Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 209307, 12 pages
http://dx.doi.org/10.1155/2015/209307

Research Article

Reducing Subspaces of Some Multiplication Operators on the Bergman Space over Polydisk

Yanyue Shi and Na Zhou

School of Mathematical Sciences, Ocean University of China, Qingdao 266000, China

Correspondence should be addressed to Yanyue Shi; shiyanyue@gmail.com

Received 27 November 2014; Revised 14 February 2015; Accepted 9 March 2015

Academic Editor: Sung G. Kim

We consider the reducing subspaces of M_{z^N} on $A^2_\alpha(\mathbb{D}^k)$, where $k \geq 3$, $z^N = z_1^{N_1} \cdots z_k^{N_k}$, and $N_i \neq N_j$ for $i \neq j$. We prove that each reducing subspace of M_{z^N} is a direct sum of some minimal reducing subspaces. We also characterize the minimal reducing subspaces in the cases that $\alpha = 0$ and $\alpha \in (-1, +\infty) \setminus \mathbb{Q}$, respectively. Finally, we give a complete description of minimal reducing subspaces of M_{z^N} on $A^2_\alpha(\mathbb{D}^3)$ with $\alpha > -1$.

1. Introduction

Denote by $\mathbb{D}$ the open unit disk in the complex plane and $\mathrm{d}A$ the normalized area measure on $\mathbb{D}$. For $-1 < \alpha < \infty$, denote $\mathrm{d}A_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \mathrm{d}A(z)$. For a positive integer k, the weighted Bergman space $A^2_\alpha(\mathbb{D}^k)$ is the space of all holomorphic functions on $\mathbb{D}^k$ which are square integrable with respect to the measure $\mathrm{d}\nu_\alpha(z) = \mathrm{d}A_\alpha(z_1) \cdots \mathrm{d}A_\alpha(z_k)$. $A^2_\alpha(\mathbb{D}^k, \mathrm{d}\nu_\alpha)$ is the Hilbert space with inner product

$$\langle f, g \rangle_\alpha = \int_{\mathbb{D}^k} f(z) \overline{g(z)} \mathrm{d}\nu_\alpha, \tag{1}$$

and $\|f\|^2_\alpha = \langle f, f \rangle_\alpha$. In particular, if $k = 1$, then $A^2_\alpha(\mathbb{D})$ is the weighted Bergman space on $\mathbb{D}$. Denote by $\mathbb{N}_0$ the set of all the nonnegative integers. For a k-dimension multi-index $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{N}_0^k$ ($\beta \geq 0$ means that $\beta_i \geq 0$ for any $i = 1, 2, \ldots, k$), write $z^\beta = z_1^{\beta_1} \cdots z_k^{\beta_k}$ and $\gamma_\beta = \|z^\beta\|^2_\alpha$. Then $\gamma_\beta = \omega_{\beta_1} \cdots \omega_{\beta_k}$, where $\omega_{\beta_i} = \|z_i^{\beta_i}\|^2_\alpha = \beta_i!\Gamma(2+\alpha)/\Gamma(2+\alpha+\beta_i)$. Obviously, $\{z^\beta / \sqrt{\gamma_\beta}\}_{\beta \in \mathbb{N}_0^k}$ is an orthogonal basis of $A^2_\alpha(\mathbb{D}^k)$.

For every bounded analytic function φ on $\mathbb{D}^k$, the multiplication operator M_φ is defined by

$$M_\varphi(h) = \varphi h, \quad \forall h \in A^2_\alpha\left(\mathbb{D}^k\right). \tag{2}$$

Recall that, in a Hilbert space $\mathscr{H}$, a (closed) subspace $\mathscr{M}$ is called reducing subspace of an operator T if $T(\mathscr{M}) \subset \mathscr{M}$ and $T^*(\mathscr{M}) \subset \mathscr{M}$. Moreover, $\mathscr{M}$ is called minimal if $\mathscr{M}$ does not contain any proper reducing subspaces other than $\{0\}$.

Although the definition of multiplication operator M_φ seems simple, the invariant subspace lattice $\mathrm{Lat}M_\varphi$ is very complicated. Even on the Bergman space $A^2(\mathbb{D})$, the characterization of invariant subspaces for the Bergman shift M_z remains a very fascinating open problem in operator theory. To get some deeper information about $\mathrm{Lat}M_z$, much effort has been devoted to studying the structure of the reducing subspaces of M_φ on $A^2(\mathbb{D})$ (see [1] and its references).

Firstly, it is proved that the multiplication operator M_B, where B is the product of two Blaschke factors, has exactly two nontrivial reducing subspaces by Sun and Wang [2] and Zhu [3] independently. On the weighted sequence space H^2_ω, Stessin and Zhu [4] gave a complete description of the reducing subspaces of weighted unilateral shift operators. In particular, they show that M_{z^n} has n distinct minimal reducing subspaces on $A^2(\mathbb{D})$. For finite Blaschke product B, Hu et al. [5] obtained that M_B has at least a reducing subspace on which the restriction of M_B is unitary equivalent to M_z. Later on, Xu and Yan [6] generalized this result to the weighted Bergman space $A^2_\alpha(\mathbb{D})$ with $\alpha \in \mathbb{N}_0$. In 2009, Guo et al. [7] proved that if B is a Blaschke product of degree 3, then the number of minimal reducing subspaces of M_B is at most 3. For finite Blaschke product B, they also raised a conjecture that the number of nontrivial minimal reducing subspaces of M_B equals the number of connected components of the Riemann surface of $B^{-1} \circ B$ over $\mathbb{D}$. By different techniques, some partial results are obtained in [8–10]. Finally, an affirmative answer to the conjecture is given by Douglas et al. [11]. Furthermore, when B is an infinite Blaschke product, some relative results are obtained by Guo and Huang in [12, 13].

On $A^2_\alpha(\mathbb{D}^2)$, known results about the reducing subspaces of M_φ are quite few. If φ is a monomial, the reducing subspaces of M_φ are characterized in [14–17]. If $\varphi = z_1^m + z_2^n$, Dan and Huang [18] described the minimal reducing subspaces of M_φ and the commutant algebra $\{M_\varphi, M_\varphi^*\}'$.

Let $\mathscr{M}$ be a nonzero reducing subspace of $T_{z_1^m z_2^n}$ on $A^2_\alpha(\mathbb{D}^2)$ with $\alpha \neq 0$. Suppose $f \in \mathscr{M}$ satisfies $\langle f, z_1^p z_2^q \rangle \neq 0$. By [16], we know that every $z_1^p z_2^q$ must be in $\mathscr{M}$. However, on the unweighted Bergman space $A^2(\mathbb{D}^2)$, it is not true. For example, $\mathscr{M} = \overline{\operatorname{span}}\{(z_1 z_2^5 + z_1^3 z_2^2) z_1^{2h} z_2^{3h} : h = 0, 1, \ldots\}$ is a reducing subspace of $M_{z_1^2 z_2^3}$. But $z_1 z_2^5$ does not belong to $\mathscr{M}$.

To know more about how α influences the structure of reducing subspaces, we consider the reducing subspaces of M_{z^N} over $\mathbb{D}^k$ for $k \geq 3$.

Fix integer $k \geq 3$ and distinct positive integers N_i for $i = 1, \ldots, k$. Denote $M_{z^N} = M_{z_1^{N_1} z_2^{N_2} \cdots z_n^{N_k}}$ for $N = (N_1, \ldots, N_k)$. In Section 2, we prove that each reducing subspace of M_{z^N} is a direct sum of some minimal reducing subspaces. To classify the minimal reducing subspaces, we consider three cases: (i) α is irrational; (ii) $\alpha = 0$; (iii) α is rational and $\alpha \neq 0$. For cases (i) and (ii), we describe the minimal reducing subspaces of M_{z^N}. For case (iii), we find that the minimal reducing subspaces of M_{z^N} are varied. In Section 3, we give a complete characterization of the reducing subspaces of M_{z^N} when the dimension $k = 3$.

2. Reducing Subspaces on $A^2_\alpha(\mathbb{D}^k)$

The aim of this section is to give a complete description of the reducing subspaces of M_{z^N} on $A^2_\alpha(\mathbb{D}^k)$. Denote

$$\Omega = \left\{ n = (n_1, \ldots, n_k) \in \mathbb{N}_0^k : 0 \leq n_i < N_i \text{ for some } i \right\}. \quad (3)$$

Define an equivalence on Ω by

$$q \sim n \Longleftrightarrow \gamma_{q+hN} = \gamma_{n+hN}, \quad \forall h \in \mathbb{N}_0. \quad (4)$$

Write $\gamma_{n+hN} = \prod_{i=1}^k \omega_{n_i + hN_i} = \prod_{i=1}^k ((n_i + hN_i)!\Gamma(2+\alpha)/\Gamma(2 + \alpha + n_i + hN_i))$ for $h \in \mathbb{N}_0$. For $n \in \Omega$, let

$$\mathfrak{I}_n := \{q \in \Omega : q \sim n\}, \qquad \mathscr{H}_n := \overline{\operatorname{span}}\left\{ z^J : J \in \mathfrak{I}_n \right\}. \quad (5)$$

Clearly, $\bigcup_{n \in F} \mathfrak{I}_n = \Omega$ and $\bigoplus_{n \in F} \mathscr{H}_n = \overline{\operatorname{span}}\{z^J : J \in \Omega\}$, where F is the partition of Ω by the equivalence $\sim$. Let P_n be the orthogonal projection from $A^2_\alpha(\mathbb{D}^k)$ onto $\mathscr{H}_n$.

Theorem 1. *Let $\mathscr{M}$ be a nonzero reducing subspace of M_{z^N} on $A^2_\alpha(\mathbb{D}^k)$. Then, $\mathscr{M}$ contains a minimal reducing subspace*

$$[f] = \overline{\operatorname{span}}\left\{ f z^{hN} : h \in \mathbb{N}_0 \right\}, \quad (6)$$

where $n \in \Omega$ and $f = \sum_{J \in \mathfrak{I}_n} b_J z^J$ with coefficients $b_J \in \mathbb{C}$.

Proof. Let $P_{\mathscr{M}}$ be the orthogonal projection from $A^2_\alpha(\mathbb{D}^k)$ onto $\mathscr{M}$. For abbreviation, we denote $M = M_{z^N} = M_{z_1^{N_1} z_2^{N_2} \cdots z_k^{N_k}}$.

Firstly, we show that $P_{\mathscr{M}}(z^m) \in \mathscr{H}_n$ for every $m \in \mathfrak{I}_n$. Let $l = (l_1, \ldots, l_k) \in \mathbb{N}_0^k$. We only need to prove that if $\langle P_{\mathscr{M}} z^m, z^l \rangle \neq 0$, then $l \in \mathfrak{I}_n$. If $l \notin \Omega$, then $l \geq N$; that is, $l_i \geq N_i$ for $1 \leq i \leq k$. Therefore,

$$\left\langle P_{\mathscr{M}} z^m, z^l \right\rangle = \left\langle P_{\mathscr{M}} z^m, M z^{l-N} \right\rangle = \left\langle P_{\mathscr{M}} M^* z^m, z^{l-N} \right\rangle = 0. \quad (7)$$

If $l \in \Omega$, we find that

$$M^{*h} M^h \left(z^J \right) = \prod_{i=1}^k \frac{\omega_{j_i + hN_i}}{\omega_{j_i}} z^{j_i} = \frac{\gamma_{J+hN}}{\gamma_J} z^J \quad (8)$$

for $J = (j_1, \ldots, j_k) \in \mathbb{N}_0^k$. Then,

$$\begin{aligned} \frac{\gamma_{m+hN}}{\gamma_m} \left\langle P_{\mathscr{M}} z^m, z^l \right\rangle &= \left\langle P_{\mathscr{M}} M^{*h} M^h z^m, z^l \right\rangle \\ &= \left\langle M^{*h} M^h P_{\mathscr{M}} z^m, z^l \right\rangle \\ &= \left\langle P_{\mathscr{M}} z^m, M^{*h} M^h z^l \right\rangle \\ &= \frac{\gamma_{l+hN}}{\gamma_l} \left\langle P_{\mathscr{M}} z^m, z^l \right\rangle. \end{aligned} \quad (9)$$

Thus, we get that if $\langle P_{\mathscr{M}} z^m, z^l \rangle \neq 0$, then

$$\prod_{i=1}^k \frac{\omega_{m_i + hN_i}}{\omega_{m_i}} = \prod_{i=1}^k \frac{\omega_{l_i + hN_i}}{\omega_{l_i}}, \quad \forall h \in \mathbb{N}_0. \quad (10)$$

Since $\lim_{h \to +\infty} (\omega_{m_i + hN_i} / \omega_{l_i + hN_i}) = 1$, we have $\prod_{i=1}^k (\omega_{l_i} / \omega_{m_i}) = 1$. Therefore,

$$\prod_{i=1}^k \omega_{l_i + hN_i} = \prod_{i=1}^k \omega_{m_i + hN_i}, \quad \forall h \in \mathbb{N}_0, \quad (11)$$

which implies $l \in \mathfrak{I}_m = \mathfrak{I}_n$.

Thus $P_{\mathscr{M}}(z^m) \in \mathscr{H}_n$. We also obtain that $P_{\mathscr{M}}(z^l) \perp \mathscr{H}_n$ for any $l \notin \mathfrak{I}_n$.

Next, we claim that there is a nonzero function f_0 in $\mathscr{H}_{n_0} \cap \mathscr{M}$ for some positive integer n_0.

Choose a nonzero function f in $\mathscr{M}$. Let h_0 be the minimal integer such that $P_\Omega M^{*h_0}(f) \neq 0$, where P_Ω is the orthogonal projection from $A^2_\alpha(\mathbb{D}^k)$ onto $\overline{\operatorname{span}}\{z^J : J \in \Omega\}$. Namely, there exists $n_0 \in \Omega$ such that $f_0 = P_{n_0} M^{*h_0} f = P_{n_0} P_\Omega M^{*h_0} f = \sum_{J \in \mathfrak{I}_{n_0}} b_J z^J \neq 0$. Then, we can prove that $f_0 = P_{\mathscr{M}} f_0 \in \mathscr{M}$. In fact,

(i) if $m \in \mathfrak{I}_{n_0}$, then

$$\begin{aligned} \left\langle P_{\mathscr{M}} f_0, z^m \right\rangle &= \left\langle P_{\mathscr{M}} P_{n_0} M^{*h_0} f, z^m \right\rangle \\ &= \left\langle M^{*h_0} f, P_{\mathscr{M}} P_{n_0} z^m \right\rangle \\ &= \left\langle P_{n_0} P_{\mathscr{M}} M^{*h_0} f, z^m \right\rangle \\ &= \left\langle f_0, z^m \right\rangle, \end{aligned} \quad (12)$$

where the second equality comes from $z^m, P_{\mathscr{M}}(z^m) \in \mathscr{H}_n$ and the last equality comes from $M^{*h_0} f \in \mathscr{M}$;

(ii) if m is out of $\mathfrak{J}_{n_0}$, then $\langle P_{\mathscr{M}} f_0, z^m\rangle = 0 = \langle f_0, z^m\rangle$.

Therefore, we get $[f_0] \subset \mathscr{M}$, where $[f_0]$ is the reducing subspace of M induced by f_0. Notice that

(a) $M^q\left(f_0 z^{hN}\right) = f_0 z^{(h+q)N}$ for $h, q \geq 0$;

(b) $M^{*q}\left(f_0 z^{hN}\right) = \begin{cases} \dfrac{\gamma_{n_0+hN}}{\gamma_{n_0+(h-q)N}} f_0 z^{(h-q)N}, & \text{if } h \geq q \geq 1 \\ 0, & \text{if } 0 \leq h < q; \end{cases}$

(c) $f_0 z^{h_1 N} \perp f_0 z^{h_2 N}$ with $h_1 \neq h_2$, since

$$\begin{aligned} &\left\langle f_0 z^{h_1 N}, f_0 z^{h_2 N}\right\rangle \\ &= \left\langle M^{h_1} f_0, M^{h_2} f_0\right\rangle \\ &= \begin{cases} \dfrac{\gamma_{n_0+h_2N}}{\gamma_{n_0}} \left\langle f_0 z^{(h_1-h_2)N}, f_0\right\rangle, & \text{if } h_1 > h_2 \geq 0 \\ \dfrac{\gamma_{n_0+h_1N}}{\gamma_{n_0}} \left\langle f_0, f_0 z^{(h_2-h_1)N}\right\rangle, & \text{if } h_2 > h_1 \geq 0. \end{cases} \end{aligned} \tag{13}$$

Hence, we conclude that $[f_0] = \overline{\text{span}}\{f_0 z^{hN} : h \in \mathbb{N}_0\} = \bigoplus_{h=0}^{+\infty} \text{span}\{f_0 z^{hN}\} \subset \mathscr{M}$ is a minimal reducing subspace of M. □

In the following, we will prove that each nonzero reducing subspace of M_{z^N} is the orthogonal sum of some minimal reducing subspaces.

Theorem 2. *Let $\mathscr{M}$ be a nonzero reducing subspace of M_{z^N} on $A_\alpha^2(\mathbb{D}^k)$. Then,*

$$\mathscr{M} = \bigoplus_{n\in F} [P_n \mathscr{M}], \tag{14}$$

where $[P_n\mathscr{M}]$ is the reducing subspace of M_{z^N} induced by $P_n\mathscr{M}$. If $P_n\mathscr{M} \neq \{0\}$, then

$$[P_n\mathscr{M}] = \bigoplus_{h=0}^{+\infty} z^{hN} P_n\mathscr{M} = \bigoplus_{j=1}^{q} \left[e_{n,j}\right], \tag{15}$$

where $\{e_{n,j}\}_{j=1}^q$ is the orthogonal basis of $P_n\mathscr{M}$ and $1 \leq q \leq +\infty$.

Proof. Denote $M = M_{z^N}$. Firstly, we know that $[P_n\mathscr{M}] = \bigoplus_{h=0}^{+\infty} z^{hN} P_n\mathscr{M}$, since

(i) $z^{h_1 N} P_n\mathscr{M} \perp z^{h_2 N} P_n\mathscr{M}$;

(ii) $M(z^{hN} P_n\mathscr{M}) = z^{(h+1)N} P_n\mathscr{M}$;

(iii) $M^*(P_n\mathscr{M}) = \{0\}$, $M^*(z^{hN} P_n\mathscr{M}) = z^{(h-1)N} P_n\mathscr{M}$ for $h \geq 1$;

(iv) $M^*M(z^{hN} P_n\mathscr{M}) = z^{hN} P_n\mathscr{M}$, $MM^*(z^{hN} P_n\mathscr{M}) = z^{hN} P_n\mathscr{M}$ for $h \geq 1$.

Secondly, we prove that $\mathscr{M} = \bigoplus_{n\in F}[P_n\mathscr{M}]$. On the one hand, in the proof of Theorem 1, we get $P_n\mathscr{M} \subset \mathscr{M}$. Then, $[P_n\mathscr{M}] \subset \mathscr{M}$. On the other hand, if $\mathscr{M} \neq \bigoplus_{n\in F}[P_n\mathscr{M}]$, choose a nonzero function $f \perp \bigoplus_{n\in F}[P_n\mathscr{M}]$ in $\mathscr{M}$. Theorem 1 shows that there are n_0 and h_0 such that $0 \neq f_0 = P_{n_0} M^{*h_0} f \in \mathscr{M}$. However, $\langle f_0, g\rangle = \langle f, M^{h_0} P_{n_0} g\rangle = 0$ for $g \in \mathscr{M}$, which is a contradiction.

Finally, we prove that if $P_n\mathscr{M} \neq \{0\}$, then $[P_n\mathscr{M}] = \bigoplus_{j=1}^q [e_{n,j}]$. Choose an orthogonal basis $\{e_{n,j}\}_{j=1}^q$ ($q \leq +\infty$) of the subspace $P_n\mathscr{M}$. Theorem 1 shows that $[e_{n,j}] = \overline{\text{span}}\{e_{n,j} z^{hN} : h \in \mathbb{N}_0\} \subset \mathscr{M}$. We have that $[e_{n,p_1}] \perp [e_{m,p_2}]$ for $n \neq m$, since

$$\begin{aligned} &\left\langle e_{n,p_1} z^{h_1 N}, e_{m,p_2} z^{h_2 N}\right\rangle \\ &= \left\langle M^{h_1} e_{n,p_1}, M^{h_2} e_{m,p_2}\right\rangle \\ &= \begin{cases} \dfrac{\gamma_{m+h_2N}}{\gamma_m} \left\langle e_{n,p_1} z^{(h_1-h_2)N}, e_{m,p_2}\right\rangle, & \text{if } h_1 \geq h_2 \geq 0 \\ \dfrac{\gamma_{n+h_1N}}{\gamma_n} \left\langle e_{n,p_1}, e_{m,p_2} z^{(h_2-h_1)N}\right\rangle, & \text{if } h_2 > h_1 \geq 0. \end{cases} \end{aligned} \tag{16}$$

Let $\mathscr{M}_n = \bigoplus_{j=1}^q [e_{n,j}]$. Clearly, $\mathscr{M}_n \subset [P_n\mathscr{M}]$. Assume that $\mathscr{M}_n \neq [P_n\mathscr{M}]$. Take a nonzero function $g \in [P_n\mathscr{M}] \ominus \mathscr{M}_n$. As in Theorem 1, there is an integer h_0 such that $g_0 := P_n M^{*h_0} g \in P_n\mathscr{M}$ and $g_0 \neq 0$. Since $g \perp \mathscr{M}_n$, we have $\langle g_0, e_{n,j}\rangle = \langle g, M^{h_0} P_n e_{n,j}\rangle = 0$, which is in contradiction with $g_0 \neq 0$. So we finish the proof. □

From this theorem, we know that the reducing subspaces of M_{z^N} are determined by the sets $\{\mathfrak{J}_n\}_{n\in\Omega}$. There arises the following question: what are the elements in the set $\mathfrak{J}_n$ exactly? We begin the research with the case that α is irrational.

Lemma 3. *If α is irrational, then $\mathfrak{J}_n = \{n\}$ for every $n \in \Omega$.*

Proof. Suppose $m \in \mathfrak{J}_n$; that is, $\gamma_{n+hN} = \gamma_{m+hN}$, for all $h \in \mathbb{N}_0$. Then, we have

$$\frac{\gamma_{n+hN}}{\gamma_{n+(h+1)N}} = \frac{\gamma_{m+hN}}{\gamma_{m+(h+1)N}}, \quad \forall h \in \mathbb{N}_0. \tag{17}$$

This is equivalent to

$$\begin{aligned} &\prod_{i=1}^{k} \prod_{j=1}^{N_i} \frac{n_i + hN_i + \alpha + 1 + j}{n_i + hN_i + j} \\ &= \prod_{i=1}^{k} \prod_{j=1}^{N_i} \frac{m_i + hN_i + \alpha + 1 + j}{m_i + hN_i + j}, \quad \forall h \in \mathbb{N}_0. \end{aligned} \tag{18}$$

Write

$$g(\lambda) = \prod_{i=1}^{k}\prod_{j=1}^{N_i}\left[\left(n_i + \lambda N_i + \alpha + 1 + j\right)\left(m_i + \lambda N_i + j\right)\right] - \prod_{i=1}^{k}\prod_{j=1}^{N_i}\left[\left(m_i + \lambda N_i + \alpha + 1 + j\right)\left(n_i + \lambda N_i + j\right)\right]. \tag{19}$$

Clearly, g is a polynomial over $\mathbb{C}$ and $g(h) = 0$ for any $h \in \mathbb{N}_0$. Fundamental theorem of algebra shows that $g(\lambda) \equiv 0$, for all $\lambda \in \mathbb{C}$. Denote

$$\begin{aligned} E_1 &= \left\{\frac{n_i + \alpha + 1 + j}{N_i} : j = 1, \ldots, N_i,\ i = 1, \ldots, k\right\}, \\ E_2 &= \left\{\frac{n_i + j}{N_i} : j = 1, \ldots, N_i,\ i = 1, \ldots, k\right\}, \\ F_1 &= \left\{\frac{m_i + \alpha + 1 + j}{N_i} : j = 1, \ldots, N_i,\ i = 1, \ldots, k\right\}, \\ F_2 &= \left\{\frac{m_i + j}{N_i} : j = 1, \ldots, N_i,\ i = 1, \ldots, k\right\}. \end{aligned} \tag{20}$$

Since α is irrational, $E_1 \cap E_2 = F_1 \cap F_2 = \emptyset$. Then, $g(\lambda) \equiv 0$ implies $E_1 \cup F_2 = E_2 \cup F_1$. So we get $E_1 = F_1$ and $E_2 = F_2$.

Without loss of generality, we may assume $\max F_2 = (m_k + N_k)/N_k$. Then there exist nonnegative integers i and j making

$$\frac{m_i + \alpha + 1 + j}{N_i} = \frac{n_k + \alpha + 2}{N_k}. \tag{21}$$

If $i \neq k$, then $\alpha = ((n_k+2)N_i - (m_i+1+j)N_k)/(N_k - N_i) \in \mathbb{Q}$, which is in contradiction with the assumption. So $i = k$.

Equality (21) implies $(m_k + j)/N_k = (n_k + 1)/N_k$. Then, $\max E_2 \geq (n_k + N_k)/N_k = (m_k + N_k + j - 1)/N_k \geq \max F_2$. Hence, we get $j = 1$ and $m_k = n_k$.

Therefore,

$$\prod_{i=1}^{k-1}\prod_{j=1}^{N_i}\frac{n_i + hN_i + \alpha + 1 + j}{n_i + hN_i + j} = \prod_{i=1}^{k-1}\prod_{j=1}^{N_i}\frac{m_i + hN_i + \alpha + 1 + j}{m_i + hN_i + j}, \quad \forall h \in \mathbb{N}_0. \tag{22}$$

Let

$$\begin{aligned} \widetilde{E}_1 &= \left\{\frac{n_i + \alpha + 1 + j}{N_i} : j = 1, \ldots, N_i,\ i = 1, \ldots, k-1\right\}, \\ \widetilde{E}_2 &= \left\{\frac{n_i + j}{N_i} : j = 1, \ldots, N_i,\ i = 1, \ldots, k-1\right\}, \\ \widetilde{F}_1 &= \left\{\frac{m_i + \alpha + 1 + j}{N_i} : j = 1, \ldots, N_i,\ i = 1, \ldots, k-1\right\}, \\ \widetilde{F}_2 &= \left\{\frac{m_i + j}{N_i} : j = 1, \ldots, N_i,\ i = 1, \ldots, k-1\right\}. \end{aligned} \tag{23}$$

Without loss of generality, assume $\max \widetilde{F}_2 = (m_{k-1} + N_{k-1})/N_{k-1}$. As above, it is easy to get $m_{k-1} = n_{k-1}$. Applying this process again, we can prove that $m_i = n_i$ for $i = 1, \ldots, k$. □

By Theorems 1 and 2 and Lemma 3, we obtain the following theorem.

Theorem 4. *If α is irrational, then each reducing subspace $\mathscr{M}$ of M_{z^N} on $A_\alpha^2(\mathbb{D}^k)$ is a direct sum of some minimal reducing subspaces of the form*

$$\overline{\operatorname{span}}\left\{z^{n+hN} : h \in \mathbb{N}_0\right\}, \tag{24}$$

where $n \in \mathscr{A} = \{n \in \Omega : z^n \in \mathscr{M}\}$.

Proof. Lemma 3 shows that $\mathfrak{I}_n = \{n\}$. In light of Theorem 1, we have $\mathscr{A} \neq \emptyset$. For $n \in \mathscr{A}$, Theorem 2 implies that $P_n\mathscr{M} = \operatorname{span}\{z^n\}$, $[z^n] = \overline{\operatorname{span}}\{z^{n+hN} : h \in \mathbb{N}_0\}$. Thus, $\mathscr{M} = \bigoplus_{n\in\mathscr{A}}[z^n]$. □

Next, we consider the case that $\alpha = 0$. Denote by S_k the permutation group of the set $\{1, 2, \ldots, k\}$. Let $\rho_{ij}(x) = (x + 1)N_i/N_j - 1$ for $x \in \mathbb{R}$.

Lemma 5. *If $\alpha = 0$, then*

$$\mathfrak{I}_n = \left\{\left(\rho_{1\sigma(1)}\left(n_{\sigma(1)}\right), \rho_{2\sigma(2)}\left(n_{\sigma(2)}\right), \ldots, \rho_{k\sigma(k)}\left(n_{\sigma(k)}\right)\right) : \sigma \in S_k\right\}. \tag{25}$$

Proof. Suppose $m \in \mathfrak{I}_n$. By the definition of $\mathfrak{I}_n$, we have

$$\prod_{i=1}^{k}\left(hN_i + m_i + 1\right) = \prod_{i=1}^{k}\left(hN_i + n_i + 1\right), \quad \forall h \in \mathbb{N}_0. \tag{26}$$

Let $g(\lambda) = \prod_{i=1}^{k}(\lambda + (m_i+1)/N_i) - \prod_{i=1}^{k}(\lambda + (n_i+1)/N_i)$. We have $g(\lambda) \equiv 0$, since g is a polynomial on $\mathbb{C}$ with infinitely many roots. Therefore,

$$\left\{\frac{n_i + 1}{N_i} : i = 1, \ldots, k\right\} = \left\{\frac{m_i + 1}{N_i} : i = 1, \ldots, k\right\}. \tag{27}$$

For each $j \in \{1, 2, \ldots, k\}$, there is only one integer $i \in \{1, 2, \ldots, k\}$ such that

$$\frac{n_j + 1}{N_j} = \frac{m_i + 1}{N_i}; \tag{28}$$

that is, $m_i = (n_j + 1)N_i/N_j - 1 = \rho_{ij}(n_j)$. Let

$$E = \left\{\left(\rho_{1\sigma(1)}\left(n_{\sigma(1)}\right), \rho_{2\sigma(2)}\left(n_{\sigma(2)}\right), \ldots, \rho_{k\sigma(k)}\left(n_{\sigma(k)}\right)\right) : \sigma \in S_k\right\}. \tag{29}$$

Hence $m \in E$.

Conversely, for every $m \in E$, $(m_i + 1)/N_i = (\rho_{i\sigma(i)}(n_{\sigma(i)}) + 1)/N_i$. By definition of $\rho_{ij}(n_j)$, we have

$$\frac{\rho_{i\sigma(i)}\left(n_{\sigma(i)}\right) + 1}{N_i} = \frac{n_{\sigma(i)} + 1}{N_{\sigma(i)}}. \tag{30}$$

Therefore, equality (27) holds, implying $m \in \mathfrak{I}_n$. Therefore, $\mathfrak{I}_n = E$. □

From this result, we find $\operatorname{Card}(\mathfrak{I}_n) \leq n!$.

Example 6. Let

$$f(z) = a_1 z_1 z_3 + a_2 z_1^2 z_2 z_3^3 + a_3 z_1^2 z_2^3 z_3 + a_4 z_1 z_2 z_3^5 + a_5 z_1 z_3^{11} + a_6 z_1^5 z_3^3 + a_7 z_1^{11} z_3. \tag{31}$$

Denote by $\mathcal{M} = [f]$ the reducing subspace of $M_{z_1^3 z_2 z_3^6}$ on $A^2(\mathbb{D}^3)$ induced by f. Let

(i) $\mathcal{M}_1 = \overline{\operatorname{span}}\{f_1(z) z_1^{3h} z_2^h z_3^{6h} : h = 0, 1, 2, \ldots\}$ for $f_1(z) = z_1 z_3$;

(ii) $\mathcal{M}_2 = \overline{\operatorname{span}}\{f_2(z) z_1^{3h} z_2^h z_3^{6h} : h = 0, 1, 2, \ldots\}$ for $f_2(z) = a_2 z_1^2 z_2 z_3^3 + a_4 z_1 z_2 z_3^5 + a_5 z_1 z_3^{11} + a_6 z_1^5 z_3^3$;

(iii) $\mathcal{M}_3 = \overline{\operatorname{span}}\{f_3(z) z_1^{3h} z_2^h z_3^{6h} : h = 0, 1, 2, \ldots\}$ for $f_3(z) = a_3 z_1^2 z_2^3 z_3 + a_7 z_1^{11} z_3$.

Then, $\mathcal{M} = \bigoplus_{i=1}^3 \mathcal{M}_i$.

Proof. Let $n = (n_1, n_2, n_3) = (2, 1, 3)$ and let $m = (m_1, m_2, m_3) = (11, 0, 1)$. It is easy to check that

$$\begin{aligned} (1,1,5) &= (\rho_{13}(n_3), \rho_{22}(n_2), \rho_{31}(n_1)), \\ (5,0,3) &= (\rho_{12}(n_2), \rho_{21}(n_1), \rho_{33}(n_3)), \\ (1,0,11) &= (\rho_{13}(n_3), \rho_{21}(n_1), \rho_{32}(n_2)). \end{aligned} \tag{32}$$

That is,

$$\mathfrak{I}_n = \{(2,1,3), (1,1,5), (1,0,11), (5,0,3)\}. \tag{33}$$

Similarly,

$$\mathfrak{I}_m = \{(11,0,1), (2,3,1), (0,3,5), (0,0,23)\}. \tag{34}$$

By Lemma 5, we get $f_2(z) = a_2 z_1^2 z_2 z_3^3 + a_4 z_1 z_2 z_3^5 + a_5 z_1 z_3^{11} + a_6 z_1^5 z_3^3 \in [f]$ and $f_3(z) = a_3 z_1^2 z_2^3 z_3 + a_7 z_1^{11} z_3 \in [f]$. Therefore, $a_1 z_1 z_3 \in [f]$. Notice that

$$M^{*h} M^q f(z) = \begin{cases} \sum_{i=1}^3 \mu_i f_i(z) z^{(q-h)N}, & \text{if } 0 \le h \le q \\ 0, & \text{if } 0 \le q < h, \end{cases} \tag{35}$$

where $\mu_1 = \gamma_{(1,0,1)+qN}/\gamma_{(1,0,1)+(q-h)N}$, $\mu_2 = \gamma_{(2,3,1)+qN}/\gamma_{(2,3,1)+(q-h)N}$, and $\mu_3 = \gamma_{(11,0,1)+qN}/\gamma_{(11,0,1)+(q-h)N}$. So

$$[f] = \bigoplus_{i=1}^3 [f_i] = \bigoplus_{i=1}^3 \mathcal{M}_i. \tag{36}$$

□

If α is a nonzero rational number, the structure of minimal reducing subspace turns to be more complicated. In particular, we will study the reducing subspaces of M_{z^N} on $A_\alpha^2(\mathbb{D}^3)$ in the next section.

3. Reducing Subspaces on $A_\alpha^2(\mathbb{D}^3)$

Let $\alpha \neq 0$ be rational. We consider the reducing subspaces of M_{z^N} on $A_\alpha^2(\mathbb{D}^3)$. Recall

$$\Omega = \{n = (n_1, n_2, n_3) \in \mathbb{N}_0^3 : 0 \le n_i < N_i \text{ for some } i\}, \tag{37}$$

and $\mathfrak{I}_n = \{q \in \Omega : q \sim n\}$; that is, $m \in \mathfrak{I}_n$ if and only if $\gamma_{m+hN} = \gamma_{n+hN}$ for $h \in \mathbb{N}_0$. For every $n \in \Omega$, if $m \in \mathfrak{I}_n$, we assume that $m_i \neq n_i$ for $i = 1, 2, 3$. Otherwise, if there exists j such that $m_j = n_j$, we can prove that $m_i = n_i$ for $i = 1, 2, 3$ as in [16]. Since $\gamma_\beta = \prod_{i=1}^3 \omega_{\beta_i} = \prod_{i=1}^3 (\beta_i! \Gamma(2+\alpha)/\Gamma(2+\alpha+\beta_i))$ and ω_{β_i} are decreasing as β_i is increasing, there exist i and j satisfying $n_i > m_i$ and $n_j < m_j$.

This section is organized as follows. Firstly, we consider $m \in \mathfrak{I}_n$ under the assumption that $n_1 > m_1$, $n_2 > m_2$, and $m_3 > n_3$. Let $Q_1 = \{1, 2, \ldots, n, \ldots\}$, $Q_2 = \{\alpha \in \mathbb{Q} \setminus Q_1 : \alpha > 0\}$, and $Q_3 = (-1, 0) \cap \mathbb{Q}$. We give a description of $m \in \mathfrak{I}_n$ in the cases that α is in Q_1, Q_2, and Q_3, respectively. Secondly, we get all the possible cases by symmetry (see Corollaries 11 and 13). Finally, we obtain $\operatorname{Card}(\mathfrak{I}_n) \le 2$ and Theorem 14.

Lemma 7. *Let $\alpha \in Q_2$ and let $n \in \Omega$. If $m \in \mathfrak{I}_n$ satisfies $n_1 > m_1$, $n_2 > m_2$, and $m_3 > n_3$, then one of the following statements holds:*

(1) $m = n - (1, 1 - 1)$;

(2) $m = n - (1, 1, -2)$.

Proof. Let $m \in \mathfrak{I}_n$. By definition of $\mathfrak{I}_n$, as in Lemma 3, we have

$$\begin{aligned} \prod_{i=1}^{2} \prod_{j=1}^{n_i - m_i} \frac{n_i + \alpha + 2 - j + \lambda N_i}{m_i + j + hN_i} = \prod_{j=1}^{m_3 - n_3} \frac{m_3 + \alpha + 2 - j + \lambda N_3}{n_3 + j + hN_3} \end{aligned} \tag{38}$$

for any $\lambda \in \mathbb{C}$. Denote

$$\begin{aligned} E_i &= \left\{ \frac{n_i + \alpha + 1}{N_i}, \frac{n_i + \alpha}{N_i}, \ldots, \frac{m_i + \alpha + 2}{N_i} \right\}, \\ E_3 &= \left\{ \frac{m_3}{N_3}, \frac{m_3 - 1}{N_3}, \ldots, \frac{n_3 + 1}{N_3} \right\}, \\ F_i &= \left\{ \frac{n_i}{N_i}, \frac{n_i - 1}{N_i}, \ldots, \frac{m_i + 1}{N_i} \right\}, \\ F_3 &= \left\{ \frac{m_3 + \alpha + 1}{N_3}, \frac{m_3 + \alpha}{N_3}, \ldots, \frac{n_3 + \alpha + 2}{N_3} \right\}, \end{aligned} \tag{39}$$

for $i = 1, 2$. Then,

$$\bigsqcup_{i=1}^3 E_i = \bigsqcup_{i=1}^3 F_i, \tag{40}$$

where $\bigsqcup$ denotes the disjoint union. Since $\alpha > 0$ is not an integer, $E_i \cap F_i = \emptyset$ for $i = 1, 2, 3$.

It is easy to see that

$$\max\left\{\frac{n_1+\alpha+1}{N_1},\frac{n_2+\alpha+1}{N_2},\frac{m_3}{N_3}\right\} = \max\left\{\frac{n_1}{N_1},\frac{n_2}{N_2},\frac{m_3+\alpha+1}{N_3}\right\}. \tag{41}$$

Since $(n_i+\alpha+1)/N_i > n_i/N_i$ for $i=1,2$ and $(m_3+\alpha+1)/N_3 > m_3/N_3$, we have

$$\frac{m_3+\alpha+1}{N_3} \in \left\{\frac{n_1+\alpha+1}{N_1},\frac{n_2+\alpha+1}{N_2}\right\}. \tag{42}$$

Without loss of generality, assume

$$\frac{n_1+\alpha+1}{N_1} = \frac{m_3+\alpha+1}{N_3}. \tag{43}$$

Firstly, we prove that $n_1-m_1=1$ by contradiction. Otherwise, if $n_1-m_1\ge 2$, then

$$\max\left(\bigsqcup_{i=1}^{3} E_i \setminus \left\{\frac{n_1+\alpha+1}{N_1}\right\}\right) = \max\left(\bigsqcup_{i=1}^{3} F_i \setminus \left\{\frac{m_3+\alpha+1}{N_3}\right\}\right). \tag{44}$$

Since $\max\{(n_1+\alpha)/N_1,(n_2+\alpha+1)/N_2\} > \max\{n_1/N_1,n_2/N_2\}$, we have $m_3-n_3\ge 2$ and

$$\max\left\{\frac{n_1+\alpha}{N_1},\frac{n_2+\alpha+1}{N_2},\frac{m_3}{N_3}\right\} = \max\left\{\frac{n_1}{N_1},\frac{n_2}{N_2},\frac{m_3+\alpha}{N_3}\right\} = \frac{m_3+\alpha}{N_3} > \frac{m_3}{N_3}. \tag{45}$$

Therefore, $(m_3+\alpha)/N_3 \in \{(n_1+\alpha)/N_1,(n_2+\alpha+1)/N_2\}$. Since $N_1\ne N_3$, it holds that

$$\frac{m_3+\alpha}{N_3} = \frac{n_2+\alpha+1}{N_2}. \tag{46}$$

We will find the contradictions under the assumptions (a) $n_2-m_2\ge 2$ and (b) $n_2-m_2=1$, respectively.

(a) If $n_2-m_2\ge 2$, then $\max\{(n_1+\alpha)/N_1,(n_2+\alpha)/N_2,m_3/N_3\} > \max\{n_1/N_1,n_2/N_2\}$. So $m_3-n_3\ge 3$ and

$$\max\left\{\frac{n_1+\alpha}{N_1},\frac{n_2+\alpha}{N_2},\frac{m_3}{N_3}\right\} = \max\left\{\frac{n_1}{N_1},\frac{n_2}{N_2},\frac{m_3+\alpha-1}{N_3}\right\}. \tag{47}$$

Since $N_2\ne N_3$, we have

$$\frac{m_3+\alpha-1}{N_3} = \frac{n_1+\alpha}{N_1}. \tag{48}$$

By (43) and (48), we get $1/N_1 = 2/N_3$ and $m_3/N_3 > n_1/N_1$. It follows that

$$\max\left\{\frac{n_1+\alpha-1}{N_1},\frac{n_2+\alpha}{N_2},\frac{m_3}{N_3}\right\} \ge \max\left\{\frac{n_2+\alpha}{N_2},\frac{m_3}{N_3}\right\} > \max\left\{\frac{n_1}{N_1},\frac{n_2}{N_2}\right\}. \tag{49}$$

Thus, $m_3-n_3\ge 4$ and

$$\frac{m_3+\alpha-2}{N_3} = \max\left\{\frac{n_1}{N_1},\frac{n_2}{N_2},\frac{m_3+\alpha-2}{N_3}\right\}. \tag{50}$$

By $N_1\ne N_3$ and equality (48), we conclude that

$$\frac{m_3+\alpha-2}{N_3} = \frac{n_2+\alpha}{N_2}. \tag{51}$$

Equalities (46) and (51) imply $2/N_3 = 1/N_2$. Thus, $N_2 = N_1$, which is in contradiction with the assumption.

(b) Suppose $n_2-m_2=1$. Notice that $E_2=\{(n_2+\alpha+1)/N_2\}\subset F_3$.

If $F_2=\{n_2/N_2\}\subset E_1$, then equality (40) implies $F_1=E_3$. Equivalently, $n_1/N_1=m_3/N_3$ and $(n_1-1)/N_1=(m_3-1)/N_3$. Hence, $N_1=N_3$, which is impossible.

If $F_2=\{n_2/N_2\}\subset E_3$, then $F_1\cup F_2=E_3$. It follows that $\max\{n_1/N_1,n_2/N_2\}=m_3/N_3$. Since $N_1\ne N_3$, equality (43) implies $m_3/N_3=n_2/N_2$. Therefore, $n_1/N_1=(m_3-1)/N_3$ and $(n_1-1)/N_1=(m_3-2)/N_3$. Then, $N_3=N_1$, which is a contradiction.

Summing up, we must have $n_1-m_1=1$.

Next, we prove that $n_2-m_2=1$.

If $F_1\subset E_2$, then $F_2=E_3$; that is,

$$\frac{n_2}{N_2} = \frac{m_3}{N_3}. \tag{52}$$

In this case, $n_2-m_2=m_3-n_3=1$. Otherwise, $n_2/N_2=m_3/N_3$ and $(n_2-1)/N_2=(m_3-1)/N_3$, which is in contradiction with $N_2\ne N_3$.

If $F_1\subset E_3$, then $E_2=F_3\setminus\{(m_3+\alpha+1)/N_3\}$. It follows that

$$\frac{n_2+\alpha+1}{N_2} = \frac{m_3+\alpha}{N_3}. \tag{53}$$

In this case, $n_2-m_2=1$ and $m_3-n_3=2$. Or else, $(n_2+\alpha+1)/N_2=(m_3+\alpha)/N_3$ and $(n_2+\alpha)/N_2=(m_3+\alpha-1)/N_3$, which is in contradiction with $N_2\ne N_3$. So we get the desired results. $\square$

Lemma 8. *Fix $\alpha\in Q_1$ and $n\in\Omega$. If $m\in\mathfrak{J}_n$ satisfies $n_1>m_1$, $n_2>m_2$, and $m_3>n_3$, then one of the following statements holds:*

(1) $m=n-(1,1-1)$;

(2) $m=n-(1,1,-2)$.

Proof. Let $k_i = \min\{\alpha+1, |n_i - m_i|\}$ for $i = 1, 2, 3$. Then,

$$\prod_{i=1}^{2}\prod_{j=1}^{k_i}\frac{n_i+\alpha+2-j+\lambda N_i}{m_i+j+\lambda N_i} = \prod_{j=1}^{k_3}\frac{m_3+\alpha+2-j+\lambda N_3}{n_3+j+\lambda N_3} \tag{54}$$

for $\lambda \in \mathbb{C}$. Let

$$\begin{aligned}\widetilde{E}_i &= \left\{\frac{n_i+\alpha+1}{N_i}, \frac{n_i+\alpha}{N_i}, \dots, \frac{n_i+\alpha+2-k_i}{N_i}\right\};\\ \widetilde{E}_3 &= \left\{\frac{n_3+1}{N_3}, \frac{n_3+2}{N_3}, \dots, \frac{n_3+k_3}{N_3}\right\};\\ \widetilde{F}_i &= \left\{\frac{m_i+1}{N_i}, \frac{m_i+2}{N_i}, \dots, \frac{m_i+k_i}{N_i}\right\};\\ \widetilde{F}_3 &= \left\{\frac{m_3+\alpha+1}{N_3}, \frac{m_3+\alpha}{N_3}, \dots, \frac{m_3+\alpha+2-k_3}{N_3}\right\}\end{aligned} \tag{55}$$

for $i = 1, 2$. Then,

$$\bigsqcup_{i=1}^{3}\widetilde{E}_i = \bigsqcup_{i=1}^{3}\widetilde{F}_i, \tag{56}$$

and $\widetilde{E}_i \cap \widetilde{F}_i = \emptyset$ for $i = 1, 2, 3$. As in Lemma 7, we assume equality (43) holds. Then, we can prove that $(k_1, k_2, k_3) = (1, 1, 2)$ or $(k_1, k_2, k_3) = (1, 1, 1)$.

Since $\alpha + 1 > 1$, we have $k_1 = n_1 - m_1 = 1$ and $k_2 = n_2 - m_2 = 1$. It means $\mathrm{Card}(\widetilde{E}_i) = \mathrm{Card}(\widetilde{F}_i) = 1$ for $i = 1, 2$.

If $k_3 = m_3 - n_3$, then statement (1) or statement (2) holds.

If $k_3 = \alpha + 1$, then $\mathrm{Card}(\widetilde{E}_3) = \mathrm{Card}(\widetilde{F}_3) = k_3 = 2$ and $\alpha = 1$. Equalities (43) and (53) imply that

$$\begin{aligned}\frac{n_1+2}{N_1} &= \frac{m_3+2}{N_3},\\ \frac{n_2+2}{N_2} &= \frac{m_3+1}{N_3}.\end{aligned} \tag{57}$$

Since N_1, N_2, and N_3 are distinct, equality (56) shows that

$$\begin{aligned}\frac{n_2}{N_2} &= \frac{n_3+2}{N_3},\\ \frac{n_1}{N_1} &= \frac{n_3+1}{N_3}.\end{aligned} \tag{58}$$

Then, we have $N_1 : N_2 : N_3 = 2 : 6 : 3$ and $(n_1, n_2, m_3, n_3) = r(N_1, N_2, N_3, N_3) + (0, 2, 1, -1)$. In this case, $m_3 - n_3 = 2 = k_3$. Hence, statement (2) holds. □

Lemma 9. *Fix a rational number $\alpha > 0$ and a vector $n \in \Omega$. If $m = n - (1, 1, -1) \in \mathfrak{J}_n$, then one of the following statements holds:*

(1) $1/N_1 + 1/N_2 - 1/N_3 = 0$ *and*

$$n \in \left\{r(N_1, N_2, N_3) + \left((\alpha+2)\frac{N_1}{N_3} - \alpha - 1, \frac{N_2}{N_3}, 0\right) : r \in \mathbb{R}\right\}, \tag{59}$$

(2) $1/N_1 + 1/N_2 - 1/N_3 = 0$ *and*

$$n \in \left\{r(N_1, N_2, N_3) + \left(\frac{N_1}{N_3}, (\alpha+2)\frac{N_2}{N_3} - \alpha - 1, 0\right) : r \in \mathbb{R}\right\}. \tag{60}$$

Proof. As in Lemma 7, we have $(m_3 + \alpha + 1)/N_3 \in \{(n_1 + \alpha + 1)/N_1, (n_2 + \alpha + 1)/N_2\}$.

If $(n_1 + \alpha + 1)/N_1 = (m_3 + \alpha + 1)/N_3$, Lemmas 7 and 8 show that

$$\begin{aligned}\frac{n_1+\alpha+1}{N_1} &= \frac{m_3+\alpha+1}{N_3},\\ \frac{m_3}{N_3} &= \frac{n_2}{N_2},\\ \frac{n_1}{N_1} &= \frac{n_2+\alpha+1}{N_2}.\end{aligned} \tag{61}$$

Therefore,

$$\begin{aligned}(n_1, n_2, m_3) &= r(N_1, N_2, N_3)\\ &\quad + \left((\alpha+2)\frac{N_1}{N_3} - \alpha - 1, \frac{N_2}{N_3}, 1\right),\end{aligned} \tag{62}$$

where $r \in \mathbb{R}$ and $1/N_1 + 1/N_2 - 1/N_3 = 0$. Statement (1) holds.

If $(n_2 + \alpha + 1)/N_2 = (m_3 + \alpha + 1)/N_3$, then we have

$$\begin{aligned}\frac{n_2+\alpha+1}{N_2} &= \frac{m_3+\alpha+1}{N_3},\\ \frac{m_3}{N_3} &= \frac{n_1}{N_1},\\ \frac{n_2}{N_2} &= \frac{n_1+\alpha+1}{N_1}.\end{aligned} \tag{63}$$

In this case,

$$\begin{aligned}(n_1, n_2, m_3) &= r(N_1, N_2, N_3)\\ &\quad + \left(\frac{N_1}{N_3}, (\alpha+2)\frac{N_2}{N_3} - \alpha - 1, 1\right),\end{aligned} \tag{64}$$

where $r \in \mathbb{R}$ and $1/N_1 + 1/N_2 - 1/N_3 = 0$. So statement (2) holds. □

Lemma 10. *Fix a rational number $\alpha > 0$ and a vector $n \in \Omega$. If $m = n - (1, 1, -2) \in \mathfrak{J}_n$, then one of the following statements holds:*

(1) $N_1 : N_2 : N_3 = \alpha(\alpha+1) : (\alpha+1)(\alpha+2) : \alpha(\alpha+2)$ *and*

$$n \in \left\{r(N_1, N_2, N_3) + \left(0, \frac{N_2}{N_3}, -1\right) : r \in \mathbb{R}\right\}, \tag{65}$$

(2) $N_1 : N_2 : N_3 = (\alpha+1)(\alpha+2) : \alpha(\alpha+1) : \alpha(\alpha+2)$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + \left(\frac{N_1}{N_3}, 0, -1\right) : r \in \mathbb{R} \right\}. \quad (66)$$

In this case, $1/N_1 + 1/N_2 - 2/N_3 = 0$.

Proof. As in Lemma 8, if $(n_1+\alpha+1)/N_1 = (m_3+\alpha+1)/N_3$, then $(n_2+\alpha+1)/N_2 = (m_3+\alpha)/N_3$. Since $N_1 \neq N_3$, we have $n_1/N_1 \neq m_3/N_3$. Similarly, equality (53) implies that $n_2/N_2 \neq (m_3-1)/N_3$. Therefore,

$$\begin{gathered} \frac{n_1+\alpha+1}{N_1} = \frac{m_3+\alpha+1}{N_3}, \\ \frac{n_2+\alpha+1}{N_2} = \frac{m_3+\alpha}{N_3}, \\ \frac{n_2}{N_2} = \frac{m_3}{N_3}, \\ \frac{n_1}{N_1} = \frac{m_3-1}{N_3}. \end{gathered} \quad (67)$$

Hence,

$$(n_1, n_2, m_3) = r(N_1, N_2, N_3) + \left(0, \frac{N_2}{N_3}, 1\right), \quad (68)$$

where $r \in \mathbb{R}$ and $N_1 : N_2 : N_3 = \alpha(\alpha+1) : (\alpha+1)(\alpha+2) : \alpha(\alpha+2)$. In this case, $1/N_1 + 1/N_2 - 2/N_3 = 0$. Thus, we get (1).

If $(n_2+\alpha+1)/N_2 = (m_3+\alpha+1)/N_3$, then it is easy to check that (2) holds. □

By symmetry, we can get the following corollary.

Corollary 11. *Fix a rational number* $\alpha > 0$ *and* $n \in \Omega$. *If* $m \in \mathfrak{I}_n \setminus \{n\}$, *then one of the following statements holds.*

(1) $m_{\sigma(1)} = n_{\sigma(1)} - 1, m_{\sigma(2)} = n_{\sigma(2)} - 1, m_{\sigma(3)} = n_{\sigma(3)} + 1$, *where* $\sigma \in S_k$. *In this case,* $1/N_{\sigma(1)} + 1/N_{\sigma(2)} - 1/N_{\sigma(3)} = 0$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + (k_1, k_2, k_3) : \begin{cases} k_{\sigma(1)} = \dfrac{(\alpha+2)N_{\sigma(1)}}{N_{\sigma(3)}} - \alpha - 1 \\ k_{\sigma(2)} = \dfrac{N_{\sigma(2)}}{N_{\sigma(3)}} \\ k_{\sigma(3)} = 0, \end{cases} r \in \mathbb{R} \right\}. \quad (69)$$

(2) $m_{\sigma(1)} = n_{\sigma(1)} + 1, m_{\sigma(2)} = n_{\sigma(2)} + 1, m_{\sigma(3)} = n_{\sigma(3)} - 1$, *where* $\sigma \in S_k$. *In this case,* $1/N_{\sigma(1)} + 1/N_{\sigma(2)} - 1/N_{\sigma(3)} = 0$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + (k_1, k_2, k_3) : \begin{cases} k_{\sigma(1)} = \dfrac{(\alpha+2)N_{\sigma(1)}}{N_{\sigma(3)}} - \alpha - 2 \\ k_{\sigma(2)} = \dfrac{N_{\sigma(2)}}{N_{\sigma(3)}} - 1 \\ k_{\sigma(3)} = 1, \end{cases} r \in \mathbb{R} \right\}. \quad (70)$$

(3) $m_{\sigma(1)} = n_{\sigma(1)} - 1, m_{\sigma(2)} = n_{\sigma(2)} - 1, m_{\sigma(3)} = n_{\sigma(3)} + 2$, *where* $\sigma \in S_k$. *In this case,* $N_{\sigma(1)} : N_{\sigma(2)} : N_{\sigma(3)} = \alpha(\alpha+1) : (\alpha+1)(\alpha+2) : \alpha(\alpha+2)$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + (k_1, k_2, k_3) : \begin{cases} k_{\sigma(1)} = 0 \\ k_{\sigma(2)} = \dfrac{N_{\sigma(2)}}{N_{\sigma(3)}} \\ k_{\sigma(3)} = -1, \end{cases} r \in \mathbb{R} \right\}. \quad (71)$$

(4) $m_{\sigma(1)} = n_{\sigma(1)} + 1, m_{\sigma(2)} = n_{\sigma(2)} + 1, m_{\sigma(3)} = n_{\sigma(3)} - 2$, *where* $\sigma \in S_k$. *In this case,* $N_{\sigma(1)} : N_{\sigma(2)} : N_{\sigma(3)} = \alpha(\alpha+1) : (\alpha+1)(\alpha+2) : \alpha(\alpha+2)$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + (k_1, k_2, k_3) : \begin{cases} k_{\sigma(1)} = -1 \\ k_{\sigma(2)} = \dfrac{N_{\sigma(2)}}{N_{\sigma(3)}} - 1 \\ k_{\sigma(3)} = 1, \end{cases} r \in \mathbb{R} \right\}. \quad (72)$$

Lemma 12. *Fix a rational number* $\alpha \in (-1, 0)$ *and* $n \in \Omega$. *If there exists* $m \in \mathfrak{I}_n$ *such that* $n_1 > m_1$, $n_2 > m_2$, *and* $m_3 > n_3$, *then one of the following statements holds:*

(1) $m = n - (1, 1, -1)$. *In this case,* $1/N_1 + 1/N_2 + 1/N_3 = 0$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + \left((\alpha+2)\frac{N_1}{N_3} - \alpha - 1, \frac{N_2}{N_3}, 0\right) : r \in \mathbb{R} \right\}, \quad (73)$$

or

$$n \in \left\{ r(N_1, N_2, N_3) + \left(\frac{N_1}{N_3}, (\alpha+2)\frac{N_2}{N_3} - \alpha - 1, 0 \right) : r \in \mathbb{R} \right\}, \tag{74}$$

(2) $m = n - (2, 1, -1)$. *In this case,* $N_1 : N_2 : N_3 = -\alpha(\alpha + 2) : (\alpha+1)(\alpha+2) : -\alpha(\alpha+1)$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + \left(1, (\alpha+1)\frac{N_2}{N_1}, -1 \right) : r \in \mathbb{R} \right\}. \tag{75}$$

(3) $m = n - (1, 2, -1)$. *In this case,* $N_1 : N_2 : N_3 = (\alpha + 1)(\alpha+2) : -\alpha(\alpha+2) : -\alpha(\alpha+1)$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + \left((\alpha+1)\frac{N_1}{N_2}, 1, -1 \right) : r \in \mathbb{R} \right\}. \tag{76}$$

Proof. Define E_i and F_i as in the proof of Lemma 7. Then, $\bigsqcup_{i=1}^{3} E_i = \bigsqcup_{i=1}^{3} F_i$ and $E_i \cap F_i = \emptyset$ for $i = 1, 2, 3$. Assume that

$$\frac{n_1 + \alpha + 1}{N_1} = \frac{m_3 + \alpha + 1}{N_3} \in F_3. \tag{77}$$

Then,

$$\max\left(\bigsqcup_{i=1}^{3} E_i \setminus \left\{ \frac{n_1 + \alpha + 1}{N_1} \right\} \right) = \max\left(\bigsqcup_{i=1}^{3} F_i \setminus \left\{ \frac{m_3 + \alpha + 1}{N_3} \right\} \right). \tag{78}$$

Since $(n_2 + \alpha + 1)/N_2 > n_2/N_2$, $(n_1 + \alpha)/N_1 < n_1/N_1$, and $(m_3 + \alpha)/N_3 < m_3/N_3$, we have

$$\max\left\{ \frac{n_2 + \alpha + 1}{N_2}, \frac{m_3}{N_3} \right\} = \frac{n_1}{N_1}. \tag{79}$$

By (77) and $N_1 \neq N_3$, we get

$$\frac{n_1}{N_1} = \frac{n_2 + \alpha + 1}{N_2} \in E_2. \tag{80}$$

(a) If $n_1 - m_1 = 1$, then $F_2 = E_3$. Therefore, $n_2 - m_2 = m_3 - n_3 = 1$. Otherwise, $n_2/N_2 = m_3/N_3$ and $(n_2 - 1)/N_2 = (m_3 - 1)/N_3$, which is in contradiction with $N_2 \neq N_3$.

In this case, we have

$$\frac{n_1 + \alpha + 1}{N_1} = \frac{m_3 + \alpha + 1}{N_3}, \quad \frac{n_1}{N_1} = \frac{n_2 + \alpha + 1}{N_2}, \quad \frac{n_2}{N_2} = \frac{m_3}{N_3}. \tag{81}$$

Hence,

$$(n_1, n_2, m_3) = r(N_1, N_2, N_3) + \left((\alpha+2)\frac{N_1}{N_3} - \alpha - 1, \frac{N_2}{N_3}, 1 \right), \tag{82}$$

where $r \in \mathbb{R}$ and

$$\frac{1}{N_1} + \frac{1}{N_2} = \frac{1}{N_3} \tag{83}$$

or, equivalently,

$$n = r(N_1, N_2, N_3) + \left((\alpha+2)\frac{N_1}{N_3} - \alpha - 1, \frac{N_2}{N_3}, 0 \right); \quad m = (n_1 - 1, n_2 - 1, n_3 + 1). \tag{84}$$

So (i) holds.

(b) Assume $n_1 - m_1 \geq 2$.

Since $(n_1 - 1)/N_1 < (n_1 + \alpha)/N_1$, $(n_2 + \alpha)/N_2 < n_2/N_2$, and $(m_3 + \alpha)/N_3 < m_3/N_3$, we get

$$\max\left\{ \frac{n_1 + \alpha}{N_1}, \frac{n_2 + \alpha}{N_2}, \frac{m_3}{N_3} \right\} = \max\left\{ \frac{n_1 + \alpha}{N_1}, \frac{m_3}{N_3} \right\} = \frac{n_2}{N_2} = \max\left\{ \frac{n_1 - 1}{N_1}, \frac{n_2}{N_2}, \frac{m_3 + \alpha}{N_3} \right\}. \tag{85}$$

We prove that $(n_1 + \alpha)/N_1 = n_2/N_2$ by contradiction. Otherwise, if $m_3/N_3 = n_2/N_2$, then equalities (77) and (80) imply equality (83). Therefore, $1/N_3 > 1/N_1$ and $(n_1 + \alpha)/N_1 > (m_3 + \alpha)/N_3$. Since $-1 < \alpha < 0$, we have

$$\max\left\{ \frac{n_1 + \alpha}{N_1}, \frac{m_3 - 1}{N_3} \right\} > \max\left\{ \frac{n_1 - 1}{N_1}, \frac{m_3 + \alpha}{N_3} \right\},$$
$$\max\left\{ \frac{n_1 + \alpha}{N_1}, \frac{n_2 + \alpha}{N_2}, \frac{m_3 - 1}{N_3} \right\} > \max\left\{ \frac{n_1 - 1}{N_1}, \frac{n_2 - 1}{N_2}, \frac{m_3 + \alpha}{N_3} \right\}, \tag{86}$$

which contradicts $\bigsqcup_{i=1}^{3} E_i = \bigsqcup_{i=1}^{3} F_i$. Thus,

$$\frac{n_1 + \alpha}{N_1} = \frac{n_2}{N_2}. \tag{87}$$

Combining equality (80), we get

$$-\frac{\alpha}{N_1} = \frac{\alpha + 1}{N_2}. \tag{88}$$

In the following, we prove that $n_2 - m_2 = 1$ by contradiction. Assume $n_2 - m_2 \geq 2$. By $(n_1 - 1)/N_1 > (n_1 + \alpha - 1)/N_1$, $(n_2 + \alpha)/N_2 > (n_2 - 1)/N_2$, and $m_3/N_3 > (m_3 + \alpha)/N_3$, we get

$$\max\left\{ \frac{n_2 + \alpha}{N_2}, \frac{m_3}{N_3} \right\} = \max\left\{ \frac{n_1 - 1}{N_1}, \frac{n_2 - 1}{N_2} \right\} = \frac{n_1 - 1}{N_1}. \tag{89}$$

If $(n_1-1)/N_1 = (n_2+\alpha)/N_2$, combining the equalities (87) and (88), we get $N_1 = N_2$, which is a contradiction.

If $(n_1-1)/N_1 = m_3/N_3$, equality (77) shows that $N_1 : N_3 = (\alpha+2) : (\alpha+1)$. We also have $m_3 - n_3 \geq 2$, since

$$\max\left\{\frac{n_1+\alpha-1}{N_1}, \frac{n_2+\alpha}{N_2}\right\} > \max\left\{\frac{n_1-2}{N_1}, \frac{n_2-1}{N_2}\right\}. \quad (90)$$

Therefore,

$$\begin{aligned}\max&\left\{\frac{n_1+\alpha-1}{N_1}, \frac{n_2+\alpha}{N_2}, \frac{m_3-1}{N_3}\right\}\\ &= \frac{m_3+\alpha}{N_3} = \max\left\{\frac{n_1-2}{N_1}, \frac{n_2-1}{N_2}, \frac{m_3+\alpha}{N_3}\right\}.\end{aligned} \quad (91)$$

We conclude that

$$\frac{m_3+\alpha}{N_3} = \max\left\{\frac{n_2+\alpha}{N_2}, \frac{n_1+\alpha-1}{N_1}\right\}. \quad (92)$$

If $(m_3+\alpha)/N_3 = (n_1+\alpha-1)/N_1$, equality (77) shows that

$$N_1 : N_3 = 2:1. \quad (93)$$

So $(\alpha+2):(\alpha+1) = 2:1$; that is, $\alpha = 0$ which is in contradiction with $\alpha < 0$.

If $(m_3+\alpha)/N_3 = (n_2+\alpha)/N_2$, the fact

$$\begin{aligned}\max&\left\{\frac{n_1+\alpha-1}{N_1}, \frac{n_2+\alpha-1}{N_2}, \frac{m_3-1}{N_3}\right\}\\ &= \max\left\{\frac{n_1-2}{N_1}, \frac{n_2-1}{N_2}, \frac{m_3+\alpha-1}{N_3}\right\},\end{aligned} \quad (94)$$

implies that

$$\max\left\{\frac{n_1+\alpha-1}{N_1}, \frac{m_3-1}{N_3}\right\} = \frac{n_2-1}{N_2}. \quad (95)$$

However, $(n_2-1)/N_2 \neq (m_3-1)/N_3$ because of $(m_3+\alpha)/N_3 = (n_2+\alpha)/N_2$ and $N_2 \neq N_3$ and $(n_2-1)/N_2 \neq (n_1+\alpha-1)/N_1$, because of equality (87) and $N_1 \neq N_2$. We also get a contradiction. Hence, $n_2 - m_2 = 1$.

Therefore, $E_2 = \{(n_2+\alpha+1)/N_2\} \subset F_1$ and $F_2 = \{n_2/N_2\} \subset E_1$. Further, $F_1 \setminus \{n_1/N_1\} = E_3$. It follows that $m_3 - n_3 = 1$, $n_1 - m_1 = 2$, and $m_3/N_3 = (n_1-1)/N_1$. Summing up, we have

$$\begin{aligned}\frac{n_1+\alpha+1}{N_1} &= \frac{m_3+\alpha+1}{N_3},\\ \frac{n_1}{N_1} &= \frac{n_2+\alpha+1}{N_2},\\ \frac{n_2}{N_2} &= \frac{n_1+\alpha}{N_1},\\ \frac{m_3}{N_3} &= \frac{n_1-1}{N_1}.\end{aligned} \quad (96)$$

Hence, $(n_1, n_2, m_3) = r(N_1, N_2, N_3) + (1, (\alpha+1)N_2/N_1, 0)$, where $r \in \mathbb{R}$ and

$$N_1 : N_2 : N_3 = -\alpha(\alpha+2) : (\alpha+1)(\alpha+2) : -\alpha(\alpha+1). \quad (97)$$

Thus, (2) holds. In this case, $2/N_1 + 1/N_2 - 1/N_3 = 0$.

Similarly, if $(n_2+\alpha+1)/N_2 = (m_3+\alpha+1)/N_3$, by symmetry, we can get (3) and another part of (1). □

By symmetry, we also have the following corollary.

Corollary 13. *Let $\alpha \in (-1,0) \cap \mathbb{Q}$ and let $n \in \Omega$. If there exists $m \in \mathfrak{J}_n$ such that $n \neq m$, then there is a permutation $\sigma \in S_k$ such that one of the following statements holds.*

(1) $1/N_{\sigma(1)} + 1/N_{\sigma(2)} - 1/N_{\sigma(3)} = 0$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + (k_1, k_2, k_3) : \begin{cases} k_{\sigma(1)} = \dfrac{(\alpha+2)N_{\sigma(1)}}{N_{\sigma(3)}} - \alpha - 1 \\ k_{\sigma(2)} = \dfrac{N_{\sigma(2)}}{N_{\sigma(3)}} \\ k_{\sigma(3)} = 0, \end{cases} r \in \mathbb{R} \right\}. \quad (98)$$

In this case, $m_{\sigma(1)} = n_{\sigma(1)} - 1$, $m_{\sigma(2)} = n_{\sigma(2)} - 1$, and $m_{\sigma(3)} = n_{\sigma(3)} + 1$.

(2) $1/N_{\sigma(1)} + 1/N_{\sigma(2)} - 1/N_{\sigma(3)} = 0$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + (k_1, k_2, k_3) : \begin{cases} k_{\sigma(1)} = \dfrac{(\alpha+2)N_{\sigma(1)}}{N_{\sigma(3)}} - \alpha - 2 \\ k_{\sigma(2)} = \dfrac{N_{\sigma(2)}}{N_{\sigma(3)}} - 1 \\ k_{\sigma(3)} = 1, \end{cases} r \in \mathbb{R} \right\}. \quad (99)$$

In this case, $m_{\sigma(1)} = n_{\sigma(1)} + 1$, $m_{\sigma(2)} = n_{\sigma(2)} + 1$, and $m_{\sigma(3)} = n_{\sigma(3)} - 1$.

(3) $N_{\sigma(1)} : N_{\sigma(2)} : N_{\sigma(3)} = -\alpha(\alpha+2) : (\alpha+1)(\alpha+2) : -\alpha(\alpha+1)$ *and*

$$n \in \left\{ r(N_1, N_2, N_3) + (k_1, k_2, k_3) : \begin{cases} k_{\sigma(1)} = 1 \\ k_{\sigma(2)} = \dfrac{(\alpha+1)N_{\sigma(2)}}{N_{\sigma(1)}} \\ k_{\sigma(3)} = -1, \end{cases} r \in \mathbb{R} \right\}. \quad (100)$$

In this case, $m_{\sigma(1)} = n_{\sigma(1)} - 2$, $m_{\sigma(2)} = n_{\sigma(2)} - 1$, and $m_{\sigma(3)} = n_{\sigma(3)} + 1$.

(4) $N_{\sigma(1)} : N_{\sigma(2)} : N_{\sigma(3)} = -\alpha(\alpha+2) : (\alpha+1)(\alpha+2) : -\alpha(\alpha+1)$ *and*

$$n \in \left\{ r\left(N_1, N_2, N_3\right) + \left(k_1, k_2, k_3\right) : \left\{ \begin{array}{l} k_{\sigma(1)} = -1 \\ k_{\sigma(2)} = \dfrac{(\alpha+1) N_{\sigma(2)}}{N_{\sigma(1)}} - 1 \\ k_{\sigma(3)} = 0, \end{array} \right. \quad r \in \mathbb{R} \right\}. \tag{101}$$

In this case, $m_{\sigma(1)} = n_{\sigma(1)} + 2$, $m_{\sigma(2)} = n_{\sigma(2)} + 1$, *and* $m_{\sigma(3)} = n_{\sigma(3)} - 1$.

By careful computation, we find that each choice of n and N cannot simultaneously satisfy two of the statements in Corollary 11 or Corollary 13. So $\mathrm{Card}(\mathfrak{J}_n) \leq 2$. Moreover, there are finite numbers of $n \in \Omega$ such that $\mathrm{Card}(\mathfrak{J}_n) = 2$. Denote

(i) $\mathscr{M}_n = \overline{\mathrm{span}}\{z^{n+hN} : h \in \mathbb{N}_0\}$;

(ii) $\mathscr{H}_n = \overline{\mathrm{span}}\{(az^n + bz^{n-(i,j,k)})z^{hN} : h \in \mathbb{N}_0\}$, (i, j, k) is a permutation of $\{1, 1, -1\}$ or $\{1, 1, -2\}$;

(iii) $\mathscr{N}_n = \overline{\mathrm{span}}\{(az^n + bz^{n-(i,j,k)})z^{hN} : h \in \mathbb{N}_0\}$, (i, j, k) is a permutation of $\{1, 1, -1\}$ or $\{2, 1, -1\}$.

Theorem 2 implies the following statements.

(a) If $\alpha > 0$, then each reducing subspace of M_{z^N} is the direct sum of some minimal reducing as in (i) and (ii), where the number of reducing subspaces as (ii) is finite.

(b) If $-1 < \alpha < 0$, then each reducing subspace of M_{z^N} is the direct sum of some minimal reducing as in (i) and (iii), where the number of reducing subspaces as (iii) is finite.

Finally, we consider the reducing subspaces from the viewpoint of von Neumann algebras. Denote by $\mathscr{W}^*(z^N)$ the von Neumann algebra generated by M_{z^N} and $\nu^*(z^N)$ the commutant of $\mathscr{W}^*(z^N)$. Then $\nu^*(z^N)$ is a von Neumann algebra, and it is generated by its self-adjoint projections. For each reducing subspace $\mathscr{M}$ of M_{z^N}, denote by $P_{\mathscr{M}}$ the orthogonal projection from $A^2_\alpha(\mathbb{D}^3)$ onto $\mathscr{M}$. It is known that $P_{\mathscr{M}}$ is a self-adjoint projection in $\nu^*(z^N)$. Conversely, if P is a self-adjoint projection in $\nu^*(z^N)$, then the range of P is a reducing subspace of M_{z^N}. So our results can be written in the following form.

Theorem 14. *The von Neumann algebra $\nu^*(z^N)$ is $*-$ isomorphic to*

$$\bigoplus_{n=1}^{m} M_2(\mathbb{C}) \bigoplus \left(\bigoplus_{n=1}^{+\infty} \mathbb{C} \right), \tag{102}$$

where $0 \leq m < +\infty$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This paper is supported by the Fundamental Research Funds for the Central Universities (Grant no. 201213011), NSFC (nos. 11201438, 11171315), and Shandong Province Young Scientist Research Award Fund (BS2012SF031).

References

[1] K. Guo and H. Huang, "Commutants, reducing subspaces and von Neumann algebras: based on multiplication operators on the Bergman space," Preprint.

[2] S. L. Sun and Y. Wang, "Reducing subspaces of certain analytic Toeplitz operators on the Bergman space," *Northeastern Mathematical Journal*, vol. 14, no. 2, pp. 147–158, 1998.

[3] K. Zhu, "Reducing subspaces for a class of multiplication operators," *Journal of the London Mathematical Society*, vol. 62, no. 2, pp. 553–568, 2000.

[4] M. Stessin and K. Zhu, "Reducing subspaces of the weighted shift operators," *Proceedings of the American Mathematical Society*, vol. 130, no. 9, pp. 2631–2639, 2002.

[5] J. Hu, S. Sun, X. Xu, and D. Yu, "Reducing subspace of analytic Toeplitz operators on the Bergman space," *Integral Equations and Operator Theory*, vol. 49, no. 3, pp. 387–395, 2004.

[6] A. Xu and C. Yan, "Reducing subspace of analytic Toeplitz operators on weighted Bergman spaces," *Chinese Annals of Mathematics, Series A*, vol. 30, pp. 639–646, 2009.

[7] K. Guo, S. Sun, D. Zheng, and C. Zhong, "Multiplication operators on the Bergman space via the Hardy space of the bidisk," *Journal für Die Reine und Angewandte Mathematik*, vol. 2009, no. 628, pp. 129–168, 2009.

[8] R. G. Douglas, S. Sun, and D. Zheng, "Multiplication operators on the Bergman space via analytic continuation," *Advances in Mathematics*, vol. 226, no. 1, pp. 541–583, 2011.

[9] K. Guo and H. Huang, "On multiplication operators of the Bergman space: similarity, unitary equivalence and reducing subspaces," *Journal of Operator Theory*, vol. 65, no. 2, pp. 355–378, 2011.

[10] S. Sun, D. Zheng, and C. Zhong, "Classification of reducing subspaces of a class of multiplication operators on the Bergman space via the hardy space of the bidisk," *Canadian Journal of Mathematics*, vol. 62, no. 2, pp. 415–438, 2010.

[11] R. G. Douglas, M. Putinar, and K. Wang, "Reducing subspaces for analytic multipliers of the Bergman space," *Journal of Functional Analysis*, vol. 263, no. 6, pp. 1744–1765, 2012.

[12] K. Guo and H. Huang, "Multiplication operators defined by covering maps on the Bergman space: the connection between operator theory and von Neumann algebras," *Journal of Functional Analysis*, vol. 260, no. 4, pp. 1219–1255, 2011.

[13] K. Guo and H. Huang, "Geometric constructions of thin Blaschke products and reducing subspace problem," *Proceedings of the London Mathematical Society*, vol. 109, no. 4, pp. 1050–1091, 2014.

[14] Y. Lu and X. Zhou, "Invariant subspaces and reducing subspaces of weighted Bergman space over bidisk," *Journal of the Mathematical Society of Japan*, vol. 62, no. 3, pp. 745–765, 2010.

[15] L. Shan, "Reducing subspaces for a class of analytic Toeplitz operators on bidisc," *Journal of Fudan University*, vol. 42, no. 2, pp. 196–200, 2003.

[16] Y. Shi and Y. Lu, "Reducing subspaces for Toeplitz operators on the polydisk," *Bulletin of the Korean Mathematical Society*, vol. 50, no. 2, pp. 687–696, 2013.

[17] X. Zhou, Y. Shi, and Y. Lu, "Invariant subspaces and reducing subspaces of weighted Bergman space over polydisc," *Scientia Sinica Mathematica*, vol. 41, no. 5, pp. 427–438, 2011.

[18] H. Dan and H. Huang, "Multiplication operators defined by a class of polynomials on $L_\alpha^2(D^2)$," *Integral Equations and Operator Theory*, vol. 80, no. 4, pp. 581–601, 2014.