A GENERALIZATION OF THE BERNOULLI POLYNOMIALS

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A generalization of the Bernoulli polynomials and, consequently, of the Bernoulli numbers, is defined starting from suitable generating functions. Furthermore, the differential equations of these new classes of polynomials are derived by means of the factorization method introduced by Infeld and Hull (1951).

1. Introduction

The Bernoulli polynomials have important applications in number theory and classical analysis. They appear in the integral representation of differentiable periodic functions since they are employed for approximating such functions in terms of polynomials. They are also used for representing the remainder term of the composite Euler-MacLaurin quadrature rule (see [15]).

The Bernoulli numbers [3, 13] appear in number theory, and in many mathematical expressions, such as

(i) the Taylor expansion in a neighborhood of the origin of the circular and hyperbolic tangent and cotangent functions;
(ii) the sums of powers of natural numbers;
(iii) the residual term of the Euler-MacLaurin quadrature rule.

The Bernoulli polynomials $B_n(x)$ are usually defined (see, e.g., [7, page xxix]) by means of the generating function

$$G(x,t) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi, \tag{1.1}$$

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and the Bernoulli numbers $B_n := B_n(0)$ by the corresponding equation

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.2}$$

The B_n are rational numbers. We have, in particular, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, and $B_{2k+1} = 0$, for $k = 1, 2, \ldots,$

$$B_0(x) = 1, \qquad B_1(x) = x - \frac{1}{2}, \qquad B_2(x) = x^2 - x + \frac{1}{6}. \tag{1.3}$$

The following properties are well known:

$$\begin{gathered} B_n(0) = B_n(1) = B_n, \quad n \neq 1, \\ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \qquad B_n'(x) = n B_{n-1}(x). \end{gathered} \tag{1.4}$$

The Bernoulli polynomials are easily computed by recursion since

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k(x) = n x^{n-1}, \quad n = 2, 3, \ldots. \tag{1.5}$$

Some generalized forms of the Bernoulli polynomials and numbers already appeared in literature. We recall, for example, the generalized Bernoulli polynomials $B_n^{\alpha}(x)$ recalled in the book of Gatteschi [6] defined by the generating function

$$\frac{t^{\alpha} e^{xt}}{(e^t - 1)^{\alpha}} = \sum_{n=0}^{\infty} B_n^{\alpha}(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \tag{1.6}$$

by means of which, Tricomi and Erdélyi [16] gave an asymptotic expansion of the ratio of two gamma functions.

Another generalized forms can be found in [5, 11], starting from the generating functions

$$\frac{(iz)^{\alpha} e^{(x-1/2)z}}{2^{2\alpha}\Gamma(\alpha+1) J_{\alpha}(iz/2)} = \sum_{n=0}^{\infty} B_{n,\alpha}(x) \frac{z^n}{n!}, \quad |z| < 2\left|j_1\right|, \tag{1.7}$$

where J_α is the Bessel function of the first kind of order α and $j_1 = j_1(\alpha)$ is the first zero of J_α, or

$$\frac{(ht)^\alpha (1+wt)^{x/w}}{\left[(1+wt)^{h/w}-1\right]^\alpha} = \sum_{n=0}^{\infty} B^{\alpha}_{n;h,w}(x)\frac{t^n}{n!}, \quad |t| < \left|\frac{1}{w}\right|, \tag{1.8}$$

respectively.

In this paper, we introduce a countable set of polynomials $B_n^{[m-1]}(x)$ generalizing the Bernoulli ones, which can be recovered assuming $m = 1$. To this aim, we consider a class of Appell polynomials [2], defined by using a generating function linked to the so-called Mittag-Leffler function

$$E_{1,m+1}(t) := \frac{t^m}{e^t - \sum_{h=0}^{m-1} t^h/h!}, \tag{1.9}$$

considered in the general form by Agarwal [1] (see also [12]).

Furthermore, exploiting the factorization method introduced in [10] and recalled in [8], we derive the differential equation satisfied by these polynomials. It is worth noting that the differential equation for Appell-type polynomials was derived in [14], and more recently recovered in [9] by exploiting the factorization method. It is easily checked that our differential equation matches with the general form of the above mentioned articles [9, 14]. In particular, when $m = 1$, the differential equation of the classical Bernoulli polynomials is derived again.

We will show in this paper that the differential equation satisfied by the $B_n^{[m-1]}(x)$ polynomials is of order n, so that all the considered families of polynomials can be viewed as solutions of differential operators of infinite order.

This is a quite general situation since the Appell-type polynomials, satisfying a differential operator of finite order, can be considered as an exceptional case (see [4]).

2. A new class of generalized Bernoulli polynomials

The generalized Bernoulli polynomials $B_n^{[m-1]}(x)$, $m \geq 1$, are defined by means of the generating function, defined in a suitable neighborhood of $t = 0$

$$G^{[m-1]}(x,t) := \frac{t^m e^{xt}}{e^t - \sum_{h=0}^{m-1} t^h/h!} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x)\frac{t^n}{n!}. \tag{2.1}$$

For $m = 1$, we obtain, from (2.1), the generating function $G^{(0)}(x,t) = te^{xt}/(e^t - 1)$ of classical Bernoulli polynomials $B_n^{(0)}(x)$.

Since $G^{[m-1]}(x,t)=A(t)e^{xt}$, the generalized Bernoulli polynomials belong to the class of Appell polynomials.

It is possible to define the generalized Bernoulli numbers assuming

$$B_n^{[m-1]}=B_n^{[m-1]}(0). \tag{2.2}$$

From (2.1), we have

$$e^{xt}=\sum_{h=m}^{\infty}\frac{t^{h-m}}{h!}\sum_{n=0}^{\infty}B_n^{[m-1]}(x)\frac{t^n}{n!}. \tag{2.3}$$

Since $e^{xt}=\sum_{n=0}^{\infty}x^n(t^n/n!)$, (2.3) becomes

$$\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}=\sum_{j=0}^{\infty}\frac{j!}{(j+m)!}\frac{t^j}{j!}\sum_{n=0}^{\infty}B_n^{[m-1]}(x)\frac{t^n}{n!} \tag{2.4}$$

and therefore

$$\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{h=0}^{n}\binom{n}{h}\frac{h!}{(h+m)!}B_{n-h}^{[m-1]}(x)\frac{t^n}{n!}. \tag{2.5}$$

By comparing the coefficients of (2.5), we obtain

$$x^n=\sum_{h=0}^{n}\binom{n}{h}\frac{h!}{(h+m)!}B_{n-h}^{[m-1]}(x). \tag{2.6}$$

Inverting (2.6), it is possible to find explicit expressions for the polynomials $B_n^{[m-1]}(x)$. The first ones are given by

$$\begin{aligned}&B_0^{[m-1]}(x)=m!, \qquad B_1^{[m-1]}(x)=m!\left(x-\frac{1}{m+1}\right),\\&B_2^{[m-1]}(x)=m!\left(x^2-\frac{2}{m+1}x+\frac{2}{(m+1)^2(m+2)}\right),\end{aligned} \tag{2.7}$$

and, consequently, the first generalized Bernoulli numbers are

$$B_0^{[m-1]}=m!, \qquad B_1^{[m-1]}=-\frac{m!}{m+1}, \qquad B_2^{[m-1]}=\frac{2m!}{(m+1)^2(m+2)}. \tag{2.8}$$

3. Differential equation for generalized Bernoulli polynomials

In this section, we prove the following theorem.

Theorem 3.1. *The generalized Bernoulli polynomials* $B_n^{[m-1]}(x)$ *satisfy the differential equation*

$$\begin{aligned}\frac{B_n^{[m-1]}}{n!}y^{(n)}+\frac{B_{n-1}^{[m-1]}}{(n-1)!}y^{(n-1)}+\cdots+\frac{B_2^{[m-1]}}{2!}y''\\ +(m-1)!\left(\frac{1}{m+1}-x\right)y'+n(m-1)!y=0.\end{aligned} \tag{3.1}$$

In order to prove (3.1), we first derive a recurrence relation for $B_n^{[m-1]}(x)$.

Lemma 3.2. *For any integral* $n\geq 1$, *the following linear homogeneous recurrence relation for the generalized Bernoulli polynomials holds true:*

$$B_n^{[m-1]}(x)=\left(x-\frac{1}{m+1}\right)B_{n-1}^{[m-1]}(x)-\frac{1}{n(m-1)!}\sum_{k=0}^{n-2}\binom{n}{k}B_{n-k}^{[m-1]}B_k^{[m-1]}(x). \tag{3.2}$$

This relation, starting from $n=1$, and taking into account the initial value $B_0^{[m-1]}(x)=m!$, allows a recursive formula for the generalized Bernoulli polynomials.

Proof. Differentiation of both sides of (2.1), with respect to t, yields

$$\begin{aligned}\frac{\partial}{\partial t}G^{[m-1]}(x,t)&=\frac{mt^{m-1}\left(e^t-\sum_{h=0}^{m-1}t^h/h!\right)-t^m\left(e^t-\sum_{h=1}^{m-1}t^{h-1}/(h-1)!\right)}{\left(e^t-\sum_{h=0}^{m-1}t^h/h!\right)^2}e^{xt}\\ &\quad+\frac{xt^m}{e^t-\sum_{h=0}^{m-1}t^h/h!}e^{xt}\\ &=\left[\frac{m}{t}\frac{t^m}{e^t-\sum_{h=0}^{m-1}t^h/h!}-\frac{t^m}{e^t-\sum_{h=0}^{m-1}t^h/h!}\right.\\ &\quad\left.-\frac{1}{(m-1)!}\frac{t^{2m-1}}{\left(e^t-\sum_{h=0}^{m-1}t^h/h!\right)^2}\right]e^{xt}\\ &\quad+xG^{[m-1]}(x,t)\end{aligned}$$

$$
\begin{aligned}
&= \frac{m}{t} G^{[m-1]}(x,t) + (x-1)G^{[m-1]}(x,t) \\
&\quad - \frac{t^{m-1}}{(m-1)!\left(e^t - \sum_{h=0}^{m-1} t^h/h!\right)} \\
&\quad \times \frac{t^m}{e^t - \sum_{h=0}^{m-1} t^h/h!} e^{xt} \\
&= \frac{1}{(m-1)!t}\left(m! - \frac{t^m}{e^t - \sum_{h=0}^{m-1} t^h/h!}\right) \\
&\quad \times G^{[m-1]}(x,t) + (x-1)G^{[m-1]}(x,t) \\
&= \frac{1}{(m-1)!t}\left(m! - \sum_{n=0}^{\infty} B_n^{[m-1]} \frac{t^n}{n!}\right) \\
&\quad \times G^{[m-1]}(x,t) + (x-1)G^{[m-1]}(x,t),
\end{aligned} \tag{3.3}
$$

and consequently

$$
\begin{aligned}
(m-1)!t\frac{\partial}{\partial t} G^{[m-1]}(x,t) = m!G^{[m-1]}(x,t) - \sum_{n=0}^{\infty} B_n^{[m-1]} \frac{t^n}{n!} G^{[m-1]}(x,t) \\
+ (m-1)!t(x-1)G^{[m-1]}(x,t).
\end{aligned} \tag{3.4}
$$

Recalling (2.1), the left-hand side of (3.4) becomes

$$
\begin{aligned}
(m-1)!t\frac{\partial}{\partial t} G^{[m-1]}(x,t) &= (m-1)! \sum_{n=1}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{(n-1)!} \\
&= (m-1)! \sum_{n=0}^{\infty} n B_n^{[m-1]}(x) \frac{t^n}{n!}.
\end{aligned} \tag{3.5}
$$

Furthermore, introducing $B_{-1}^{[m-1]}(x) := 0$ (but in principle $B_{-1}^{[m-1]}(x)$ could be chosen as an arbitrary constant), the following equation is obtained:

$$
\begin{aligned}
(m-1)!t(x-1)G^{[m-1]}(x,t) &= (m-1)! \sum_{n=0}^{\infty} (x-1) B_n^{[m-1]}(x) \frac{t^{n+1}}{n!} \\
&= (m-1)! \sum_{n=0}^{\infty} n(x-1) B_{n-1}^{[m-1]}(x) \frac{t^n}{n!},
\end{aligned} \tag{3.6}
$$

and moreover

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{[m-1]} \frac{t^n}{n!} G^{[m-1]}(x,t) &= \sum_{n=0}^{\infty} B_n^{[m-1]} \frac{t^n}{n!} \sum_{h=0}^{\infty} \frac{t^h}{h!} B_h^{[m-1]}(x) \\ &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x) \right] \frac{t^n}{n!}.\end{aligned} \tag{3.7}$$

Substitution of (3.5), (3.6), and (3.7) into (3.4) yields

$$\begin{aligned}(m-1)! \sum_{n=0}^{\infty} n B_n^{[m-1]}(x) \frac{t^n}{n!} = m! &\sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!} \\ &- \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x) \right] \frac{t^n}{n!} \\ &+ (m-1)! \sum_{n=0}^{\infty} n(x-1) B_{n-1}^{[m-1]}(x) \frac{t^n}{n!}.\end{aligned} \tag{3.8}$$

Then the conclusion immediately follows by the identity principle of power series, equating coefficients in the left- and right-hand side of the last equation (3.8). □

Proof of Theorem 3.1. We now use this recurrence relation to find the operator E_n^+ such that

$$E_n^+ B_n^{[m-1]}(x) = B_{n+1}^{[m-1]}(x), \quad n = 0, 1, \ldots. \tag{3.9}$$

It is easy to see that, for $k = 0, 1, \ldots, n-1$,

$$\frac{d^{n-k}}{dx^{n-k}} B_n^{[m-1]}(x) = \frac{n!}{k!} B_k^{[m-1]}(x). \tag{3.10}$$

By means of (3.10), the recurrence relation can be written as

$$B_{n+1}^{[m-1]}(x) = \left[\left(x - \frac{1}{m+1} \right) - \frac{1}{(m-1)!} \sum_{k=0}^{n-1} \frac{B_{n+1-k}^{[m-1]}}{(n+1-k)!} D_x^{n-k} \right] B_n^{[m-1]}(x), \tag{3.11}$$

and therefore

$$E_n^+ = \left(x - \frac{1}{m+1} \right) - \frac{1}{(m-1)!} \sum_{k=0}^{n-1} \frac{B_{n+1-k}^{[m-1]}}{(n+1-k)!} D_x^{n-k}. \tag{3.12}$$

We are now in a position to determine the differential equation for $B_n^{[m-1]}(x)$. Applying both operators $E_{n+1}^{-} = (1/(n+1))D_x$ and E_n^{+} to $B_n^{[m-1]}(x)$, we have

$$\left(E_{n+1}^{-}E_n^{+}\right)B_n^{[m-1]}(x) = B_n^{[m-1]}(x). \tag{3.13}$$

That is,

$$\begin{aligned}\frac{1}{n+1}D_x\left[\left(x-\frac{1}{m+1}\right)-\frac{1}{(m-1)!}\sum_{k=0}^{n-1}\frac{B_{n+1-k}^{[m-1]}}{(n+1-k)!}D_x^{n-k}\right]B_n^{[m-1]}(x)& \\ = B_n^{[m-1]}(x).&\end{aligned} \tag{3.14}$$

This leads to the differential equation with $B_n^{[m-1]}(x)$ as a polynomial solution. □

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