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Research Article

A New Method for Proving Existence Theorems for Abstract Hammerstein Equations

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An abstract Hammerstein equation is an equation of the form $u + KFu = 0$. A *new method* is introduced to prove the existence of a solution of this equation where K and F are nonlinear accretive (monotone) operators. The method does not involve the complicated technique of factorizing a linear map via a Hilbert space and does not involve the use of deep variational techniques.

1. General Introduction

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. The space E is said to have *Gâteaux differentiable norm* if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{1}$$

exists for all $x, y \in S$; in this case E is said to be smooth. E is said to have *uniformly Gâteaux differentiable norm* if, for each $y \in S$, the limit is attained uniformly for $x \in S$. Further, E is said to be *uniformly smooth* if the limit is attained uniformly for $(x, y) \in S\times S$. The *modulus of smoothness* of E, $\rho_E : [0,\infty) \to [0,\infty)$, is defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0. \tag{2}$$

E is equivalently said to be *smooth* if $\rho_E(\tau) > 0\ \forall\tau > 0$. Let $q > 1$; E is said to be *q-uniformly smooth* (or to have a *modulus of smoothness of power type q*) if there exists $c > 0$ such that $\rho_E(\tau) \le c\tau^q$.

L_p, l_p, and the Sobolev space W_m^p, $1 < p < \infty$, are all q-uniformly smooth. In fact

$$L_p \text{ or } l_p \text{ or } W_m^p \text{ is } \begin{cases} p\text{-uniformly smooth}, & 1 < p \le 2, \\ 2\text{-uniformly smooth}, & p \ge 2. \end{cases} \tag{3}$$

Furthermore (see, e.g., [1]),

$$\begin{aligned}\rho_{L_p}(\tau) &= \rho_{l_p}(\tau) = \rho_{W_m^p}(\tau) \\ &= \begin{cases} (1+\tau^p)^{1/p} - 1 < \dfrac{1}{p}\tau^p, & 1 < p \le 2, \\ \dfrac{(p-1)}{2}\tau^2 + o(\tau^2) < \dfrac{p-1}{2}\tau^2, & p \ge 2. \end{cases}\end{aligned} \tag{4}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q,\ \|f\| = \|x\|^{q-1}\right\}, \tag{5}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known (see, e.g., Xu [2]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$ where J denotes J_2 (called the *normalized duality mapping*). It is well known that if E^* is strictly convex, J is single-valued. For more information and examples concerning (generalized) duality mappings, one may see the book of Cioranescu [3] and its review by Reich [4]. In the sequel, we will denote the single-valued duality map by j.

A map $A : D(A) \subset X \to X$ is called *accretive* if, for all $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that the following inequality holds:

$$\langle Ax - Ay, j(x-y) \rangle \geq 0. \tag{6}$$

If X is a real Hilbert space, the map A is called *monotone*. In this case, A satisfies the following condition:

$$\langle Ax - Ay, x - y \rangle \geq 0. \tag{7}$$

The map A is called *strongly accretive* if there exists $c > 0$ such that, for all $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$, such that

$$\langle Ax - Ay, j(x-y) \rangle \geq c\|x-y\|^2. \tag{8}$$

A nonlinear integral equation of Hammerstein type (see, e.g., Hammerstein [5]) has the form

$$u(x) + \int_\Omega k(x,y) f(y, u(y))\, dy = h(x), \tag{9}$$

where dy is a σ-finite measure on Ω; the kernel k is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear, and h is a function on Ω. Setting

$$Kv(\cdot) := \int_\Omega k(\cdot, y) v(y)\, dy \quad \text{on } \Omega \tag{10}$$

and $Fu(\cdot) := f(\cdot, u(\cdot))$ on Ω, then integral equation (9) can be put in abstract operator form as follows:

$$u + KFu = 0, \tag{11}$$

where, without loss of generality, we have taken $h \equiv 0$.

Interest in (9) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function, can, as a rule, be transformed into the form of (9).

Furthermore, equations of Hammerstein type play crucial role in the theory of optimal control systems, in automation, and in network theory (see, e.g., Dolezale [6]).

Several existence theorems for the solution of (9) have been proved by a host of distinguished mathematicians using various techniques (see, e.g., Browder and Gupta [7, 8], Chepanovich [9], and Petryshyn and Fitzpatrick [8]). In the remaining part of this section, we highlight the techniques used by Browder and Gupta [7] and Petryshyn and Fitzpatrick [8]. To do this, we first give definitions of some terms which are required in the theorems.

In the sequel, the symbol "$\to$" denotes strong convergence while "$\rightharpoonup$" denotes weak convergence.

Definition 1 (see, e.g., [7]). A mapping $A : D(A) \subset X^* \to X$ is said to be *hemicontinuous* if it is continuous from each line segment of X^* to the weak topology of X. That is, $\forall u \in D(A)$, $\forall v \in X^*$, and $(t_n)_{n\geq 1} \subset \mathbb{R}^+$ such that $t_n \to 0^+$ and $u + t_n v \in D(A)$ for n sufficiently large and we have $A(u + t_n v) \rightharpoonup A(u)$.

Definition 2 (see, e.g., [7]). Let $A : X \to X^*$ be a bounded monotone linear mapping. A is said to be *angle-bounded* with constant $c \geq 0$ if, for all u, v in X, $|\langle Au, v \rangle - \langle Av, u \rangle| \leq 2c\{\langle Au, u \rangle\}^{1/2}\{\langle Av, v \rangle\}^{1/2}$. (This is well defined since $\langle Au, u \rangle \geq 0$ and $\langle Av, v \rangle \geq 0$ by the linearity and monotonicity of A.)

In [7] Browder and Gupta proved the following theorem.

Theorem 3 (Browder-Gupta [7]). *Let X be a real Banach space and X^* its conjugate dual space. Let K be a monotone angle-bounded continuous linear mapping of X into X^* with constant of angle-boundedness $c \geq 0$. Let F be a hemicontinuous (possibly nonlinear) mapping of X^* into X such that, for a given constant $k \geq 0$,*

$$\langle v_1 - v_2, Fv_1 - Fv_2 \rangle \geq -k\|v_1 - v_2\|_{X^*}^2 \tag{12}$$

for all v_1 and v_2 in X^*. Suppose finally that there exists a constant R with $k(1+c^2)R < 1$ such that for u in X

$$\langle Ku, u \rangle \leq R\|u\|_X^2. \tag{13}$$

Then, there exists exactly one solution w in X^* of the nonlinear equation

$$w + KFw = 0. \tag{14}$$

The main tool used by the authors in proving Theorem 3 is that of splitting the linear operator K via a Hilbert space and then applying a deep result of Minty [10]. Precisely, they proved that if X is a real Banach space, X^* is its dual space, and K is a bounded *linear* mapping of X into X^* which is monotone and angle-bounded, then there exist a Hilbert space H, a continuous linear mapping S of X into H with adjoint S^* injective, and a bounded skew-symmetric linear mapping B of H into H such that

$$K = S^*(I + B)S \tag{15}$$

(see Figure 1).

This factorization enabled the authors to transform the problem into another problem in a Hilbert space such that Hammerstein equation (11) has a solution if and only if the new problem has a solution in a real Hilbert space. They set $f = (I+B)^{-1} + KFK^*$, $D := B(0,1)$, the closed unit ball in H, and showed that f is hemicontinuous and monotone and satisfies $\langle u, f(u) \rangle \geq 0$ $\forall u \in D$. With these facts, they used the following result of Minty [10] to prove Theorem 3 (see [10] for definitions of terms).

Theorem 4 (Minty [10]). *Let $D \subset X$ be bounded and surround 0; let $C \subset X$ contain $\overline{\text{co}}(D)$ and surround every point of $\overline{\text{co}}(D)$ densely. Let*

$$f : C \longrightarrow X^* \tag{16}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
H \arrow[r, "B", "\text{Linear, bounded, skew-symmetric}"'] & H \arrow[d, "S^*", "\text{Injective}"'] \\
X \arrow[u, "S", "\text{Linear, bounded}"'] \arrow[r, "K", "\text{Given}"'] & X^*
\end{tikzcd}$$

Figure 1: Factorization of operator, K.

be monotone and hemicontinuous at every point of $\overline{\text{co}}(D)$ *and suppose*

$$u \in D \text{ implies } \langle u, f(u) \rangle \geq 0. \tag{17}$$

Then, there exists $u \in \overline{\text{co}}(D)$ *such that* $f(u) = 0$.

Petryshyn and Fitzpatrick employed deep variational techniques to prove the existence of a solution to (11). They proved the following theorems.

Theorem 5 (Petryshyn-Fitzpatrick [8]). *Let* X *be a reflexive Banach space and let* K *be a linear, monotone, and symmetric mapping of* X *into* X^*. *Suppose* f *is a weakly (sequential) lower semicontinuous functional on* X^* *such that*

$$f(u) \geq -\frac{1}{2} a_1 \|u\|^2 - a_2 \|u\|^\delta - a_3, \tag{18}$$

where $a_1\|K\| < 1$, $a_2 > 0$, $a_3 > 0$, *and* $0 < \delta < 2$. *Suppose also that* $F : X^* \to X$ *is such that* $\text{grad}(f) = F$. *Then,*

$$w + KFw = 0 \tag{19}$$

has a solution in X^*.

Theorem 6 (Petryshyn-Fitzpatrick [8]). *Let* X *be a reflexive Banach space with* $K : X \to X^*$ *linear, monotone, and symmetric. Let* $F : X^* \to X$ *be potential and have a Gâteaux derivative which satisfies the inequality*

$$DF(u, v, v) \geq -a \|v\|^2 \quad (v, u \in X^*) \tag{20}$$

and $DN(tu, v, v)$ *is continuous in* $t \in [0, 1]$ *for* u *and* v *fixed, where* $a\|K\| < 1$. *Then, (19) has a solution in* X^*.

In this paper, we introduce *a new method*, perhaps simpler than methods used so far in the literature, of proving existence of solutions of Hammerstein equation in certain cases. To achieve this, we recast (11) into a fixed point problem and use a technique recently introduced by Chidume and Zegeye [11], some existence results of Deimling [12] for zeros of accretive maps, and some surjectivity results of Browder [13] for Lipschitz strongly accretive maps. No linearity assumption is imposed on any of our maps.

2. Preliminaries

Let X be a normed linear space and let K be a convex subset of X. For $x \in X$, the *inward set*, $I_k(x)$, of x relative to K, is defined as follows:

$$I_K(x) = \{x + c(u - x) : c \geq 1, u \in K\}. \tag{21}$$

A mapping $T : K \to X$ is said to be *inward* if $Tx \in I_K(x)$ for each $x \in K$ and *weakly inward* if Tx belongs to the closure of $I_K(x)$ for each $x \in K$.

A relationship between the weak inward condition and the condition

$$\lim_{\lambda \to 0^+} \frac{\text{dist}(x - \lambda Ax, D(A))}{\lambda} = 0 \quad \forall x \in D(A) \tag{22}$$

for a map $A : D(A) \subset X \to X$ is given in Lemma 11. Further relationship between condition (22), the weak inward condition, and Lemma 11 can be found in [14].

In the sequel, X is a q-uniformly smooth real Banach space, $q > 1$, and $E := X \times X$ with

$$\|[u, v]\|_E = \left(\|u\|^q + \|v\|^q\right)^{1/q} \quad \forall [u, v] \in E. \tag{23}$$

If $X(= H)$ is a real Hilbert space, we will denote E by $E^H := H \times H$.

If F and K are maps from X to X such that range of F is contained in domain of K, that is, $R(F) \subseteq D(K)$, Chidume and Zegeye [11] defined a map $A : E \to E$ as follows:

$$A[u, v] = [Fu - v, Kv + u] \tag{24}$$

for all $u, v \in X$ and observed that $A[u, v] = 0$ if and only if

$$\begin{aligned} Fu - v &= 0, \\ Kv + u &= 0, \end{aligned} \tag{25}$$

so that u solves (11). System (25) can be recast as a fixed point problem as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -K \\ F & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{26}$$

We will use the ideas of map A on E.

In Lemmas 9 and 10, we use the following variant definition of accretive maps as given by Deimling [12].

Definition 7 (accretive map in the sense of Deimling [12]). Let X be real Banach space. A map $A : D(A) \subset X \to X$ is said to be *accretive* (*in the sense of Deimling*) if

$$\langle A(x) - A(y), x - y \rangle_+ \geq 0 \quad \forall x, y \in D(A), \tag{27}$$

where

$$\langle x, y \rangle_+ := \sup_{j(y) \in J(y)} \langle x, j(y) \rangle, \quad \forall x, y \in X. \tag{28}$$

It is evident that, in any real Banach space, an accretive map is also accretive in the sense of Deimling. The converse is true in any real Banach X whose dual X^* is strictly convex or whose normalized duality map is single-valued. This is certainly the case when X is q-uniformly smooth, $q > 1$.

Definition 8 (see, e.g., [15]). A *bounded convex* subset K of a Banach space X is said to have *normal structure* if every *convex* subset C of K having more than one element contains at least one nondiametral point; that is, there exists $x^0 \in C$ such that

$$\sup\{\|x^0 - x\| : x \in C\} < \sup\{\|x - y\| : x, y \in C\} = d(C). \tag{29}$$

The Banach space X is said to have *normal structure* if *every* bounded convex subset of X has normal structure.

Lemma 9 (Deimling [12]). *Let X be a reflexive real Banach space with normal structure and let D be a closed convex bounded subset of X. Let $A : D \to X$ be a Lipschitz and accretive map satisfying condition (22). Then, $0 \in A(D)$.*

Lemma 10 (Deimling [12]). *Let X be real Banach space and let D be a closed convex subset of X. Let $A : D \subset X \to X$ be an accretive continuous map such that $\langle Ax, x\rangle_+ \geq 0$ for all $x \in X$ with $\|x\| \geq R$ for some $R > 0$ or $\lim\|Ax\| = \infty$ as $\|x\| \to \infty$. Suppose A satisfies condition (22) and suppose that $A(D)$ is closed. Then, $0 \in A(D)$.*

Lemma 11 (Caristi [16]). *Let D be a convex subset of a normed linear space X and let $A : D \to X$ be a map. Then condition (22) holds if and only if $(I - A)$ is weakly inward and I is the identity map on D.*

Remark 12. In view of Lemma 11, if $D = H$ in Lemma 10, then condition (22) can be dropped.

Lemma 13 (Xu [2]). *Let $q > 1$ and E a smooth real Banach space. Then the following are equivalent.*

(i) *E is q-uniformly smooth.*

(ii) *There exists a constant $d_q > 0$ such that, for all $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q. \tag{30}$$

(iii) *There exists a constant $c_q > 0$ such that for all $x, y \in E$ and $\lambda \in [0, 1]$*

$$\begin{aligned}\|(1-\lambda)x + \lambda y\|^q &\geq (1-\lambda)\|x\|^q + \lambda\|y\|^q \\ &\quad - w_q(\lambda)c_q\|x - y\|^q,\end{aligned} \tag{31}$$

where $w_q(\lambda) := \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$.

From now on, c_q and d_q denote the constants appearing in Lemma 13.

Lemma 14 (Chidume [15], p. 173). *Let X be a q-uniformly smooth real Banach space. Let $F, K : X \to X$ be maps with F surjective such that the following conditions hold:*

(i) *there exists $\alpha > 0$ such that, for each $u_1, u_2 \in D(F)$,*

$$\langle Fu_1 - Fu_2, j_q(u_1 - u_2)\rangle \geq \alpha\|u_1 - u_2\|^q; \tag{32}$$

(ii) *there exists $\beta > 0$ such that, for each $u_1, u_2 \in D(K)$,*

$$\langle Ku_1 - Ku_2, j_q(u_1 - u_2)\rangle \geq \beta\|u_1 - u_2\|^q; \tag{33}$$

(iii) $(1 + d_q)(1 + c_q) \geq 2^q$, $\min\{\alpha, \beta\} =: \gamma > ((1 + d_q)(1 + c_q) - 2^q)/q(1 + c_q)$.

Let a map $A : E \to E$ be defined by (24). Then, for each z_1, $z_2 \in E$,

$$\begin{aligned}&\langle Az_1 - Az_2, j_q(z_1 - z_2)\rangle \\ &\quad \geq \left[\gamma - q^{-1}\left(\left(1 + d_q\right) - \frac{2^q}{\left(1 + c_q\right)}\right)\right]\|z_1 - z_2\|^q.\end{aligned} \tag{34}$$

Lemma 15. *Let H be a real Hilbert space. Let $K : D(K) \subset H \to H$, $F : D(F) \subset H :\to H$ be two monotone maps such that $R(F) \subset D(K)$. Then the map $A : D(F) \times D(K) \subset E^H \to E^H$ defined by (24) is monotone.*

Proof. The proof follows from the lines of argument of the proof of Lemma 14 (see Chidume and Zegeye [11]). □

Lemma 16 (Chidume [15], p. 173). *Let X be a q-uniformly smooth real Banach space and let $K : D(K) \subset X \to X$, $F : D(F) \subset X \to X$ be two Lipschitz maps such that $R(F) \subset D(K)$. Let $A : D(A) \subset E$ be a map such that $D(F) \times D(K) = D(A)$ and defined by (24). Then, A is Lipschitz.*

We need the following definition which was given by Browder [17].

Definition 17 (Browder [17]). Let X and Y be real Banach spaces with Y^* the conjugate space of Y. Let ϕ be a mapping of X into Y^* such that $\phi(X)$ is dense in Y^* with

$$\begin{aligned}\|\phi(x)\|_{Y^*} &= \|x\|, \\ \phi(\xi x) &= \xi\phi(x)\end{aligned} \tag{35}$$

for all $x \in X$, $\xi \geq 0$. The mapping $f : X \to Y$ is said to be *strongly ϕ-accretive* if there exists $c > 0$ such that, for all x and u in X,

$$\langle f(x) - f(u), \phi(x - u)\rangle \geq c\|x - u\|^2. \tag{36}$$

It follows from this definition that if X is a real Banach space such that the normalized duality map J is single-valued and $J(X)$ is dense in X^* (e.g., when X is a reflexive and smooth real Banach space), then a strongly accretive map $A : X \to X$ is J-strongly accretive.

Theorem 18 (Browder [13]). *Let X and Y be Banach spaces with Y^* uniformly convex and suppose $f : X \to Y$ is a strongly ϕ-accretive mapping satisfying a Lipschitz condition on each bounded subset of X. Then, $f(X) = Y$.*

The following corollary follows from Theorem 18.

Corollary 19. *Let X be a real Banach space with uniformly convex dual X^* and suppose $f : X \to X$ is a strongly accretive Lipschitz mapping. Then, $f(X) = X$.*

3. Main Results

Let $X := L_p$, $1 < p < 2$, and let $E := X \times X$ with $\|z\|_E^2 := \|[u, v]\|_E^2 = \|u\|_X^2 + \|v\|_E^2$ for arbitrary $z = [u, v] \in E$. For L_p spaces, $1 < p < 2$, the following estimate has been established (see, e.g., Chidume [15], p. 183):

$$\begin{aligned} &A(u_1, u_2, v_1, v_2) \\ &\quad := [\langle v_1 - v_2, j(u_1 - u_2)\rangle + \langle u_1 - u_2, j(u_2 - u_1)\rangle] \\ &\quad \leq p(2 - p)\left(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2\right) \\ &\qquad\qquad \forall u_1, u_2, v_1, v_2 \in X. \end{aligned} \tag{37}$$

We begin with a proof of the following theorem for L_p spaces, $1 < p < 2$, *which is new.*

Theorem 20. *Let $X = L_p$ $(1 < p < 2)$; let $F, K : X \to X$ be mappings such that $D(K) = F(X) = X$ and the following conditions hold:*

(a) *there exists $\alpha > 0$ such that, for each $u_1, u_2 \in X$,*

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \alpha \|u_1 - u_2\|^2; \tag{38}$$

(b) *there exists $\beta > 0$ such that, for each $u_1, u_2 \in X$,*

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \beta \|u_1 - u_2\|^2; \tag{39}$$

(c) $\gamma := \min\{\alpha, \beta\}$ *with* $\gamma > p(2 - p)$.

Let $E := X \times X$ and define $A : E \to E$ by (24) for all $[u, v] \in E$. Then, for arbitrary $z_1, z_2 \in E$, the following inequality holds:

$$\begin{aligned} &\left\langle Az_1 - Az_2, j^E(z_1 - z_2)\right\rangle \\ &\quad \geq [\gamma - p(2 - p)] \|z_1 - z_2\|^2. \end{aligned} \tag{40}$$

Proof. We compute as follows:

$$\begin{aligned} &\left\langle Az_1 - Az_2, j^E(z_1 - z_2)\right\rangle \\ &\quad = \langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle - \langle v_1 - v_2, j(u_1 - u_2)\rangle \\ &\qquad + \langle Kv_1 - Kv_2, j(v_1 - v_2)\rangle \\ &\qquad + \langle u_1 - u_2, j(v_1 - v_2)\rangle \\ &\quad \geq \alpha \|u_1 - u_2\|^2 + \beta \|v_1 - v_2\|^2 \\ &\qquad - \langle v_1 - v_2, j(u_1 - u_2)\rangle + \langle u_1 - u_2, j(v_1 - v_2)\rangle \\ &\quad \geq \gamma\left(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2\right) \\ &\qquad - [\langle v_1 - v_2, j(u_1 - u_2)\rangle - \langle u_1 - u_2, j(v_1 - v_2)\rangle] \\ &\quad \geq \gamma \|z_1 - z_2\|^2 - A(u_1, u_2, v_1, v_2) \\ &\quad \geq \gamma \|z_1 - z_2\|^2 \\ &\qquad - p(2 - p)\left(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2\right) \\ &\quad = (\gamma - p(2 - p)) \|z_1 - z_2\|^2 \quad \forall z_1, z_2 \in E, \end{aligned} \tag{41}$$

completing proof of the theorem. □

Remark 21. Observe that the condition $1 + \sqrt{1 - \gamma} < p < 2$ implies $\gamma > p(2 - p)$.

We now prove the following existence theorems.

3.1. The Case of Hilbert Spaces

Theorem 22. *Let H be a real Hilbert space and let $K : D(K) \subset H \to H$, $F : D(F) \subset H \to H$ be two Lipschitz monotone maps such that $D(F)$ and $D(K)$ are closed, convex, and bounded and $R(F) \subset D(K)$. Let $A : D(A) \subset E^H \to E^H$ be a map such that $D(F) \times D(K) =: D(A)$ and A is defined by (24). Suppose that A satisfies condition (22). Then, Hammerstein equation (11) has a solution.*

Proof. The fact that K and F are Lipschitz and monotone implies that A is Lipschitz and monotone (Lemmas 15 and 16). Since the normalized duality map is the identity map in real Hilbert spaces, monotonicity of A is equivalent to accretivity in the sense of Deimling. Also $D(A)$ is closed and convex since $D(F)$ and $D(K)$ are. Therefore, by Lemma 9, $0 \in A(D)$; that is, there exists $[u, v] \in D$ such that $Fu - v = 0$ and $Kv + u = 0$. So u solves (11). This completes the proof. □

Theorem 23. *Let H be a real Hilbert space and let $K : D(K) \subset H \to H$, $F : D(F) \subset H \to H$ be two continuous monotone maps such that $D(F)$ and $D(K)$ are closed and convex and $R(F) \subset D(K)$. Let $A : D(A) \subset E^H \to E^H$ be a map such that $D(F) \times D(K) =: D(A)$ and A is defined by (24). Suppose that $\langle Aw, w\rangle \geq 0$ for all $w \in D(A)$ with $\|w\| \geq R$ for some $R > 0$ or $\lim \|Aw\| = \infty$ as $\|w\| \to \infty$ and suppose that A satisfies condition (22). Suppose that $A(D(A))$ is closed. Then, Hammerstein equation (11) has a solution.*

Proof. The fact that K and F are monotone implies that A is monotone (Lemma 15). The fact that $D(F)$ and $D(K)$ are closed and convex implies that $D(A)$ is closed and convex. Also since E^H is a real Hilbert space and the normalized duality map of any real Hilbert space is the identity map, we have $\langle Aw, w\rangle_+ = \langle Aw, w\rangle$ for all $w \in D(A)$. Therefore, the assumptions on A and $D(A)$ together with Lemma 10 give that $0 \in A(D)$; that is, there exists $[u, v] \in D$ such that $Fu - v = 0$ and $Kv + u = 0$. So u solves (11). This completes the proof. □

Corollary 24. *Let H be a real Hilbert space and let $K, F : H \to H$ be two continuous monotone maps defined on H. Let $A : E^H \to E^H$ be a map defined by (24). Suppose that*

$\langle Aw, w\rangle \geq 0$ for all $w \in E^H$ with $\|w\| \geq R$ for some $R > 0$ or $\lim \|Aw\| = \infty$ as $\|w\| \to \infty$. Suppose that $A(E^H)$ is closed. Then, Hammerstein equation (11) has a solution.

Proof. Since A is defined on E^H, it satisfies condition (22). Therefore, the result follows from Theorem 23. □

3.2. The Case of L^p Spaces, $1 < p < \infty$

Theorem 25. *Let $K : D(K) \subset L^p \to L^p$ and $F : D(F) \subset L^p \to L^p$ be two Lipschitz mappings satisfying the following conditions:*

(a) *there exists $\alpha > 0$ such that, for each $u_1, u_2 \in D(F)$,*

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \alpha \|u_1 - u_2\|^2; \tag{42}$$

(b) *there exists $\beta > 0$ such that, for each $u_1, u_2 \in D(K)$,*

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \beta \|u_1 - u_2\|^2. \tag{43}$$

Let $D(F)$ and $D(K)$ be closed, convex, and bounded such that $R(F) \subset D(K)$. Let $E := L^p \times L^p$ and let $A : D(A) \subset E \to E$ be a map such that $D(F) \times D(K) =: D(A)$ and A is defined by (24). Suppose that A satisfies condition (22). Let $\gamma := \min\{\alpha, \beta\}$. If $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ or $1 + \sqrt{1-\gamma} < p \leq 2$, then Hammerstein equation (11) has a solution.

Proof. The fact that K and F are Lipschitz implies that A is Lipschitz by Lemma 16. Also $D(A)$ is closed and convex since $D(F)$ and $D(K)$ are.

Case 1 ($2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$). In this case L^p is 2-uniformly smooth space and $c_q = d_q = p - 1$ (see, e.g., [2]). Therefore, $(1 + c_q)(1 + d_q) = p^2 \geq 4 = 2^q$ and

$$\gamma > \frac{1}{2p}\left(p^2 - 4\right) = \frac{\left(1 + d_q\right)\left(1 + c_q\right) - 2^q}{q\left(1 + c_q\right)} \tag{44}$$

for $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$. This implies by Lemma 14 that A is accretive. Therefore, A is accretive in the sense of Deimling. Hence, using Lemma 9, we have that $0 \in A(D)$; that is, there exists $[u, v] \in D$ such that $Fu - v = 0$ and $Kv + u = 0$. So u solves (11).

Case 2 ($1 + \sqrt{1-\gamma} < p \leq 2$). The condition $1 + \sqrt{1-\gamma} < p \leq 2$ implies that $\gamma > p(2 - p)$. Hence, by Theorem 20, A is accretive. We conclude as in Case 1. This completes the proof. □

Theorem 26. *Let $K : D(K) \subset L^p \to L^p$, $F : D(F) \subset L^p \to L^p$ be two continuous mappings satisfying the following conditions:*

(a) *there exists $\alpha > 0$ such that, for each $u_1, u_2 \in D(F)$,*

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \alpha \|u_1 - u_2\|^2; \tag{45}$$

(b) *there exists $\beta > 0$ such that, for each $u_1, u_2 \in D(K)$,*

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \beta \|u_1 - u_2\|^2. \tag{46}$$

Let $D(F)$ and $D(K)$ be closed and convex, such that $R(F) \subset D(K)$. Let $E := L^p \times L^p$ and let $A : D(A) \subset E \to E$ be a mapping such that $D(F) \times D(K) =: D(A)$ and A is defined by (24) for $[u, v] \in D(A)$. Suppose that $\langle Aw, w\rangle_+ \geq 0$ for all $w \in D(A)$ with $\|w\| \geq R$ for some $R > 0$ or $\lim \|Aw\| = \infty$ as $\|w\| \to \infty$ and suppose A satisfies condition (22). Let $\gamma := \min\{\alpha, \beta\}$. If $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ or $1 + \sqrt{1-\gamma} < p \leq 2$, then Hammerstein equation (11) has a solution.

Proof. Evidently, continuity of K and F gives the continuity of A. Also $D(A)$ is closed and convex since $D(F)$ and $D(K)$ are. The rest follows as in the proof of Theorem 25. This completes the proof. □

Corollary 27. *Let $K : D(K) \subset L^p \to L^p$, $F : D(F) \subset L^p \to L^p$ be two continuous accretive mappings satisfying the following conditions:*

(a) *there exists $\alpha > 0$ such that, for each $u_1, u_2 \in D(F)$,*

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \alpha \|u_1 - u_2\|^2; \tag{47}$$

(b) *there exists $\beta > 0$ such that, for each $u_1, u_2 \in D(K)$,*

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \beta \|u_1 - u_2\|^2. \tag{48}$$

Let $D(F) = L^p = D(K)$. Let $E := L^p \times L^p$ and let $A : E \to E$ be a mapping defined by (24) $[u, v] \in D(A)$. Suppose that $\langle Aw, w\rangle_+ \geq 0$ for all $w \in E$ with $\|w\| \geq R$ for some $R > 0$ or $\lim \|Aw\| = \infty$ as $\|w\| \to \infty$. Let $\gamma := \min\{\alpha, \beta\}$. If $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ or $1 + \sqrt{1-\gamma} < p \leq 2$, then Hammerstein equation (11) has a solution.

Proof. Since A is defined on E, it satisfies condition (22) of Theorem 26. Also $D(A)$ is closed and convex. Therefore, the result follows from Theorem 26. □

3.3. The Case of Hilbert Spaces with Lipschitz Strongly Monotone Mappings

Theorem 28. *Let H be a real Hilbert space and let $K : H \to H$, $F : H \to H$ be two Lipschitz strongly monotone mappings with constants α, β, respectively. Let $A : E^H \to E^H$ be a mapping defined by (24) for $[u, v] \in E^H$. Then, Hammerstein equation (11) has a solution.*

Proof. Using Lemma 16 we have that A is Lipschitz. Also since every real Hilbert space is q-uniformly smooth with $q = 2$, $d_q = c_q = 1$, we have that $(1 + c_q)(1 + d_q) = 4 = 2^q$. Also $\min\{\alpha, \beta\} > 0 = ((1 + c_q)(1 + d_q) - 2^q)/q$. Therefore, A is strongly monotone by Lemma 14. Since E^H is a real Hilbert space and every real Hilbert space is uniformly convex, we invoke Corollary 19 to obtain that $A(E^H) = E^H$. So there exists $[u, v] \in E^H$ such that $A[u, v] = 0$; that is, $Fu - v = 0, Kv + u = 0$. Hence u solves (11). This completes the proof. □

3.4. The Case of L_p Spaces, $1 < p < \infty$, with Lipschitz Strongly Accretive Mappings

Theorem 29. *Let $K : L^p \to L_p$, $F : L^p \to L_p$ be two Lipschitz mappings satisfying the following conditions:*

(a) *there exists $\alpha > 0$ such that, for each $u_1, u_2 \in L^p$,*

$$\langle Fu_1 - Fu_2, j(u_1 - u_2) \rangle \geq \alpha \|u_1 - u_2\|^2; \tag{49}$$

(b) *there exists $\beta > 0$ such that, for each $u_1, u_2 \in L^p$,*

$$\langle Ku_1 - Ku_2, j(u_1 - u_2) \rangle \geq \beta \|u_1 - u_2\|^2. \tag{50}$$

Let $E := L^p \times L^p$ and let $A : E \to E$ be a mapping defined by (24). Let $\gamma := \min\{\alpha, \beta\}$. If $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ or $1 + \sqrt{1 - \gamma} < p \leq 2$, then, Hammerstein equation (11) has a solution.

Proof. Using Lemma 16 we have that A is Lipschitz.

Case 1 ($2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$). In this case L^p is 2-uniformly smooth space and $c_q = d_q = p - 1$ (see, e.g., [2]). Therefore, $(1 + c_q)(1 + d_q) = p^2 \geq 4 = 2^q$ and

$$\gamma > \frac{1}{2p}\left(p^2 - 4\right) = \frac{\left(1 + d_q\right)\left(1 + c_q\right) - 2^q}{q\left(1 + c_q\right)} \tag{51}$$

for $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$. This implies by Lemma 14 that A is *strongly* accretive. Since every L_p space, $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$, is uniformly convex, by Corollary 19, $A(L_p) = L_p$. Therefore there exists $[u, v] \in D$ such that $A[u, v] = 0$; that is, $Fu - v = 0$ and $Kv + u = 0$. So u solves (11).

Case 2 ($1 + \sqrt{1 - \gamma} < p \leq 2$). The inequality $1 + \sqrt{1 - \gamma} < p \leq 2$ implies that $\gamma > p(2 - p)$. Hence by Theorem 20 A is *strongly* accretive. The result now follows as in Case 1 since every L_p space, $1 + \sqrt{1 - \gamma} < p \leq 2$, is uniformly convex. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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