with $i = 0$, provides us with an expression for $\mathbb{E}\{\chi(M\cap f^{-1}(D))\}$, which is what is needed to approximate exceedance probabilities, as explained in the following section.

For a simple but extremely important example, suppose that $k = 1$ and F is the identity. Then it is easy to compute the $\mathcal{M}_j^1([u,\infty))$ to see that, with $i = 0$, and $f: M\to\mathbb{R}$ satisfying the conditions above, (2.14) becomes

$$E\{\chi(A_u(f,M))\} = e^{-u^2/2}\sum_{j=0}^{\dim M}(2\pi)^{-(j+1)/2}\mathcal{L}_j(M)H_{j-1}(x), \tag{2.16}$$

where $A_u(f,M) = \{t\in M : f(t)\ge u\}$ is an excursion set, the H_n, $n\ge 0$, are Hermite polynomials, and $H_{-1}(x) \overset{\Delta}{=} [\varphi(x)]^{-1}\int_x^\infty \varphi(u)\,du$, where φ is the standard Gaussian density. Since the structure of M affects only the LKCs, and neither the

Minkowski functionals nor the flag coefficients, computing LKCs is an independent problem. This is what this paper is about, for scale and rotation space fields.

(A word to the purely applied statistician who needs only a numerical, data driven method for estimating LKCs from data and wishes to avoid theory. Two different such methods [2, 15] are currently available.)

Throughout, we assume that M is a regular (in the sense of [3]) stratified manifold. These are basically sets that can be partitioned into a disjoint union of C^2 manifolds, so that we can write

$$M = \bigsqcup_{j=0}^{\dim M} \partial_j M, \tag{2.17}$$

where each nonempty stratum, $\partial_j M$, $0 \le j \le \dim(M)$, is a j-dimensional manifold whose closure contains $\partial_i M$ for all $i < j$. For a N-dimensional cube, $\partial_N M$ is its interior, $\partial_{n-1} M$ is the union of its $(N-1)$-dimensional (open) faces and so on, down to $\partial_0 M$, which is the collection of its vertices.

If such a space is locally convex, then a simple way to define a Euclidean version $\mathcal{L}_j^E$ of the LKCs is via a *tube formula*, which states that, for small enough ρ, and $M \subset \mathbb{R}^N$,

$$\mathcal{H}_N^E\Big(\Big\{t \in \mathbb{R}^N : \inf_{s \in M} \|t - s\| \le \rho\Big\}\Big) = \sum_{j=0}^{N} \omega_{N-j} \rho^{N-j} \mathcal{L}_j^E(M), \tag{2.18}$$

where (for reasons that will become clearer later) we write $\mathcal{H}_N^E$ for the Lebesgue (Hausdorff) measure on $\mathbb{R}^N$. It follows that, for rectangles, $\mathcal{L}_j^E(\prod_{i=1}^N [0, T_i]) = \sum T_{i_1} \cdots T_{i_j}$, where the sum is taken over all distinct choices of subscripts.

Note that it will always be true that $\mathcal{L}_N^E(M)$ is the volume of M, while $\mathcal{L}_{N-1}^E(M)$ is half the surface measure of M. Furthermore, $\mathcal{L}_0^E(M) = \chi(M)$.

In what follows, however, we shall need to work in a non-Euclidean setting, as the Lipschitz–Killing curvatures in the GKF, and thus in (2.16), are computed relative to a Riemannian metric induced by a Gaussian field. This metric g on M is defined by

$$g_x(X_x, Y_x) \stackrel{\Delta}{=} \mathbb{E}\{(X_x f_x^i)(Y_x f_x^i)\} \tag{2.19}$$

for any i and for $X_x, Y_x \in T_x M$, the tangent space to M at $x \in M$. This can also be written in terms of the covariance function as

$$g_x(X_x, Y_x) = X_x Y_y C(x, y)|_{x=y}. \tag{2.20}$$

Both of the Gaussian fields of interest to us—viz. the scale and rotation space fields—which we defined on compact sets, can easily be extended, to $\mathbb{R}^N \times \mathbb{R}^+$ and $\mathbb{R}^N \times \mathrm{Sym}_N^+$, respectively, and so (2.19) defines metrics on these spaces. In all that follows, unless stated otherwise, these metrics, or their restrictions, are what we shall work with, and it will be in relation to them that Hausdorff measures $\mathcal{H}_j$ and Lipschitz–Killing curvatures $\mathcal{L}_j$ are defined. When we consider a submanifold L of either of these spaces the notation $\mathcal{H}_j^L$, $\mathcal{L}_j^L$ will be used. When the standard Euclidean metric is used these will be replaced by $\mathcal{H}_j^E$ and $\mathcal{L}_j^E$.

A Riemannian metric defines a volume form, a Riemannian curvature and a second fundamental form on M (and, indeed, on each the $\partial_j M$). Then $\mathcal{L}_{\dim M}(M)$ is its Riemannian volume and so is an integral of the volume form. $\mathcal{L}_{\dim M-1}(M)$ is closely related to the (Riemannian) area of $\partial_{\dim M-1}(M)$, the $(\dim M - 1)$-dimensional boundary of M. The remaining Lipschitz–Killing curvatures involve integrals of both curvatures and second fundamental forms, and are more complicated. Rather than give the general form (for which you can see Chapter 10 of [3]), we shall give specific definitions for each of the cases we treat, in the following sections. Now, however, we shall give (relatively) simple expressions for the cases of interest to us. These are the main results of this paper.

2.6. *Main results*. We shall present the results of this section in stages. In the first we shall represent the Lipschitz–Killing curvatures as an integral for a situation more general than that needed to handle rotation space fields. This is Theorem 2.1, and it shows how to write these integrals as products of Lebesgue measures of subsets of $\mathbb{R}^N$ and integrals over submanifolds of Sym_N^+. In Theorem 2.2 we shall compute these integrals for certain situations, which will lead to Corollaries 2.3 and 2.4. These give relatively simple, quite explicit, expressions for the highest-order Lipschitz–Killing curvatures for rotation and scale space fields, respectively. More detailed information for scale space fields is given in Section 8.

We say that a manifold $L \subset \mathrm{Sym}_N^+$ is orthogonally invariant if $S \in L \Rightarrow QSQ' \in L$ for every $Q \in O(N)$, the set of orthogonal matrices. For such L of dimension m, we define

$$F_{m,j}(L) = \int_L \mathcal{R}_j(S)\, d\mathcal{H}_m^L(S), \tag{2.21}$$

where $\mathcal{R}_j(S) = \int_{O(N)} |(QSQ')_{j\times j}|^{1/2}\, d\omega(Q)$, ω is the Haar probability measure on $O(N)$, and for a matrix A, $A_{j\times j}$ denotes the upper left submatrix of size $j \times j$. The measure $\mathcal{H}_m^L$ is the Haar measure induced by the restriction of the metric induced by the field g to a fiber $t \times \mathrm{Sym}_N^+$, which, as we shall see, is independent of the choice of t. With these definitions, we can now state the following:

THEOREM 2.1. *Let $f : A_l \times B_k \subset \mathbb{R}^N \times \mathrm{Sym}_N^+ \to \mathbb{R}$ be the restriction of the rotation space random field defined in Section 2.3, to the product of an l-dimensional regular stratified manifold $A_l \subset \mathbb{R}^N$ and a k-dimensional regular stratified manifold $B_k \subset \mathrm{Sym}_N^+$. Furthermore, suppose that each stratum $\partial_j B_k$, $0 \le j \le k$, is orthogonally invariant and that the regularity conditions of Sec-*

tion 2.4 vis-a-vis f all hold. Then,

$$\mathcal{L}_{l+k}(A_l \times B_k) = \gamma^{l/2}\mathcal{H}_l^E(\partial_l A_l)F_{k,l}(\partial_k B_k),$$
$$\mathcal{L}_{l+k-1}(A_l \times B_k) = \tfrac{1}{2}\big[\gamma^{(l-1)/2}\mathcal{H}_{l-1}^E(\partial_{l-1}A_l)F_{k,l-1}(\partial_k B_k) + \gamma^{l/2}\mathcal{H}_l^E(\partial_l A_l)F_{k-1,l}(\partial_{k-1}B_k)\big],$$

where γ is given by (2.3) *or* (2.7).

It is not hard to see that $\mathcal{R}_j(S)$ is a symmetric function of the eigenvalues of S. Furthermore, as we shall see, as each stratum of B_m is orthogonally invariant, its Hausdorff measure is in fact invariant under $S \mapsto QSQ'$, for any orthogonal Q. Hence, the $F_{m,j}$ are expressible as integrals over certain functions of the eigenvalues of $S \in \partial_m B_m$. In particular, for rotation space, with search region $T \times \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})$, we arrive at the following explicit expressions.

THEOREM 2.2. *With the notation defined above, under the conditions of Theorem 2.1, and for a spherically symmetric kernel,*

$$F_{K,N}\big(\partial_K \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})\big) = C_{k,N}\int_{\sigma_2^{-2}\le\lambda_1\le\cdots\le\lambda_N\le\sigma_1^{-2}} |\triangle(\lambda)| \prod_{j=1}^N \lambda_j^{-N/2}\,d\lambda,$$

$$F_{K,N-1}\big(\partial_K \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})\big) = C_{k,N}\int_{\sigma_2^{-2}\le\lambda_1\le\cdots\le\lambda_N\le\sigma_1^{-2}} |\triangle(\lambda)| \prod_{j=1}^N \lambda_j^{-(N+1)/2}\mathcal{R}_{N-1}\big(\operatorname{diag}(\lambda)\big)\,d\lambda,$$

$$F_{K-1,N}\big(\partial_{K-1} \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})\big) = \frac{C_{k,N}}{D_{k,N}}\int_{\sigma_2^{-2}\le\lambda_1\le\cdots\le\lambda_{N-1}\le\sigma_1^{-2}} |\triangle(\lambda)| \times \prod_{j=1}^{N-1} \lambda_j^{-N/2}\Bigg[\sigma_2^{N-2}\prod_{j=1}^{N-1}|\lambda_j - \sigma_2^{-2}| + \sigma_1^{N-2}\prod_{j=1}^{N-1}|\sigma_1^{-2} - \lambda_j|\Bigg]\,d\lambda,$$

where $C_{k,N}$ and $D_{k,N}$ are given in (2.11) *and* (2.13).

Theorem 2.1 implies the following corollary, with all $F_{m,j}$'s as above.

COROLLARY 2.3. *Under the conditions of Theorem 2.2 and the regularity conditions of Section 2.4, the top two Lipschitz–Killing curvatures for the rotation random field on* $T \times \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})$ *are*

$$\begin{aligned}
&\mathcal{L}_D\big(T \times \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})\big) \\
&\quad = \gamma^{N/2} \mathcal{H}_N^E(\partial_N T) F_{K,N}\big(\partial_K \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})\big), \\
&\mathcal{L}_{D-1}\big(T \times \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})\big) \\
&\quad = \tfrac{1}{2}\big(\gamma^{(N-1)/2} \mathcal{H}_{N-1}^E(\partial_{N-1} T) F_{K,N-1}\big(\partial_K \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})\big) \\
&\qquad\qquad + \gamma^{N/2} \mathcal{H}_N^E(\partial_N T) F_{K-1,N}\big(\partial_{K-1} \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})\big)\big).
\end{aligned}$$

To obtain a parallel result for scale space fields, we use the fact that the scale space random field on $T \times [\sigma_1, \sigma_2]$, with a spherically symmetric kernel, has the same distribution as the restriction of the rotation space random field defined in Section 2.3 to $T \times \mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})$, where

$$\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2}) = \{\nu I : \sigma_2^{-2} \le \nu \le \sigma_1^{-2}\} \subset \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2}).$$

Once we study the metric induced by the rotation field, Theorem 2.1 will effortlessly yield the following result.

COROLLARY 2.4. *Under the conditions of Theorem 2.2 and the regularity conditions of Section 2.4, the top two Lipschitz–Killing curvatures for the scale space random field on* $T \times [\sigma_1, \sigma_2]$ *are*

$$\begin{aligned}
\mathcal{L}_{N+1}\big(T \times \mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})\big) &= \gamma^{N/2} \mathcal{H}_N^E(\partial_N T) F_{1,N}\big(\partial_1 \mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})\big) \\
&= \widetilde{C}_{k,N} \gamma^{N/2} \mathcal{H}_N^E(\partial_N T) \int_{\sigma_2^{-2}}^{\sigma_1^{-2}} \nu^{N/2-1}\, d\nu \\
&= \widetilde{C}_{k,N} \gamma^{N/2} \mathcal{H}_N^E(\partial_N T) \frac{1}{N/2}(\sigma_1^{-N} - \sigma_2^{-N})
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
&\mathcal{L}_N\big(T \times \mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})\big) \\
&\quad = \frac{1}{2}\big(\gamma^{(N-1)/2} \mathcal{H}_{N-1}^E(\partial_{N-1} T) F_{1,N-1}\big(\partial_1 \mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})\big) \\
&\qquad\qquad + \gamma^{N/2} \mathcal{H}_N^E(\partial_N T) F_{0,N}\big(\partial_0 \mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})\big)\big) \\
&\quad = \frac{1}{2}\bigg(\widetilde{C}_{k,N} \gamma^{(N-1)/2} \mathcal{H}_{N-1}^E(\partial_{N-1} T) \frac{(\sigma_1^{-(N-1)} - \sigma_2^{-(N-1)})}{(N-1)/2} \\
&\qquad\qquad + \gamma^{N/2} \mathcal{H}_N^E(\partial_N T)(\sigma_1^{-N} + \sigma_2^{-N})\bigg),
\end{aligned} \tag{2.23}$$

where $\widetilde{C}_{k,N}$ *is as in* (2.8).

In Section 8 we analyze scale space fields independently of the rotation space case for a manifold with a boundary and without assuming spherical symmetry. The formulae for all Lipschitz–Killing curvatures are given via (8.2), the terms of which are all computed in Section 8.7.

2.7. *Comparison with existing results*. While the formulae of Theorem 2.2 may look rather forbidding, one should note two facts. The first is that, for dimension $N > 2$, they are the first of their kind. The second is that although for dimension $N = 2$ they are implicit in [9], one computation relied on computer algebra and gave a large, unstructured formula, while the second, based on tube formulae, does not precisely give the LKCs. (This is not unreasonable, since that computation was directed explicitly at approximating exceedance probabilities, and not computing LKCs.) To recoup the results of [9], we treat, as an example, the case $N = 2$ (so that $D = 5$). In that case, it is possible to carry out the integrations needed to compute $\mathcal{L}_5$ and $\mathcal{L}_4$. A page or so of calculus shows that, for the Gaussian kernel,

$$\mathcal{L}_5(T \times \mathbb{S}) = 2^{-4}\pi\sigma_1^{-2}|T|\left[r^2 - 1 - \left(r^2 + 1\right)\ln r\right], \tag{2.24}$$

where $|T|$ is the simple Euclidean area of T and $r = \sigma_1/\sigma_2$.

The LKC $\mathcal{L}_4$ is a little harder to calculate, since while the first term inside the brackets is again amenable to simple calculus, the second term leads us to special functions. However, a simple answer is attainable, and

$$\begin{aligned}\mathcal{L}_4(T \times \mathbb{S}) &= 2^{-7/2}\pi\sigma_1^{-2}|T|\left(r^2 - 1\right)\ln r \\ &\quad + 2^{-9/2}\pi|\partial T|\int_{\sigma_2^{-2}\le\lambda_1\le\lambda_2\le\sigma_1^{-2}}(\lambda_2 - \lambda_1)(\lambda_1\lambda_2)^{-3/2}\mathcal{R}(\lambda_1, \lambda_2)\,d\lambda,\end{aligned}$$

where $|\partial T|$ is the length of ∂T, and, with E the elliptic integral $E(y) = \int_0^{\pi/2}\sqrt{1 - y^2\sin^2\theta}\,d\theta$, we have $\mathcal{R}(\lambda_1, \lambda_2) = \pi\lambda_1 E((\lambda_1^2 - \lambda_2^2)/\lambda_1^2)/2$.

Note also that, as stated, Theorem 2.2 is weaker than the result for the scale case in Section 8 in that it gives only the highest two LKCs, rather than all of them. However, while more would be desirable, it is the higher ones that are most important for the Euler characteristic heuristic discussed in the following section. We shall explain there why this is the case.

3. The Euler characteristic heuristic: ECH. As intimated in the Introduction, signal plus noise problems often boil down to being able to compute the exceedance probabilities (1.1). For smooth Gaussian processes, there is typically no way to directly compute these probabilities, and they must be approximated, using approximations that are accurate for the tail 5% or so of the distribution.

There are a number of ways to carry out such approximations, and two of these are at the center of the papers [9] and [10], discussed in the Introduction. One relies on *tube formulae* and moves the problem from one about Gaussian processes on Euclidean sets M to one involving computation of volumes of tubes around embeddings of M in high-dimensional spheres.

The other is the Euler characteristic heuristic [1, 3] which provides a generally easier route, along with the advantage that it also provides information on sample paths at all levels. The ECH states that for smooth random fields f, on nice, locally convex, compact sets M (which will be the case in our scenarios if T is locally convex, since $\mathbb{D}$ and $\mathbb{S}$ are convex) and large u,

$$\mathbb{P}\Big\{\sup_{t\in M} f(t) \geq u\Big\} \approx \mathbb{E}\{\chi(A_u(f,M))\}. \tag{3.1}$$

In the case of smooth, Gaussian, f, with zero mean and constant unit variance (but with no stationarity assumptions) the approximation in (3.1) can be quantified (cf. [3, 11]), and it is known that

$$\liminf_{u\to\infty} -u^{-2}\log\big|\mathrm{Diff}_{f,M}(u)\big| \geq \frac{1}{2}\bigg(1+\frac{1}{\sigma_c^2(f)}\bigg), \tag{3.2}$$

where we write $\mathrm{Diff}_{f,M}(u)$ for the difference of the two terms in (3.1), and $\sigma_c^2(f) > 0$ is an f-dependent constant, that is, in many cases, computable.

The importance of (3.2) is that it shows that the two terms in (3.1), both of which are known from the general theory of Gaussian processes to be of order $u^{\dim M-1}\exp(-u^2/2)$, differ by a term of exponentially smaller order, viz. of order $\exp(-u^2/2(1+\sigma_c^2(f))$. This fact is what justifies the use of the ECH for Gaussian fields (in which case it is no longer heuristic) and in large part motivates the computations of this paper, and also explains the claim of the previous section, that the leading Lipschitz–Killing curvatures are the important ones in applying the ECH.

4. Before we start the proofs. As we wrote earlier, it should have been possible to read the paper up to this point without a knowledge of differential geometry. This will not be true for the proofs.

The derivations of the leading Lipschitz–Killing curvatures for the rotation space case, which is what we have concentrated on so far, and all LKCs for the scale space case, which will be treated in Section 8, begin in the same fashion. Each starts by taking the parameter space M, writing it as a stratified manifold, and carefully identifying the strata $\partial_j M$ of (2.17). The next step will be to take the generic formulae for LKCs, given in (necessary, but sometimes painful) detail in [3], and rewrite them for the current setting, that is, for the specific stratification, and for the induced Riemannian metric (2.19). We shall not explain in detail how this is done other than to say, at each stage, which generic formula we are using. Then, the Riemannian metric induced by the field is computed.

At this point the proofs diverge. For scale space we compute all LKCs, and so geometric objects such as curvature and second fundamental forms arise. For rotation space, with only the two leading LKCs, we can avoid them. The problem lies not in the the higher dimension of rotation space, but in its more complicated Riemannian metric. Nevertheless, in Section 6 we shall see that once the geometry of rotation space is understood, the two leading LKCs of scale space are effortlessly computed.

Throughout, we have tried to carry out computations as carefully as possible, so that not only can the interested reader follow the calculations, but also can see how they need to be adjusted for other cases.

5. Proof of Theorem 2.1. We shall break the proof of Theorem 2.1 into stages, the first of which relates to the crucial orthogonality properties mentioned in the Introduction.

5.1. *Orthogonality and the Riemannian metric on* $\mathbb{R}^N \times \{S\}$. Our aim in this subsection is to establish the orthogonality of the two subspaces

$$T_t\mathbb{R}^N, T_S\mathrm{Sym}_N^+ \subset T_t\mathbb{R}^N \oplus T_S\mathrm{Sym}_N^+ \simeq T_{(t,S)}\mathbb{R}^N \times \mathrm{Sym}_N^+,$$

under the inner product defined by the Riemannian metric induced by the process. Another factor that comes into play is a structured dependence of the restrictions of the Riemannian metric on $\mathbb{R}^N \times \mathrm{Sym}_N^+$ to fibers of the form $\mathbb{R}^N \times \{S\}$. This will turn out to be the key that allows us to carry out further detailed computations.

To establish this independence, let (t, S) be a point in $\mathbb{R}^N \times \mathrm{Sym}_N^+$ and let X, Y be two derivations in $T_t\mathbb{R}^N, T_S\mathrm{Sym}_N^+ \subset T_{(t,S)}\mathbb{R}^N \times \mathrm{Sym}_N^+$, respectively. Using the connection to the covariance function given in (2.20), and writing Xt and YS to denote the elementwise derivatives of the identity functions on $\mathbb{R}^N$ and Sym_N^+, respectively, we have

$$\begin{aligned} g(X,Y) &= X_{(t,S)}Y_{(\widetilde{t},\widetilde{S})}C\big((t,S),(\widetilde{t},\widetilde{S})\big)\big|_{(t,S)=(\widetilde{t},\widetilde{S})} \\ &= 2|S|^{1/4}\big(Y|S|^{1/4}\big)\int \big((Xt)'Su\big)\dot{k}\big(u'Su\big)k\big(u'Su\big)\,du \\ &\qquad + 2|S|^{1/2}\int \big((Xt)'Su\big)\big(u'(YS)u\big)\big[\dot{k}\big(u'Su\big)\big]^2\,du \\ &= 0, \end{aligned}$$

where the last equality follows since both integrands are odd functions. This shows that the subspaces $T_t\mathbb{R}^N$ and $T_S\mathrm{Sym}_N^+$ are orthogonal at each point.

We now compute the restriction of the Riemannian metric g to a fiber $\mathbb{R}^N \times \{S\}$, again based on (2.20). Writing, again, $\nabla_t f$ for the spatial derivative of f with respect to the elements of t, the Gram matrix of the Riemannian metric on $\mathbb{R}^N \times \{S\}$, relative to the standard Euclidean basis, is

$$\begin{aligned}
&\mathrm{Var}\{\nabla_t f(t,S)\}\\
&\quad = |S|^{1/2}\int_{\mathbb{R}^N} \nabla_t k\big((t-u)'S(t-u)\big)\big(\nabla_t k\big((t-u)'S(t-u)\big)\big)'\,du\\
&\quad = 4|S|^{1/2}\int_{\mathbb{R}^N} \dot{k}\big(w'Sw\big)Sw\big(\dot{k}\big(w'Sw\big)Sw\big)'\,dw\\
&\quad = 4|S|^{1/2}S\bigg[\int_{\mathbb{R}^N} \big(\dot{k}\big(w'Sw\big)\big)^2 ww'\,dw\bigg]S\\
&\quad = 4S^{1/2}\bigg[\int_{\mathbb{R}^N} \big(\dot{k}\big(v'v\big)\big)^2 vv'\,dv\bigg]S^{1/2}\\
&\quad = \gamma S.
\end{aligned}$$

5.2. *The Riemannian metric on* $\{t\}\times \mathrm{Sym}_N^+$. Relying on (2.20) to express the Riemannian metric and using a simple change of variables, it can easily be seen that the restriction of g to $\{t\}\times\mathrm{Sym}_N^+$ is independent of t. Denote this metric on $\mathrm{Sym}_N^+ \simeq \{t\}\times \mathrm{Sym}_N^+$ by $g^{\mathrm{Sym}_N^+}$. Using the identity $\nabla_S|S| = |S|S^{-1}$, valid on Sym_N^+, we have

$$\begin{aligned}
&\nabla_S\big(|S|^{1/4}k\big((t-u)'S(t-u)\big)\big)\\
&\qquad = \tfrac{1}{4}|S|^{1/4}S^{-1}k\big((t-u)'S(t-u)\big)\\
&\qquad\quad + |S|^{1/4}\dot{k}\big((t-u)'S(t-u)\big)(t-u)(t-u)'.
\end{aligned}$$

Therefore, for tangent vectors $A, B\in \mathrm{Sym}_N \simeq T_S\mathrm{Sym}_N^+$,

$$\begin{aligned}
g_S^{\mathrm{Sym}_N^+}(A,B) &= \int_{\mathbb{R}^N}\big\langle \nabla_S\big(|S|^{1/4}k\big((t-u)'S(t-u)\big)\big), A\big\rangle\\
&\qquad\qquad \times \big\langle \nabla_S\big(|S|^{1/4}k\big((t-u)'S(t-u)\big)\big), B\big\rangle\,du\\
&= |S|^{1/2}\int_{\mathbb{R}^N}\bigg[\frac{1}{4}k\big(u'Su\big)\langle S^{-1}, A\rangle + \dot{k}\big(u'Su\big)\langle uu', A\rangle\bigg]\\
&\qquad\qquad\qquad \times \bigg[\frac{1}{4}k\big(u'Su\big)\langle S^{-1}, B\rangle + \dot{k}\big(u'Su\big)\langle uu', B\rangle\bigg]\,du.
\end{aligned}$$

After some slightly tedious calculus, and based on the fact that $\kappa_4 - 3\kappa_{2,2} = 0$ [see (2.9)], all of the terms above can be expressed in terms of moments of the kernel k and its derivative $\dot{k}$ as follows:

$$g_S^{\mathrm{Sym}_N^+}(A,B) = \bigg(\frac{1}{16}+\frac{\rho}{2}+\kappa_{2,2}\bigg)\mathrm{Tr}\big(S^{-1}A\big)\,\mathrm{Tr}\big(S^{-1}B\big) + 2\kappa_{2,2}\,\mathrm{Tr}\big(S^{-1}AS^{-1}B\big).$$

When h is the Gaussian kernel $h(u) = e^{-\|u\|_2^2/2}/\pi^{N/4}$, and therefore $k(x) = e^{-x/2}/\pi^{N/4}$, simple calculations show that $\frac{1}{16}+\frac{\rho}{2}+\kappa_{2,2} = 0$, leaving only the term $2\kappa_{2,2}\,\mathrm{Tr}(S^{-1}AS^{-1}B)$ above.

5.3. *Putting everything together.* Let $\mathcal{H}_l^{A_l \times \{S\}}$ denote the Riemannian measures on $A_l \times \{S\} \simeq A_l$ induced by g and let $\mathcal{H}_l^E$ denote the Hausdorff measure, that is, Riemannian measure under the standard Euclidean metric on T. A standard calculation gives

$$\frac{d\mathcal{H}_l^{A_l \times \{S\}}}{d\mathcal{H}_l^E}(t) = \gamma^{l/2} |S|_{T_t \partial_l A_l}^{1/2},$$

where $|S|_{T_t \partial_l A_l}$ is defined to be the determinant of PSP' for some $l \times N$ matrix P, the rows of which form an orthonormal basis of $T_t \partial_l A_l$ (under the Euclidean inner product). The determinant, of course, is independent of the choice of basis.

Working in a chart and relying on the orthogonality established above, it is easy to see that

$$\begin{aligned} \mathcal{L}_{k+l}(A_l \times B_k) &= \mathcal{H}_{k+l}(A_l \times B_k) \\ &= \int_{\partial_k B_k} \int_{\partial_l A_l} d\mathcal{H}_l^{A_l \times \{S\}}(t)\, d\mathcal{H}_k^{B_k}(S) \qquad (5.1) \\ &= \gamma^{l/2} \int_{\partial_l A_l} \int_{\partial_k B_k} |S|_{T_t \partial_l A_l}^{1/2}\, d\mathcal{H}_k^{B_k}(S)\, d\mathcal{H}_l^E(t). \end{aligned}$$

The equality above relies on the expression for the Radon–Nikodym derivative above and a Fubini theorem.

We recall now our assumption that B_k is orthogonally invariant, that is, $S \in B_k \Rightarrow QSQ' \in B_k$ for any $Q \in O(N)$. Thus, the mapping $S \mapsto QSQ'$ is a diffeomorphism from B_k to itself. Moreover, for the covariance function of the rotation field, we have

$$C\big((t, S), (t, \widetilde{S})\big) = C\big((t, QSQ'), (t, Q\widetilde{S}Q')\big).$$

Therefore, $S \mapsto QSQ'$ is also a Riemannian isometry. Thus, the measure $\mathcal{H}_k^{B_k}$, induced by the Riemannian metric, is also invariant under $S \mapsto QSQ'$. This implies that we can drop the dependence of the integral in (5.1) on the tangent space $T_t \partial_l A_l$. That is, we can replace it, for all t, with a fixed j-dimensional subspace of our choice or a random one. Therefore,

$$\begin{aligned} \mathcal{H}_{k+l}(A_l \times B_k) &= \gamma^{l/2} \mathcal{H}_l(\partial_l A_l) \int_{\partial_k B_k} \mathcal{R}_l(S)\, d\mathcal{H}_k^{B_k}(S) \qquad (5.2) \\ &= \gamma^{l/2} \mathcal{H}_l(\partial_l A_l) F_{k,l}(\partial_k B_k). \end{aligned}$$

To complete the proof of the theorem, one must repeat essentially the same calculation for the codimension part of $A_l \times B_k$.

6. Proof of Corollary 2.4. In this section we perform the computations necessary to evaluate the expression in Corollary 2.4. These calculations are much simpler than those for rotation space. The set $\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})$ can be stratified as

$$\begin{aligned}
\partial_1\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2}) &= \{\nu I : \sigma_2^{-2} < \nu < \sigma_1^{-2}\},\\
\partial_0\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2}) &= \{\sigma_2^{-2} I\} \cup \{\sigma_1^{-2} I\}.
\end{aligned}$$

Clearly, each stratum is orthogonally invariant. Consequently, Theorem 2.1 implies we need only compute the following three terms, each of which is straightforward:

$$\begin{aligned}
F_{0,N}(\partial_0\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})) &= \sigma_1^{-N} + \sigma_2^{-N},\\
F_{1,N}(\partial_1\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})) &= \int_{\sigma_2^{-2}}^{\sigma_1^{-2}} \nu^{N/2} f(\nu)\, d\nu,\\
F_{1,N-1}(\partial_1\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})) &= \int_{\sigma_2^{-2}}^{\sigma_1^{-2}} \nu^{(N-1)/2} f(\nu)\, d\nu,
\end{aligned}$$

where $f(\nu)$ is the appropriate density with respect to the Lebesgue measure.

Above, we expressed the necessary integrals in terms of the parametrization of $\partial_1\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})$ by the curve $\nu \mapsto \nu \cdot I$. Hence,

$$\begin{aligned}
f(\nu) &= \sqrt{g_{\nu\cdot I}^{\mathrm{Sym}_N^+}(I, I)}\\
&= \nu^{-1}\sqrt{N^2\left(\frac{1}{16} + \frac{\rho}{2} + \kappa_{2,2}\right) + 2N\kappa_{2,2}}\\
&= \widetilde{C}_{k,N}\nu^{-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
F_{1,N}(\partial_1\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})) &= \widetilde{C}_{k,N}\frac{1}{N/2}(\sigma_1^{-N} - \sigma_2^{-N}),\\
F_{1,N-1}(\partial_1\mathbb{D}(\sigma_2^{-2}, \sigma_1^{-2})) &= \widetilde{C}_{k,N}\frac{1}{(N-1)/2}(\sigma_1^{-(N-1)} - \sigma_2^{-(N-1)}).
\end{aligned}$$

7. Proof of Theorem 2.2. We start the proof of Theorem 2.2 by describing a stratification of $\mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})$, which is one of the regularity conditions required for applying the GKF. In fact, we shall actually only use the strata of codimension 0 and 1.

7.1. *Stratification of the space*. Our first step is to express the manifold $M = T \times \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})$ as a Whitney stratified manifold. Note that a stratification for $\mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})$ yields a stratification for M as the product stratification, given that a Whitney stratification of T is assumed given. Thus, our first goal is to stratify $\mathbb{S}(\tau_1, \tau_2)$ for arbitrary $0 < \tau_1 < \tau_2 < \infty$.

Semialgebraic orbits of the Lie group action $GL_N(\mathbb{R}) \times \mathrm{Sym}_N \to \mathrm{Sym}_N$, $(P, S) \mapsto PSP'$,

$$\{S \in \mathrm{Sym}_N \,|\, \mathrm{rank}(S) = k,\, S \geq 0\}, \qquad 0 \leq k \leq N, \tag{7.1}$$

constitute a Whitney stratification of $\{S \in \mathrm{Sym}_N | S \geq 0\}$ (cf. [7], page 21).

Since

$$A^k(\tau_1) \triangleq \{S \in \mathrm{Sym}_N \,|\, \mathrm{rank}(S - \tau_1 I) = N - k,\, \tau_1 I \leq S\},$$
$$B^k(\tau_2) \triangleq \{S \in \mathrm{Sym}_N \,|\, \mathrm{rank}(S - \tau_2 I) = N - k,\, S \leq \tau_2 I\}$$

are obtained by applying affine transformations to the strata of (7.1), they form Whitney stratifications of

$$\{S \in \mathrm{Sym}_N | \tau_1 I \leq S\} \quad \text{and} \quad \{S \in \mathrm{Sym}_N | S \leq \tau_2 I\}. \tag{7.2}$$

The final step in stratifying $\mathbb{S}(\tau_1, \tau_2)$ is to verify that these stratifications intersect transversely. For this, it suffices to show that for any $j, k \geq 0$ such that $j + k \leq N$, a matrix $S \in A^j(\tau_1) \cap B^k(\tau_2)$ and nonzero matrix $C \in \mathrm{Sym}_N$, there exist smooth paths $A(t) : (-\delta, \delta) \longrightarrow A^j(\tau_1)$, $B(t) : (-\delta, \delta) \longrightarrow B^k(\tau_2)$ with $A(0) = B(0) = S$ such that for any $t \in (-\delta, \delta)$,

$$A(t) - B(t) = \big(A(t) - S\big) - \big(B(t) - S\big) = tC. \tag{7.3}$$

Based on the fact that the group action above preserves rank, the problem of finding the paths $A(t)$, $B(t)$ is equivalent to defining them by

$$A(t) = X(t)(S - \tau_1 I)\big(X(t)\big)' + \tau_1 I, \tag{7.4}$$

$$B(t) = Y(t)(S - \tau_2 I)\big(Y(t)\big)' + \tau_2 I \tag{7.5}$$

and finding appropriate invertible matrices $X(t)$, $Y(t)$.

Even under the restriction $X(t) = Y(t)$ a solution exists. Substitution in (7.3) gives

$$X(t)(S - \tau_1 I)\big(X(t)\big)' + \tau_1 I - X(t)(S - \tau_2 I)\big(X(t)\big)' - \tau_2 I = tC,$$

which is equivalent to

$$X(t)\big(X(t)\big)' = \frac{t}{\tau_2 - \tau_1} C + I.$$

For small t, define $X(t)$ to be the unique positive definite square root of $\frac{t}{\tau_2 - \tau_1} C + I$. Substituting $X(t)$ in (7.4) and (7.5) defines, for small δ, the required paths $A(t)$, $B(t)$, and proves transversality.

The transversality of the intersections, together with the stratifications in (7.2), yield the Whitney stratification

$$\mathbb{S}(\tau_1, \tau_2) = \bigcup_{\substack{0 \le j,k \\ j+k \le N}} \mathbb{S}^{j,k}(\tau_1, \tau_2),$$

where $\mathbb{S}^{j,k}(\tau_1, \tau_2) = A^j(\tau_1) \cap B^k(\tau_2)$. That is, we have found that the partition of $\mathbb{S}(\tau_1, \tau_2)$ according to the geometrical multiplicity of τ_1 and τ_2 is a Whitney stratification.

Note for later use that the submanifold of Sym_N of matrices of rank $N - k$, and thus $A^k(\tau_1)$ and $B^k(\tau_2)$, are of codimension $k(k+1)/2$. Since, for $j, k \ge 0$ with $j + k \le N$, $A^j(\tau_1)$ and $B^k(\tau_2)$ intersect tranversely, the codimension of $\mathbb{S}^{j,k}(\tau_1, \tau_2)$ is $[j(j+1) + k(k+1)]/2$.

7.2. *Volume density on* Sym_N^+. Having computed the Riemannian metric on Sym_N^+, in order to apply Theorem 2.1 to the rotation space random field on $T \times \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})$, we first explicitly compute $d\mathcal{H}_K^{\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}/d\mathcal{H}_K^E$, that is, the density with respect to the Lebesgue measure on Sym_N^+ when viewed as $\mathbb{R}^K$ via the choice of a basis for the Hilbert–Schmidt inner product on Sym_N.

Before turning to the calculation we need some notation. Denote the standard basis of Sym_N by $e = (e_{ij})_{i=1,j=i}^{N,N}$. Normalizing the elements e_{ij} with $i \ne j$ by a factor of $1/\sqrt{2}$ yields an orthonormal basis relative to the Hilbert–Schmidt inner product which we denote by $\widetilde{e} = (\widetilde{e}_{ij})_{i=1,j=i}^{N,N}$.

Working with the chart defined by mapping a matrix to its coordinates relative to $\widetilde{e}$, we see that the density is given by

$$\frac{d\mathcal{H}_K^{\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}}{d\mathcal{H}_K^E}(S) = \sqrt{\det\{g_S^{\mathrm{Sym}_N^+}(\widetilde{e}_m, \widetilde{e}_n)\}_{m=1,n=1}^{K,K}}, \tag{7.6}$$

where $(\widetilde{e}_m)_{m=1}^K = (\widetilde{e}_{i(m),j(m)})_{m=1}^K$ is obtained by vectorizing $(\widetilde{e}_{ij})_{i=1,j=i}^{N,N}$.

As already stated in Section 5.3, for any $Q \in O(N)$, the mapping $S \mapsto QSQ'$ is a Riemannian isometry of $g_S^{\mathrm{Sym}_N^+}$, and thus preserves the measure $\mathcal{H}_K^{\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}$. Clearly, it also preserves the Euclidean Hausdorff measure $\mathcal{H}_K^E$. Therefore, the density in (7.6) is invariant under $S \mapsto QSQ'$ as well. Since any real symmetric matrix is orthogonally diagonalizable, it follows that the density only depends on the eigenvalues of S. It is therefore sufficient to compute it for diagonal matrices.

Let $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$. For any two basis elements of $\widetilde{e}$,

$$\begin{aligned} g_\Lambda^{\mathrm{Sym}_N^+}(\widetilde{e}_{ij}, \widetilde{e}_{kl}) &= \left(\tfrac{1}{16} + \kappa_{2,2} + \tfrac{1}{2}\rho\right) \mathrm{Tr}(\Lambda^{-1}\widetilde{e}_{ij}) \, \mathrm{Tr}(\Lambda^{-1}\widetilde{e}_{kl}) \\ &\quad + 2\kappa_{2,2} \, \mathrm{Tr}(\Lambda^{-1}\widetilde{e}_{ij}\Lambda^{-1}\widetilde{e}_{kl}) \\ &= \left(\tfrac{1}{16} + \kappa_{2,2} + \tfrac{1}{2}\rho\right)\lambda_i^{-1}\lambda_k^{-1}\delta_{ij}\delta_{kl} + 2\kappa_{2,2}\lambda_i^{-1}\lambda_j^{-1}\delta_{ik}\delta_{jl}. \end{aligned}$$

Suppose we order the basis in such a way that its first N elements are $\widetilde{e}_{1,1},\dots,\widetilde{e}_{N,N}$. Then, the corresponding $K\times K$ matrix has the form

$$\begin{aligned}\{g_\Lambda^{\mathrm{Sym}_N^+}(\widetilde{e}_m,\widetilde{e}_n)\}_{m=1,n=1}^{K,K} &= 2\kappa_{2,2}\operatorname{diag}(\{\lambda_{i(m)}^{-1}\lambda_{j(m)}^{-1}\}_{m=1}^K)\\ &\quad+\left(\tfrac{1}{16}+\kappa_{2,2}+\tfrac{1}{2}\rho\right)VV',\end{aligned}$$

where $V=(\{\lambda_i^{-1}\}_{i=1}^N,0,\dots,0)'\in\mathbb{R}^K$.

By inspection, this is a rank-one perturbation of a diagonal matrix. Now, for a diagonal matrix D and vector u, $\det(D+uu')=\det(D)(1+u'D^{-1}u)$, which, when applied in this setting, yields

$$\frac{d\mathcal{H}_K^{\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}}{d\mathcal{H}_K^E}(\Lambda)=\sqrt{\det\{g_\Lambda^{\mathrm{Sym}_N^+}(\widetilde{e}_m,\widetilde{e}_n)\}_{m=1,n=1}^{K,K}}=c_{k,N}|\Lambda|^{-(N+1)/2},$$

where $c_{k,N}$ is defined in (2.12). By the argument above, for general $S\in\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})$,

$$\frac{d\mathcal{H}_K^{\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}}{d\mathcal{H}_K^E}(S)=c_{k,N}|S|^{-(N+1)/2}.$$

7.3. *The push-forward to eigenvalues.* Rewriting the integral defining $F_{K,N}(\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2}))$, (2.21), in terms of the density $d\mathcal{H}_K^{\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}/d\mathcal{H}_K^E$, gives a relatively complicated integral. Fortunately, there is a simple way to convert it to an integral against a simpler density and, more importantly, over a much simpler domain. To do this, we need first to recall a classical result from the theory of random matrices.

The starting point of the computation of the joint density of the eigenvalues in the Gaussian orthogonal ensemble (GOE) model (cf. [5]) is similar to our situation. We begin with a measure μ on Sym_N with density (relative to $\mathcal{H}_K^E$) that depends strictly on the eigenvalues. The goal is to compute the density of the push-forward of μ, denoted by ν, under the mapping $\alpha:\mathrm{Sym}_N\to\mathbb{R}^N$ which maps a matrix to the vector composed of its ordered eigenvalues.

In the case of the GOE, the density of μ is simply the product of the densities of the independent entries of the (real symmetric) random matrix. The entries of the matrix are zero-mean Gaussian variables with variance 2 on the diagonal and variance 1 off the diagonal. That is,

$$\begin{aligned}\frac{d\mu}{d\mathcal{H}_K^E}(S) &= 2^{N-D/2}2^{-N/2}(2\pi)^{-N(N+1)/4}\exp\left\{-\frac{1}{4}\sum_{i=1}^N S_{ii}-\frac{1}{2}\sum_{1\le i<j\le N}S_{ij}\right\}\\ &= 2^{(N-D)/2}(2\pi)^{-N(N+1)/4}\exp\left\{-\frac{1}{4}\operatorname{Tr}(S^2)\right\},\end{aligned}$$

where the factor of $2^{N-D/2}$ comes from the fact that $\mathcal{H}_K^E$ was defined via identification with the Hilbert–Schmidt basis and not the standard basis of symmetric matrices.

The classical result is that

$$\frac{d\nu}{dl}(\lambda) = (2\pi)^{-N/2}2^{-N(N+1)/4}\prod_{j=1}^{N}\frac{\Gamma(1/2)}{\Gamma(j/2)}\mathbf{1}_{\lambda_1\le\cdots\le\lambda_N}\left|\triangle(\lambda)\right|\exp\left\{-\frac{1}{4}\sum_{i=1}^{N}\lambda_i^2\right\},$$

where l is the Lebesgue measure on $\mathbb{R}^N$ and $\triangle(\lambda) \triangleq \prod_{i<j}(\lambda_j-\lambda_i)$ is the Vandermonde determinant.

The specific Gaussian nature of μ has only a limited role to play in this result, and all that is relevant for our needs is that the density is determined by $\alpha(S)$ at each point. In fact, for any measure $\widetilde{\mu}$ with such a density, we can write $d\widetilde{\mu}/d\mathcal{H}_K^E(S) = \zeta_{\widetilde{\mu}}(\alpha(S))$ and rely on the GOE argument to conclude that, for the corresponding push-forward measure $\widetilde{\nu}$,

$$\begin{aligned}\frac{d\widetilde{\nu}}{dl}(\lambda) &= \frac{d\nu}{dl}(\lambda)\left(\frac{\zeta_{\widetilde{\mu}}}{\zeta_\mu}\right)(\lambda)\\ &= (2\pi)^{D/2-N}\prod_{j=1}^{N}\frac{\Gamma(1/2)}{\Gamma(j/2)}\mathbf{1}_{\lambda_1\le\cdots\le\lambda_N}\left|\triangle(\lambda)\right|\zeta_{\widetilde{\mu}}(\lambda),\end{aligned}$$

where μ and ν are the measures corresponding to the GOE model.

In particular, denoting the push-forward of $\mathcal{H}_K^{\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}$ by α by ρ gives

$$\begin{aligned}\frac{d\rho}{dl}(\lambda) &= (2\pi)^{D/2-N}c_{k,N}\prod_{j=1}^{N}\left(\frac{\Gamma(1/2)}{\Gamma(j/2)}\lambda_j^{-(N+1)/2}\right)\mathbf{1}_{\lambda_1\le\cdots\le\lambda_N}\left|\triangle(\lambda)\right|\\ &= C_{k,N}\mathbf{1}_{\lambda_1\le\cdots\le\lambda_N}\left|\triangle(\lambda)\right|\prod_{j=1}^{N}\lambda_j^{-(N+1)/2}.\end{aligned}$$

7.4. *The two leading LKCs.* In view of Theorem 2.1, having computed the density $\frac{d\rho}{dl}(\lambda)$, the computation of $\mathcal{L}_D(M)$ simply becomes a matter of rewriting the integral of $F_{K,N}(\partial_K\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2}))$ in terms of eigenvalues. Doing so leads to

$$\begin{aligned}&F_{K,N}\bigl(\partial_K\mathbb{S}\bigl(\sigma_2^{-2},\sigma_1^{-2}\bigr)\bigr)\\ &\quad= \int_{\partial_K\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}|S|^{1/2}\,d\mathcal{H}_K^{\partial_K\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}\\ &\quad= C_{k,N}\int_{\sigma_2^{-2}\le\lambda_1\le\cdots\le\lambda_N\le\sigma_1^{-2}}\left|\triangle(\lambda)\right|\prod_{j=1}^{N}\lambda_j^{-N/2}\,d\lambda.\end{aligned}$$

We now turn to $F_{K,N-1}(\partial_K \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2}))$, for which an identical argument yields

$$F_{K,N-1}\bigl(\partial_K \mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2})\bigr) = C_{k,N} \int_{\sigma_2^{-2} \le \lambda_1 \le \cdots \le \lambda_N \le \sigma_1^{-2}} |\triangle(\lambda)| \prod_{j=1}^{N} \lambda_j^{-(N+1)/2} \mathcal{R}_{N-1}\bigl(\operatorname{diag}(\lambda)\bigr)\, d\lambda.$$

It remains to evaluate $F_{K-1,N}(\mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2}))$. Although this depends on $\mathcal{H}_{K-1}^{\partial_{K-1}\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}$, we shall not have to study this measure directly in order to compute the integral. Rather, an application of Federer's coarea formula [3, 6] with a carefully chosen map will suffice.

For any $0 < \tau < \sigma_1^{-2}$ define $\mathbb{S}^u_{\min}(\tau, \sigma_1^{-2})$ to be the set of real, symmetric matrices with unique minimal eigenvalue and with all eigenvalues in (τ, σ_1^{-2}). Consider the smooth mapping $\lambda_{\min} : \mathbb{S}^u_{\min}(0, \sigma_1^{-2}) \to \mathbb{R}$ which maps a matrix to its minimal eigenvalue. To apply the coarea formula, we need a Riemannian structure on the domain and range of $\lambda_{\min}$. For the domain we take $g_S^{\mathrm{Sym}_N^+}$ as the Riemannian metric, and on $\mathbb{R}$ consider the Euclidean metric.

Now, fix some $0 < \tau < \sigma_1^{-2}$. Let $S \in \mathbb{S}^u_{\min}(\tau, \sigma_1^{-2})$ and write $S = Q \Lambda Q'$ with $Q \in O(N)$, $\Lambda = \operatorname{diag}(\lambda)$ diagonal and $\lambda = (\lambda_1, \ldots, \lambda_N)$ ordered so that $\lambda_1 \le \cdots \le \lambda_N$.

Defining $\widetilde{e}^Q_{ij} = Q \widetilde{e}_{ij} Q'$, $\widetilde{e}^Q = (\widetilde{e}^Q_{ij})_{i=1,j=i}^{N,N}$ forms a basis of Sym_N^+. Let Proj_V be the orthogonal projection onto

$$V \triangleq \operatorname{span}\bigl(\{\widetilde{e}^Q_{ij}\}_{i=1,j=i}^{N,N} \setminus \{\widetilde{e}^Q_{11}\}\bigr).$$

Let $(b_{ij})_{i=1,j=i}^{N,N}$ be an orthonormal basis of $T_S \mathbb{S}^u_{\min}(\tau, \sigma_1^{-2})$ relative to $g_S^{\mathrm{Sym}_N^+}$, such that for all b_{ij} with $(i,j) \ne (1,1)$, $b_{ij} \in V$, and such that, denoting $\bar{b}_{11} \triangleq \widetilde{e}^Q_{11} - \mathrm{Proj}_V(\widetilde{e}^Q_{11})$,

$$b_{11} = \bar{b}_{11} / \|\bar{b}_{11}\| = (\lambda_1 / D_{k,N}) \bar{b}_{11},$$

with $D_{k,N}$ given in (2.13). The last equality follows from a page or two of calculus, using the projection formula of [3], page 173, and the Sherman-Morrison formula (see [8], Section 2.7.1).

Since the basis is orthonormal, the Jacobian can be expressed by

$$J\lambda_{\min}(S) = \left[\sum_{i=1,j=i}^{N,N} \langle \nabla \lambda_{\min}(S), b_{ij} \rangle^2 \right]^{1/2}.$$

Note that if we can show that $\langle \nabla \lambda_{\min}(S), \widetilde{e}^Q_{ij} \rangle$ is zero for any $(i,j) \ne (1,1)$, then it will follow that

$$J\lambda_{\min}(S) = (\lambda_1 / D_{k,N}) \bigl| \langle \nabla \lambda_{\min}(S), \widetilde{e}^Q_{11} \rangle \bigr|.$$

In order to show this, and also compute this value, we examine the eigenvalues of $S + t\widetilde{e}_{ij}^{Q}$, for small t. Write $S + t\widetilde{e}_{ij}^{Q} = Q(\Lambda + t\widetilde{e}_{ij})Q'$, and note that therefore we can work with the eigenvalues of $U_t \triangleq \Lambda + t\widetilde{e}_{ij}$ instead. When $i = j$, U_t is diagonal with eigenvalues $\{\lambda_1, \dots, \lambda_{i-1}, \lambda_i + t, \lambda_{i+1}, \dots, \lambda_N\}$. When $i \neq j$, U_t also takes a very simple form, with eigenvalues

$$\{\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_N\}$$
$$\cup \left\{\tfrac{1}{2}\left(\lambda_i + \lambda_j \pm \sqrt{(\lambda_i - \lambda_j)^2 + 2t^2}\right)\right\}.$$

Therefore,

$$\langle \nabla\lambda_{\min}(S), \widetilde{e}_{ij}^{Q}\rangle = \lim_{t\to 0} \frac{\lambda_{\min}(S + t\widetilde{e}_{ij}^{Q}) - \lambda_{\min}(S)}{t} = \delta_{(1,1)}(i, j),$$

which implies $J\lambda_{\min}(S) = \frac{\lambda_{\min}(S)}{D_{k,N}}$. Finally, Federer's coarea formula gives

$$\int_{\mathbb{S}^u_{\min}(\tau,\sigma_1^{-2})} J\lambda_{\min}(S)|S|^{1/2}\, d\mathcal{H}_K^{\mathbb{S}^u_{\min}(\tau,\sigma_1^{-2})}(S)$$
$$= \int_{(\tau,\sigma_1^{-2})} dx \int_{\lambda_{\min}^{-1}(x)} |S|^{1/2}\, d\mathcal{H}_{K-1}^{\mathbb{S}^u_{\min}(\tau,\sigma_1^{-2})}(S).$$

Since the integral on $\lambda_{\min}^{-1}(x)$ is continuous in x, differentiating by τ yields

$$\begin{aligned}
&\int_{\lambda_{\min}^{-1}(\sigma_2^{-2})} |S|^{1/2}\, d\mathcal{H}_{K-1}^{\mathbb{S}^u_{\min}(\tau,\sigma_1^{-2})}(S) \\
&\quad = -\frac{d}{d\tau}\bigg|_{\tau=\sigma_2^{-2}} \int_{\mathbb{S}^u_{\min}(\tau,\sigma_1^{-2})} J\lambda_{\min}(S)|S|^{1/2}\, d\mathcal{H}_K^{\mathbb{S}^u_{\min}(\tau,\sigma_1^{-2})}(S) \\
&\quad = -\frac{d}{d\tau}\bigg|_{\tau=\sigma_2^{-2}} \frac{C_{k,N}}{D_{k,N}} \int_{\sigma_2^{-2}\le\lambda_1\le\cdots\le\lambda_N\le\sigma_1^{-2}} |\triangle(\lambda)|\lambda_1 \prod_{j=1}^{N} \lambda_j^{-N/2}\, d\lambda \\
&\quad = \frac{C_{k,N}}{D_{k,N}} \sigma_2^{N-2} \int_{\sigma_2^{-2}\le\lambda_1\le\cdots\le\lambda_{N-1}\le\sigma_1^{-2}} |\triangle(\lambda)| \prod_{j=1}^{N-1} \left(\lambda_j^{-N/2}|\lambda_j - \sigma_2^{-2}|\right) d\lambda,
\end{aligned}$$

where the second equality is implied by the fact that any point of

$$\{\lambda : \tau \le \lambda_1 \le \cdots \le \lambda_N \le \sigma_1^{-2}\} \setminus \alpha\left(\mathbb{S}^u_{\min}(\tau, \sigma_1^{-2})\right)$$

is a boundary point or has zero density and the last equality follows by writing the integral as an iterated integral and differentiating.

Obviously, we can also consider the maximal eigenvalue, define $\lambda_{\max}$ and $\mathbb{S}^u_{\max}(\sigma_2^{-2}, \tau)$ accordingly, and carry out a similar computation. Combined with the last result and the fact that $\partial_{K-1}\mathbb{S}(\sigma_2^{-2}, \sigma_1^{-2}) = \lambda_{\min}^{-1}(\sigma_2^{-2}) \cup \lambda_{\max}^{-1}(\sigma_1^{-2})$, we can compute the final term needed to complete the proof:

$$
\begin{aligned}
&F_{K-1,N}\big(\partial_{K-1}\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})\big)\\
&\quad=\int_{\partial_{K-1}\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}|S|^{1/2}\,d\mathcal{H}_{K-1}^{\mathbb{S}(\sigma_2^{-2},\sigma_1^{-2})}(S)\\
&\quad=\frac{C_{k,N}}{D_{k,N}}\int_{\sigma_2^{-2}\le\lambda_1\le\cdots\le\lambda_{N-1}\le\sigma_1^{-2}}|\triangle(\lambda)|\\
&\qquad\qquad\times\prod_{j=1}^{N-1}\lambda_j^{-N/2}\Bigg[\sigma_2^{N-2}\prod_{j=1}^{N-1}|\lambda_j-\sigma_2^{-2}|\\
&\qquad\qquad\qquad+\sigma_1^{N-2}\prod_{j=1}^{N-1}|\sigma_1^{-2}-\lambda_j|\Bigg]\,d\lambda.
\end{aligned}
$$

8. Scale space fields. In this section we shall compute *all* the Lipschitz–Killing curvatures for the scale space random field, under the assumptions of Section 2.2 on the kernel h and of Section 2.4 on the parameter space T. Also, we require only that h satisfy (2.3), and not full rotational symmetry.

In the final analysis, the formulae that we shall obtain are quite simple. They involve no more than the Euclidean Lipschitz–Killing curvatures of T, the parameters γ and κ of (2.3) and (2.4), and combinatorial coefficients. On the other hand, there are many elements to each formula, so that any explicit computation of them will involve a computer. Thus, we shall proceed by setting out two general formulae—(8.1) and (8.2)—and then in Section 8.7 list explicit expressions for each of the summands in these formulae. Putting it all together to get actual numbers is a simple computing exercise.

As for the two leading Lipschitz–Killing curvatures, simple algebra shows that the results of this section coincide with those of Corollary 2.4.

For notational convenience in what follows, we make the simple scale transformation $s=-\ln\sigma$, so that our random field is now denoted by $f(s,t)$ and has covariance function

$$C\big((s_1,t_1),(s_2,t_2)\big)=e^{N(s_1+s_2)/2}\int_{\mathbb{R}^N}h\big((t_1-u)e^{s_1}\big)h\big((t_2-u)e^{s_2}\big)\,du.$$

The parameter space of f, which we shall denote by M, is now of the form $T\times[s_1,s_2]\subset\mathbb{R}^{N+1}$, for some $-\infty<s_1<s_2<\infty$.

8.1. *Stratifying the parameter space.* We start by stratifying the parameter space M into manifolds of common dimension. In particular, we write the $(N+1)$-dimensional parameter space $M=T\times[s_1,s_2]$ as the disjoint union of four types of pieces, arranged in three strata, according to their dimension (recall that T has a smooth boundary ∂T):

$$\begin{aligned}\partial_{N+1}M &= M^\circ = (s_1, s_2) \times T^\circ,\\ \partial_N M &= (s_1, s_2) \times \partial T \cup \{s_2\} \times T^\circ \cup \{s_1\} \times T^\circ\\ &\stackrel{\Delta}{=} \text{“side”} \cup \text{“top”} \cup \text{“bottom”},\\ \partial_{N-1}M &= \{s_1\} \times \partial T \cup \{s_2\} \times \partial T.\end{aligned}$$

8.2. *What we need to compute.* Ultimately, we need to compute the Lipschitz–Killing curvatures $\mathcal{L}_i(M)$ of M under the Riemannian metric induced by the random field f. Rather than doing this directly, we shall compute certain other curvatures, denoted by $\mathcal{L}_i^\alpha(M)$, specifically designed to be simpler to handle on sets of constant curvature. (We shall soon see that M has constant curvature, given by $-\kappa^{-1}$.)

There are simple relationships between the $\mathcal{L}_i(M)$ and $\mathcal{L}_i^\alpha(M)$, including the following one, which is (10.5.12) of [3]:

$$\mathcal{L}_i(M) = \sum_{n=0}^{\lfloor (N+1-i)/2 \rfloor} \frac{\alpha^n (i+2n)!}{(4\pi)^n n! i!} \mathcal{L}_{i+2n}^\alpha(M). \tag{8.1}$$

To compute the $\mathcal{L}_i^\alpha(M)$ themselves, we first write

$$\mathcal{L}_i^\alpha(M) = \sum_{j=N-1}^{N+1} \mathcal{L}_i^\alpha(M; \partial_j M) \tag{8.2}$$

and then restrict ourselves to the case $\alpha = -\kappa^{-1}$. Applying (10.7.10)[5] of [3] along with the (yet to be proven) fact that M has constant curvature $-\kappa^{-1}$ gives that, for $i \le j$,

$$\begin{aligned}&\mathcal{L}_i^{-\kappa^{-1}}(M; \partial_j M)\\ &\quad = \frac{1}{(2\pi)^{(j-i)/2}(j-i)!} \int_{\partial_j M} \mathbb{E}\{\mathrm{Tr}^{T_t \partial_j M}(S_{Z_{j,t}}^{j-i}) \mathbb{1}_{N_t M}(Z_{j,t})\} \mathcal{H}_j(dt).\end{aligned} \tag{8.3}$$

Here, for each $t \in \partial_j M$, $Z_{j,t}$ is a normally distributed random vector of dimension $N+1-j$ in the space $T_t \partial_j M^\perp$, the orthogonal complement of $T_t \partial_j M$ in $\mathbb{R}^{N+1}$, S is the Riemannian shape operator of M, $N_t M$ is the normal cone to M at t, and $\mathcal{H}_j$ is the volume measure on $\partial_j M$ corresponding to the Riemannian metric induced by the random field f.

The next step is to compute each element in (8.3).

[5]There is a small, but, for us, rather significant typo in (10.7.10) of [3], in that there is a minus sign missing before the κ in $\kappa I^2/2$.

8.3. *The induced Riemannian metric.* The next step is to identify the Riemannian metric g that the random field induces on M. In order to describe this, however, we need first to choose a family of vector fields generating the tangent bundle of M.

We do this sequentially, starting with the set T, ignoring for the moment the scale component of the parameter space. For T, let η be the (Euclidean) outward unit normal vector field on ∂T, and extend this to a full (Euclidean) C^2, orthonormal tangent vector bundle $X_1, \dots, X_N$, on T, with $X_N = \eta$. Enlarge this to a vector bundle on all of M by adding the vector field $\nu = \partial/\partial s$, the field of tangent vectors in the scale direction.

The Riemannian metric g at points $(s,t) \in M$ can now be calculated on pairs of vectors from the above vector field using the variances (2.4) and (2.5) and the independence discussed there. These yield

$$g_{(s,t)}(X_{i,(s,t)}, X_{j,(s,t)}) = \gamma e^{2s} \delta_{ij}, \tag{8.4}$$

$$g_{(s,t)}(X_{i,(s,t)}, \nu_{(s,t)}) = 0, \tag{8.5}$$

$$g_{(s,t)}(\nu_{(s,t)}, \nu_{(s,t)}) = \kappa, \tag{8.6}$$

where δ_{ij} is the Kronecker delta. We note, once again, that (8.5) implies that the metric g is actually of product form is essential to all that follows.

Note that the above three equations also imply that the structure of the normal cones to M is well described by the initial, Euclidean, choice of vector fields. There is, of course, no normal cone at points $t \in \partial_{N+1} M$, the interior of M. For the sets in $\partial_N M$, one of the vector fields η and ν describes the normal geometry. In particular, along the side the normal is an s-dependent multiple of η and along the top and bottom the normals are constant multiples of ν. Along ∂_{N-1} all normals are linear combinations of elements in the vector fields η and ν.

8.4. *The Levi-Civita connection.* The next step involves computing the second fundamental forms in (8.3) and so requires identifying the Levi-Civita connections. If we think of M as embedded in a smooth open subset of $\mathbb{R}^{N+1}$, with the same induced metric, then we shall write ∇ for the Levi-Civita connection on M and $\widetilde{\nabla}$ for the Levi-Civita connection on the ambient set.

For our purposes, we need only to know how $\widetilde{\nabla}$ operates on vectors normal to M. Furthermore, since the Lie bracket

$$[X_i, \nu] = 0 \qquad \text{for all } 1 \le i \le N, \tag{8.7}$$

we need only compute $\widetilde{\nabla}_{X_i} X_j$, $\widetilde{\nabla}_\nu X_i$ and $\widetilde{\nabla}_\nu \nu$.

To start, note that a straightforward application of Koszul's formula [which in the current situation simplifies to $2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y)$] and (8.4) and (8.5) imply, for $i, j, k = 1, \dots, N$, that

$$g(\widetilde{\nabla}_{X_i} X_j, X_k) = \gamma e^{2s} \langle \nabla_{X_i} X_j, X_k \rangle = g(\nabla_{X_i} X_j, X_k),$$

where ∇ is the standard Euclidean connection, and

$$g(\widetilde{\nabla}_{X_i} X_j, \nu) = -\frac{\gamma}{2}\frac{\partial}{\partial s} e^{2s}\delta_{ij} = -\gamma e^{2s}\delta_{ij}. \tag{8.8}$$

Now note that for any tangent vectors X, Y to T, $\widetilde{\nabla}_X Y$ is a vector in $\mathbb{R}^{N+1}$, and so can be written as $\sum_{j=1}^N a_j X_j + b\nu$ for appropriate coefficients. This fact, together with the last two equalities and (8.6), implies that for any two vector fields X, Y on T,

$$\widetilde{\nabla}_X Y = \nabla_X Y - \kappa^{-1}\gamma e^{2s}\langle X, Y\rangle \nu, \tag{8.9}$$

giving us the first computation.

As far as $\widetilde{\nabla}_\nu \nu$ is concerned, note first that another easy consequence of Koszul's formula is that

$$g(\widetilde{\nabla}_\nu \nu, \nu) = \tfrac{1}{2}\nu\big(g(\nu, \nu)\big) = 0, \tag{8.10}$$

giving us $\widetilde{\nabla}_\nu \nu \equiv 0$.

All that remains is to compute $\widetilde{\nabla}_\nu X_i$. An application of the Weingarten equation [i.e., the scalar second fundamental form is given by $S_\nu(X, Y) = g(\widetilde{\nabla}_X Y, \nu) = -g(Y, \widetilde{\nabla}_X \nu)$] and (8.9) yield

$$g(\widetilde{\nabla}_\nu X_i, X_j) = \frac{\gamma}{2}\frac{\partial}{\partial s} e^{2s}\delta_{ij} = \gamma e^{2s}\delta_{ij} = g(X_i, X_j)$$

and $g(\widetilde{\nabla}_\nu X_i, \nu) = 0$. Applying now the torsion freeness of connections (i.e., $\nabla_X Y - \nabla_Y X - [X, Y] = 0$), along with (8.7) to the above, gives

$$\widetilde{\nabla}_\nu X_i = \widetilde{\nabla}_{X_i} \nu = X_i, \tag{8.11}$$

and we have the last of the three cases we were seeking.

8.5. *The second fundamental forms and the curvature matrix.* With the relevant connections determined, we can now turn to computing second fundamental forms and curvature matrices along the three stratifications of M.

1. $\partial_{N+1} M$, *the interior* $(s_1, s_2) \times T^\circ$: Since the normal space in the interior is empty, there are no normals, no second fundamental form and no curvature matrix here.

2. $\partial_N M$: *The side* $(s_1, s_2) \times \partial T$: A convenient choice for an orthonormal (in the metric g) basis for the tangent space at any point on the side is given by

$$\left\{\frac{X_1}{\gamma^{1/2}e^s}, \dots, \frac{X_{N-1}}{\gamma^{1/2}e^s}, \frac{\nu}{\kappa^{1/2}}\right\}$$

with outward unit normal vector $\eta/\gamma^{1/2}e^s$.

The scalar second fundamental forms of interest are therefore

$$S_{\eta/(\gamma^{1/2}e^s)}\left(\frac{X_i}{\gamma^{1/2}e^s}, \frac{X_j}{\gamma^{1/2}e^s}\right), \qquad S_{\eta/(\gamma^{1/2}e^s)}\left(\frac{X_i}{\gamma^{1/2}e^s}, \frac{\nu}{\kappa^{1/2}}\right),$$
$$S_{\eta/(\gamma^{1/2}e^s)}\left(\frac{\nu}{\kappa^{1/2}}, \frac{\nu}{\kappa^{1/2}}\right),$$

all of which can be computed directly using Weingarten's equation and (8.9)–(8.11). We summarize them in a curvature matrix, writing

$$\begin{pmatrix} C = \gamma^{-1/2}e^{-s}C_t & 0 \\ 0 & 0 \end{pmatrix},$$

where C_t is the $(N-1) \times (N-1)$ Euclidean curvature matrix of ∂T in the basis $\{X_1, \dots, X_{N-1}, \eta\}$. Recall for later use that, from Section 7.2 of [3], in the current scenario, the trace in (8.3) can be replaced by

$$(j-i)!\,\mathrm{detr}_{j-i}\, C, \tag{8.12}$$

where detr_j is our usual sum of determinants of $j \times j$ principal minors.

3. $\partial_N M$, *the bottom* $\{s_1\} \times T^\circ$ *and the top* $\{s_2\} \times T^\circ$: Starting with the bottom $\{s_1\} \times T^\circ$, the outward unit normal vector is $-\kappa^{-1/2}\nu$, and a convenient orthonormal basis is given by $e^{-s}\{X_1, \dots, X_N\}$. The curvature matrix is therefore $N \times N$ with entries

$$S_{-\kappa^{-1/2}\nu}\left(\frac{X_i}{\gamma^{1/2}e^s}, \frac{X_j}{\gamma^{1/2}e^s}\right) = g\left(\widetilde{\nabla}_{X_i/(\gamma^{1/2}e^s)}(-\kappa^{-1/2}\nu), \frac{X_j}{\gamma^{1/2}e^s}\right) = \kappa^{-1/2}\delta_{ij}$$

by Weingarten's equation and (8.8).

Thus, for the bottom, the outward curvature matrix is $\kappa^{-1/2}I_{N\times N}$, while along the top the same arguments give it as $-\kappa^{-1/2}I_{N\times N}$.

4. $\partial_{N-1}M$: *The edges* $\{s_1\} \times \partial T$ *and* $\{s_2\} \times \partial T$: For the edges, we need to consider the scalar second fundamental form itself and not just the curvature matrix. As above, an orthonormal basis for the tangent space is $\{X_1/\gamma^{1/2}e^s, \dots, X_{N-1}/\gamma^{1/2}e^s, \nu/\kappa^{1/2}\}$, but now an orthonormal basis for the normal space is $\{\gamma^{-1/2}e^{-s}\eta, \kappa^{-1/2}\nu\}$.

Applying the Weingarten equation, we need to compute

$$g\big(\widetilde{\nabla}_{\gamma^{-1/2}e^{-s}X_i}(\gamma^{-1/2}e^{-s}X_j), a\gamma^{-1/2}e^{-s}\eta + b\kappa^{-1/2}\nu\big)$$

for arbitrary a, b. Applying (8.9) gives that this is $a\gamma^{-1/2}e^{-s}C_{ij,t} + b\kappa^{-1/2}\delta_{ij}$, where, with a minor abuse of notation, we now use C_t to denote the $(N-1) \times (N-1)$ Euclidean curvature matrix of ∂M at t.

8.6. *The curvature tensor.* We now have all the pieces we need to compute the Lipschitz–Killing curvatures $\mathcal{L}_j^{-\kappa^{-1}}$, and could actually proceed to the final computation. However, in justifying the formula (8.3), we used the fact that M has constant negative curvature $-\kappa^{-1}$. Now we shall take a moment to prove this. Let $R(X,Y) = \widetilde{\nabla}_X \widetilde{\nabla}_Y - \widetilde{\nabla}_Y \widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]}$ be the curvature operator. From previous calculations of the connection

$$\begin{aligned}
R(X_i, \nu)X_k &= \widetilde{\nabla}_{X_i} \widetilde{\nabla}_\nu X_k - \widetilde{\nabla}_\nu \widetilde{\nabla}_{X_i} X_k \\
&= \widetilde{\nabla}_{X_i} X_k - \widetilde{\nabla}_\nu \big(\nabla_{X_i} X_k - \delta_{ik}\kappa^{-1}\gamma e^{2s}\nu\big) \\
&= \nabla_{X_i} X_k - \delta_{ik}\kappa^{-1}\gamma e^{2s}\nu - \nabla_{X_i} X_k + \delta_{ik}\kappa^{-1}\frac{\partial}{\partial s}\gamma e^{2s}\nu \\
&= \delta_{ik}\kappa^{-1}\gamma e^{2s}\nu, \\
R(X_i, \nu)\nu &= \widetilde{\nabla}_{X_i} \widetilde{\nabla}_\nu \nu - \widetilde{\nabla}_\nu \widetilde{\nabla}_{X_i} \nu \\
&= -\widetilde{\nabla}_\nu X_i \\
&= -X_i, \\
R(X_i, X_j)\nu &= \widetilde{\nabla}_{X_i} \widetilde{\nabla}_{X_j} \nu - \widetilde{\nabla}_{X_j} \widetilde{\nabla}_{X_i} \nu - \widetilde{\nabla}_{[X_i, X_j]}\nu \\
&= \widetilde{\nabla}_{X_i} X_j - \widetilde{\nabla}_{X_j} X_i - [X_i, X_j] \\
&= \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] - \kappa^{-1}\gamma e^{2s}(\delta_{ij} - \delta_{ji})\nu \\
&= 0, \\
R(X_i, X_j)X_k &= \widetilde{\nabla}_{X_i} \widetilde{\nabla}_{X_j} X_k - \widetilde{\nabla}_{X_j} \widetilde{\nabla}_{X_i} X_k - \widetilde{\nabla}_{[X_i, X_j]} X_k \\
&= \widetilde{\nabla}_{X_i}\big(\nabla_{X_j} X_k - \delta_{jk}\kappa^{-1}\gamma e^{2s}\nu\big) - \widetilde{\nabla}_{X_j}\big(\nabla_{X_i} X_k - \delta_{ik}\kappa^{-1}\gamma e^{2s}\nu\big) \\
&\quad - \nabla_{[X_i, X_j]} X_k + \kappa^{-1}\gamma e^{2s}\langle [X_i, X_j], X_k\rangle \nu \\
&= \nabla_{X_i}\nabla_{X_j} X_k - \delta_{jk}\kappa^{-1}\gamma e^{2s} X_i - \kappa^{-1}\gamma e^{2s}\langle \nabla_{X_j} X_k, X_i\rangle \nu \\
&\quad - \nabla_{X_j}\nabla_{X_i} X_k + \delta_{ik}\kappa^{-1}\gamma e^{2s} X_j - \kappa^{-1}\gamma e^{2s}\langle \nabla_{X_i} X_k, X_j\rangle \nu \\
&\quad - \nabla_{[X_i, X_j]} X_k + \kappa^{-1}\gamma e^{2s}\langle [X_i, X_j], X_k\rangle \nu \\
&= \nabla_{X_i}\nabla_{X_j} X_k - \nabla_{X_j}\nabla_{X_i} X_k - \nabla_{[X_i, X_j]} X_k \\
&\quad + \kappa^{-1}\gamma e^{2s}\nu\big(\langle [X_i, X_j], X_k\rangle - \langle \nabla_{X_j} X_k, X_i\rangle + \langle \nabla_{X_i} X_k, X_j\rangle\big) \\
&\quad - \delta_{jk}\kappa^{-1}\gamma e^{2s} X_i + \delta_{ik}\kappa^{-1}\gamma e^{2s} X_j \\
&= -\delta_{jk}\kappa^{-1}\gamma e^{2s} X_i + \delta_{ik}\kappa^{-1}\gamma e^{2s} X_j,
\end{aligned}$$

the last equality following from the flatness of $\mathbb{R}^{N+1}$ and torsion freeness.

Since the curvature tensor is given by $R(X, Y, Z, W) = g(R(X, Y)Z, W)$, it is now easy to use the above calculations to check cases and see that

$$R(X, Y, Z, W) = -\kappa^{-1} I^2(X, Y, Z, W)/2,$$

where I is the identity form given by $I(X, Y) = g(X, Y)$, and with the usual tensor product $I^2(X, Y, Z, W) \triangleq I(X, Z)I(Y, W) - I(X, W)I(Y, Z)$.

From this follows our claim that M is a space of constant curvature $-\kappa^{-1}$.

8.7. *The Lipschitz–Killing curvatures* $\mathcal{L}^{-\kappa^{-1}}$. With all the preparation done, we can now begin the computation of the Lipschitz–Killing curvatures $\mathcal{L}_j^{-\kappa^{-1}}(M; \partial_k M)$, according to their definition in (8.3). As in the discussion of second fundamental forms, we divide the computation into separate sections, each corresponding to a different stratum in the stratification of M.

Throughout the following computation we take Z_1 and Z_2 to be two independent $N(0, 1)$ random variables. We shall associate the Z_j with the normal vector fields to obtain normal vectors of the form $Z_1\nu + Z_2\eta$.

We also adopt the notation

$$\mu_k \triangleq \mathbb{E}\{Z_j^k \mathbb{1}_{Z_j \geq 0}\} = \begin{cases} \dfrac{1}{2}, & \text{if } k = 0, \\ \dfrac{2^{n-1} n!}{\sqrt{2\pi}}, & \text{if } k = 2n+1 \text{ is odd}, \\ \dfrac{(2n-1)(2n-3)\cdots}{2}, & \text{if } k = 2n \text{ is even}. \end{cases} \tag{8.13}$$

1. $\partial_{N+1} M$, *the interior* $(s_1, s_2) \times T^\circ$: Since the second fundamental form is zero in $\partial_{N+1} M$, the only nonzero Lipschitz–Killing curvature occurs when $N+1 = j = i$ in (8.3). In this case

$$\begin{aligned} \mathcal{L}_{N+1}^{-\kappa^{-1}}(M; \partial_{N+1} M) &= \mathcal{H}_{N+1}\big((s_1, s_2) \times T^\circ\big) \\ &= \kappa^{1/2} \gamma^{N/2} \int_{s_1}^{s_2} \int_{T^\circ} e^{Ns}\, dt\, ds \\ &= \kappa^{1/2} \gamma^{N/2} \frac{e^{Ns_2} - e^{Ns_1}}{N} \mathcal{L}_N^E(T; T^\circ), \end{aligned} \tag{8.14}$$

where $\mathcal{L}_N^E(T; T^\circ)$ is computed in the standard Euclidean sense.

2. $\partial_N M$: *The side* $(s_1, s_2) \times \partial T$: For this case, all the Lipschitz–Killing curvatures need to be computed, and we replace the trace in (8.3) by the determinants

of the curvature matrix as in (8.12). Then

$$
\begin{aligned}
&\mathcal{L}_j^{-\kappa^{-1}}\bigl(M;(s_1,s_2)\times\partial T\bigr)\\
&\quad=(2\pi)^{-(N-j)/2}\mathbb{E}\bigl\{Z_2^{N-j}\mathbb{1}_{\{Z_2>0\}}\bigr\}\\
&\qquad\times\int_{(s_1,s_2)\times\partial T}\operatorname{detr}_{N-j}\begin{pmatrix} e^{-s}C_t & 0\\ 0 & 0\end{pmatrix}d\mathcal{H}_N(s,t)\\
&\quad=(2\pi)^{-(N-j)/2}\kappa^{1/2}\mathbb{E}\bigl\{Z_2^{N-j}\mathbb{1}_{\{Z_2>0\}}\bigr\}\\
&\qquad\times\int_{s_1}^{s_2}\int_{\partial T}\bigl(\gamma^{-1/2}e^{-s}\bigr)^{N-j}\bigl(\gamma^{1/2}e^{s}\bigr)^{N-1}\operatorname{detr}_{N-j}(C_t)\,dt\,ds\\
&\quad=\kappa^{1/2}\gamma^{(j-1)/2}\frac{e^{(j-1)s_2}-e^{(j-1)s_1}}{j-1}\mathcal{L}_{j-1}^{E}(T;\partial T)
\end{aligned}
\tag{8.15}
$$

in a parallel notation to (8.14).

3. $\partial_N M$, *the bottom* $\{s_1\}\times T^\circ$ *and the top* $\{s_2\}\times T^\circ$: Beginning with the bottom, $\{s_1\}\times T^\circ$, for $0\le j\le N$ we have

$$
\begin{aligned}
&\mathcal{L}_j^{-\kappa^{-1}}\bigl(M;\{s_1\}\times T^\circ\bigr)\\
&\quad=(2\pi)^{-(N-j)/2}\mathbb{E}\bigl\{Z_1^{N-j}\mathbb{1}_{\{Z_1>0\}}\bigr\}\\
&\qquad\times\int_{\{s_1\}\times T^\circ}\operatorname{detr}_{N-j}\bigl(\kappa^{1/2}I_{N\times N}\bigr)\,d\mathcal{H}_N(t)\\
&\quad=\bigl(2\pi\kappa^{-1}\bigr)^{-(N-j)/2}\mu_{N-j}\binom{N}{j}\gamma^{N/2}e^{Ns_1}\mathcal{L}_N^{E}\bigl(T;T^\circ\bigr).
\end{aligned}
\tag{8.16}
$$

A similar result holds for the top, $\{s_2\}\times T^\circ$, namely,

$$
\begin{aligned}
&\mathcal{L}_j^{-\kappa^{-1}}\bigl(M;\{s_2\}\times T^\circ\bigr)\\
&\quad=(-1)^{N-j}\bigl(2\pi\kappa^{-1}\bigr)^{-(N-j)/2}\mu_{N-j}\binom{N}{j}\gamma^{N/2}e^{Ns_2}\mathcal{L}_N^{E}\bigl(T;T^\circ\bigr).
\end{aligned}
\tag{8.17}
$$

4. $\partial_{N-1}M$: *The edges* $\{s_1\}\times\partial T$ *and* $\{s_2\}\times\partial T$: We start with the top edge, $\{s_2\}\times\partial T$,

$$
\begin{aligned}
&\mathcal{L}_j^{-\kappa^{-1}}\bigl(M;\{s_2\}\times\partial T\bigr)\\
&\quad=(2\pi)^{-(N-1-j)/2}\\
&\qquad\times\int_{\{s_2\}\times\partial T}\mathbb{E}\bigl\{\operatorname{detr}_{N-1-j}\bigl(Z_1\kappa^{-1/2}I-Z_2\gamma^{-1/2}e^{-s}C_t\bigr)\\
&\qquad\qquad\times\mathbb{1}_{\{Z_1>0\}}\mathbb{1}_{\{Z_2>0\}}\bigr\}\,d\mathcal{H}_{N-1}(s,t).
\end{aligned}
$$

Use now the easily checked expansion that, for $0 \le k \le n$,

$$\mathrm{detr}_k(\alpha I_{n\times n} + A_{n\times n}) = \sum_{m=0}^{k} \alpha^{k-m} \binom{n-m}{k-m} \mathrm{detr}_m(A),$$

to expand the detr term in the expectation above and see that

$$\begin{aligned}
&\mathcal{L}_j^{-\kappa^{-1}}\bigl(M; \{s_2\} \times \partial T\bigr)\\
&\quad = (2\pi)^{-(N-1-j)/2}\\
&\qquad \times \sum_{m=0}^{N-1-j} \kappa^{-(N-1-j-m)/2} \binom{N-1-m}{j}\\
&\qquad\qquad \times \mathbb{E}\bigl\{Z_1^{N-1-j-m} \mathbb{1}_{\{Z_1>0\}}\bigr\} \bigl(\gamma^{1/2} e^{s_2}\bigr)^{N-1} \mathbb{E}\bigl\{Z_2^m \mathbb{1}_{\{Z_2>0\}}\bigr\}\\
&\qquad\qquad \times \int_{\partial T} \mathrm{detr}_m\bigl(\gamma^{-1/2} e^{-s_2} C_t\bigr)\, d\mathcal{H}_{N-1}(t)\\
&\quad = \bigl(2\pi\kappa^{-1}\bigr)^{-(N-1-j)/2}\\
&\qquad \times \sum_{m=0}^{N-1-j} \bigl(\gamma^{1/2} e^{s_2}\bigr)^{N-1-m} \binom{N-1-m}{j} \kappa^{m/2}\\
&\qquad\qquad \times \mu_{N-1-j-m} \mathbb{E}\bigl\{Z_2^m \mathbb{1}_{\{Z_2>0\}}\bigr\} \int_{\partial T} \mathrm{detr}_m(C_t)\, d\mathcal{H}_{N-1}(t)\\
&\quad = \kappa^{(N-1-j)/2} \bigl(\gamma^{1/2} e^{s_2}\bigr)^{N-1} \sum_{m=0}^{N-1-j} \binom{N-1-m}{j} \bigl(\gamma^{-1/2} e^{-s_2} \kappa^{1/2}\bigr)^m\\
&\qquad\qquad \times \mu_{N-1-j-m} \mathcal{L}_{N-1-m}^E(T; \partial T).
\end{aligned} \tag{8.18}$$

A similar argument also works for the bottom edge $\{s_1\} \times \partial T$, the only change being that $\mathbb{1}_{\{Z_1>0\}}$ becomes $\mathbb{1}_{\{Z_1<0\}}$, giving the final form

$$\begin{aligned}
&\mathcal{L}_j^{-\kappa^{-1}}\bigl(M; \{s_1\} \times \partial T\bigr)\\
&\quad = \kappa^{(N-1-j)/2} \bigl(\gamma^{1/2} e^{s_1}\bigr)^{N-1}\\
&\qquad \times \sum_{m=0}^{N-1-j} \binom{N-1-m}{j} \bigl(\gamma^{-1/2} e^{-s_1} \kappa^{1/2}\bigr)^m\\
&\qquad\qquad \times (-1)^{N-1-j-m} \mu_{N-1-j-m} \mathcal{L}_{N-1-m}^E(T; \partial T).
\end{aligned} \tag{8.19}$$

Collecting (8.14)–(8.19) now gives us all the $\mathcal{L}_i^{-\kappa^{-1}}(M; \partial_j M)$, from which, via (8.1) and (8.2), we can compute the $\mathcal{L}_i(M)$, as promised.

REFERENCES

[1] ADLER, R. J. (2000). On excursion sets, tube formulas and maxima of random fields. *Ann. Appl. Probab.* **10** 1–74. MR1765203

[2] ADLER, R. J., BARTZ, K. and KOU, S. C. (2011). Estimating thresholding levels for random fields via Euler characteristics. Unpublished manuscript.

[3] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. Springer, New York. MR2319516

[4] ADLER, R. J., TAYLOR, J. E. and WORSLEY, K. J. (2013). *Applications of Random Fields and Geometry*: *Foundations and Case Studies*. Springer. To appear. Early chapters available at http://webee.technion.ac.il/people/adler/publications.html.

[5] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[6] FEDERER, H. (1959). Curvature measures. *Trans. Amer. Math. Soc.* **93** 418–491. MR0110078

[7] GIBSON, C. G., WIRTHMÜLLER, K., DU PLESSIS, A. A. and LOOIJENGA, E. J. N. (1976). *Topological Stability of Smooth Mappings*. *Lecture Notes in Mathematics* **552**. Springer, Berlin. MR0436203

[8] PRESS, W. H., TEUKOLSKY, S. A., VETTERLING, W. T. and FLANNERY, B. P. (2007). *Numerical Recipes*: *The Art of Scientific Computing*, 3rd ed. Cambridge Univ. Press, Cambridge. MR2371990

[9] SHAFIE, K., SIGAL, B., SIEGMUND, D. and WORSLEY, K. J. (2003). Rotation space random fields with an application to fMRI data. *Ann. Statist.* **31** 1732–1771. MR2036389

[10] SIEGMUND, D. O. and WORSLEY, K. J. (1995). Testing for a signal with unknown location and scale in a stationary Gaussian random field. *Ann. Statist.* **23** 608–639. MR1332585

[11] TAYLOR, J., TAKEMURA, A. and ADLER, R. J. (2005). Validity of the expected Euler characteristic heuristic. *Ann. Probab.* **33** 1362–1396. MR2150192

[12] TAYLOR, J. E. (2001). Euler characteristics for Gaussian fields on manifolds. Ph.D. thesis, McGill Univ., Montreal. MR2704196

[13] TAYLOR, J. E. (2006). A Gaussian kinematic formula. *Ann. Probab.* **34** 122–158. MR2206344

[14] TAYLOR, J. E. and ADLER, R. J. (2009). Gaussian processes, kinematic formulae and Poincaré's limit. *Ann. Probab.* **37** 1459–1482. MR2546751

[15] TAYLOR, J. E. and WORSLEY, K. J. (2007). Detecting sparse signals in random fields, with an application to brain mapping. *J. Amer. Statist. Assoc.* **102** 913–928. MR2354405

[16] WORSLEY, K. J. (2001). Testing for signals with unknown location and scale in a χ^2 random field, with an application to fMRI. *Adv. in Appl. Probab.* **33** 773–793. MR1875779

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