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Research Article

On Retarded Integral Inequalities for Dynamic Systems on Time Scales

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The object of this paper is to establish some nonlinear retarded inequalities on time scales which can be used as handy tools in the theory of integral equations with time delays.

1. Introduction

Integral inequalities play an important role in the qualitative analysis of differential and integral equations. The well-known Gronwall inequality provides explicit bounds for solutions of many differential and integral equations. On the basis of various initiatives, this inequality has been extended and applied to various contexts (see, e.g., [1–4]), including many retarded ones (see, e.g., [5–9]).

Recently, Ye and Gao [7] obtained the following.

Theorem A. *Let* $I=[t_0,T)\subset\mathbb{R}$, $a(t),b(t)\in C(I,\mathbb{R}^+)$, $\phi(t)\in C([t_0-r,t_0],\mathbb{R}^+)$, $a(t_0)=\phi(t_0)$, *and* $u(t)\in C([t_0-r,T),\mathbb{R}^+)$ *with*

$$u(t)\le a(t)+\int_{t_0}^{t}(t-s)^{\beta-1}b(s)\,u(s-r)\,ds,\quad t\in[t_0,T)$$
$$u(t)\le\phi(t),\quad t\in[t_0-r,t_0),\tag{1}$$

where $\beta>0$. *Then, the following assertions hold.*

(i) *Suppose that* $\beta>1/2$. *Then,*

$$u(t)\le e^{t}\left[w_1(t)+y_1(t)\right]^{1/2},\quad t\in[t_0+r,T),$$
$$u(t)\le a(t)+\int_{t_0}^{t}(t-s)^{\beta-1}b(s)\,\phi(s-r)\,ds,\quad t\in[t_0,t_0+r),\tag{2}$$

where $K_1=\Gamma(2\beta-1)e^{-2r}/4^{\beta-1}$, $C_1=\max\{2,e^{2r}\}$, $w_1(t)=C_1e^{-2t_0}a^2(t)$, $\phi_1(t)=C_1e^{-2t_0}\phi^2(t)$, *and*

$$\begin{aligned}y_1(t)&=\int_{t_0}^{t_0+r}K_1b^2(s)\,\phi_1(s-r)\,ds\cdot\exp\left(\int_{t_0+r}^{t}K_1b^2(\tau)\,d\tau\right)\\&\quad+\int_{t_0+r}^{t}w_1(s-r)\,K_1b^2(s)\exp\left(\int_{s}^{t}K_1b^2(\tau)\,d\tau\right)ds.\end{aligned}\tag{3}$$

If, in addition, $a(t)$ *and* $\phi(t)$ *are nondecreasing* C^1*-functions, then*

$$u(t)\le\sqrt{C_1}\,a(t)\exp\left(t-t_0+\frac{K_1}{2}\int_{t_0}^{t}b^2(s)\,ds\right),\quad t\in[t_0,T).\tag{4}$$

(ii) *Suppose that* $0<\beta\le1/2$. *Then,*

$$u(t)\le e^{t}\left[w_2(t)+y_2(t)\right]^{1/q},\quad t\in[t_0+r,T),$$
$$u(t)\le a(t)+\int_{t_0}^{t}(t-s)^{\beta-1}b(s)\,\phi(s-r)\,ds,\quad t\in[t_0,t_0+r),\tag{5}$$

where $K_2 = [(\Gamma(1-(1-\beta)p))/p^{1-p(1-\beta)}]^{1/p}$, $C_2 = \max\{2^{q-1}, e^{qr}\}$, $w_2(t) = C_2 e^{-qt_0} a^q(t)$, $\phi_2(t) = C_2 e^{-qt_0}\phi^q(t)$, $\psi(t) = 2^{q-1}K_2^q e^{-qr} b^q(t)$, *and*

$$y_2(t) = \int_{t_0}^{t_0+r} \psi(s)\,\phi_2(s-r)\,ds \cdot \exp\left(\int_{t_0+r}^{t}\psi(\tau)\,d\tau\right) + \int_{t_0+r}^{t} w_2(s-r)\,\psi(s)\,\exp\left(\int_{s}^{t}\psi(\tau)\,d\tau\right) ds. \tag{6}$$

If, in addition, $a(t)$ *and* $\phi(t)$ *are nondecreasing* C^1*-functions, then*

$$u(t) \le C_2^{1/q} a(t)\,\exp\left(t - t_0 + \frac{1}{q}\int_{t_0}^{t}\psi(s)\,ds\right), \quad t \in [t_0, T). \tag{7}$$

In this paper, we will further investigate functions u satisfying the following more general inequalities:

$$u(t) \le a(t) + \int_{t_0}^{t} (t-s)^{\beta-1} b(s)\,u^{n/m}(s-r)\,\Delta s, \quad t \in [t_0, T)_{\mathbb{T}}, \qquad u(t) \le \phi(t), \quad t \in [t_0 - r, t_0)_{\mathbb{T}}, \tag{8}$$

$$u(t) \le a(t) + \int_{t_0}^{t} (t-s)^{\beta-1}\left[b(s)\,u^{n/m}(s) + c(s)\,u^{n/m}(s-r)\right]\Delta s, \quad t \in [t_0, T)_{\mathbb{T}}, \qquad u(t) \le \phi(t), \quad t \in [t_0 - r, t_0)_{\mathbb{T}}, \tag{9}$$

where $\mathbb{T}$ is any time scale, $u(t)$, $a(t)$, $b(t)$, $c(t)$, and $\phi(t)$ are real-valued nonnegative rd-continuous functions defined on $\mathbb{T}$, m and n are positive constants, $m \ge n, m \ge 1$, $(1/p) + (1/m) = 1$, $\beta > (p-1)/p$, and $[t_0, T)_{\mathbb{T}} := [t_0, T) \cap \mathbb{T}$.

First, we make a preliminary definition.

Definition 1. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided that

$$1 + \mu(t)\,p(t) \ne 0, \quad \forall t \in \mathbb{T}^k \tag{10}$$

holds, where $\mu(t)$ is graininess function; that is, $\mu(t) := \sigma(t) - t$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathscr{R}$.

2. Main Results

For convenience, we first cite the following lemma.

Lemma 2 (see [10]). *Let* $a \ge 0$, $p \ge q \ge 0$, $p \ne 0$; *then*

$$a^{q/p} \le \frac{q}{p} K^{(q-p)/p} a + \frac{p-q}{p} K^{q/p} \tag{11}$$

for any $K > 0$.

Lemma 3. *Let* $a(t) \ge 0$, $b(t) > 0$, $p(t) := nb(t)/m$, $-b \in \mathscr{R}^+ := \{f \in \mathscr{R} : 1 + \mu(t)f(t) > 0, \text{for all } t \in \mathbb{T}\}$, $\phi(t) \ge 0$ *is rd-continuous on* $[t_0 - r, t_0]_{\mathbb{T}}$, *and* $r \ge 0$ *and* $m \ge n > 0$ *are real constants. If* $u(t) \ge 0$ *is rd-continuous and*

$$u^m(t) \le a(t) + \int_{t_0}^{t} b(s)\,u^n(s-r)\,\Delta s, \quad t \in [t_0, T)_{\mathbb{T}}, \qquad u(t) \le \phi(t), \quad t \in [t_0 - r, t_0)_{\mathbb{T}}, \tag{12}$$

then

$$u^m(t) \le a(t) + \int_{t_0+r}^{t} p(s)\,a(s-r)\,e_{-p}(s,t)\,\Delta s + e_{-p}(t_0+r, t)\int_{t_0}^{t_0+r} b(s)\,\phi^n(s-r)\,\Delta s + \frac{m-n}{n}\left(e_{-p}(t_0+r,t) - 1\right) \tag{13}$$

for $t \in [t_0 + r, T)_{\mathbb{T}}$ *and*

$$u^m(t) \le a(t) + \int_{t_0}^{t} b(s)\,\phi^n(s-r)\,\Delta s \tag{14}$$

for $t \in [t_0, t_0 + r)_{\mathbb{T}}$.

Furthermore, if $a(t)$ *and* $\phi(t)$ *are nondecreasing with* $a(t_0) = \phi^n(t_0)$, *then*

$$u^m(t) \le c(t)\,e_{-b}(t_0, t), \quad t \in [t_0, T)_{\mathbb{T}}, \tag{15}$$

where $c(t) := a(t) + (m-n)/n$.

Proof. Let $z(t) = \int_{t_0}^{t} b(s)u^n(s-r)\Delta s$. Then, $z(t_0) = 0, u^m(t) \le a(t) + z(t)$ and $z(t)$ is positive, nondecreasing for $t \in [t_0, T)_{\mathbb{T}}$. By Lemma 2, we get

$$\begin{aligned} z^{\Delta}(t) &= b(t)\,u^n(t-r) \le b(t)\,[a(t-r) + z(t-r)]^{n/m} \\ &\le b(t)\left[\frac{n}{m}(a(t-r) + z(t-r)) + \frac{m-n}{m}\right] \\ &\le \frac{n}{m} b(t)\,z(\sigma(t)) + \frac{n}{m} b(t)\,a(t-r) + \frac{m-n}{m} b(t) \\ &= p(t)\,z(\sigma(t)) + p(t)\,a(t-r) + \frac{m-n}{n} p(t) \end{aligned} \tag{16}$$

for $t \in [t_0 + r, T)_{\mathbb{T}}$. Multiplying (16) by $e_{-p}(t, t_0 + r) > 0$, we get

$$\left(z(t)\,e_{-p}(t, t_0 + r)\right)^{\Delta} \le p(t)\,a(t-r)\,e_{-p}(t, t_0 + r) + \frac{m-n}{n} p(t)\,e_{-p}(t, t_0 + r). \tag{17}$$

Integrating both sides from $t_0 + r$ to t, we obtain

$$\begin{aligned} z(t) \le{}& e_{-p}(t_0+r,t)\, z(t_0+r) \\ &+ e_{-p}(t_0+r,t)\int_{t_0+r}^{t} p(s)\, a(s-r)\, e_{-p}(s,t_0+r)\,\Delta s \\ &+ \frac{m-n}{n}\left(e_{-p}(t_0+r,t)-1\right). \end{aligned} \tag{18}$$

For $t \in [t_0, t_0+r)_{\mathbb{T}}$, $z^{\Delta}(t) \le b(t)\phi^n(t-r)$, so

$$z(t) \le \int_{t_0}^{t} b(s)\,\phi^n(s-r)\,\Delta s. \tag{19}$$

Using (18) and (19), we get

$$\begin{aligned} z(t) \le{}& e_{-p}(t_0+r,t)\int_{t_0}^{t_0+r} b(s)\,\phi^n(s-r)\,\Delta s \\ &+ \int_{t_0+r}^{t} p(s)\, a(s-r)\, e_{-p}(s,t)\,\Delta s \\ &+ \frac{m-n}{n}\left(e_{-p}(t_0+r,t)-1\right) \end{aligned} \tag{20}$$

for $t \in [t_0+r, T)_{\mathbb{T}}$.

Noting that $u^m(t) \le a(t) + z(t)$, inequalities (13) and (14) follow.

Finally, if $a(t)$ and $\phi(t)$ are nondecreasing, then for $t \in [t_0, t_0+r)_{\mathbb{T}}$, by (14), we have

$$\begin{aligned} u^m(t) &\le a(t) + \phi^n(t-r)\int_{t_0}^{t} b(s)\,\Delta s \\ &\le a(t)\left(1+\int_{t_0}^{t} b(s)\,\Delta s\right) \le c(t)\, e_{-b}(t_0,t). \end{aligned} \tag{21}$$

If $t \in [t_0+r, T)_{\mathbb{T}}$, by (13),

$$\begin{aligned} u^m(t) \le{}& a(t) + e_{-p}(t_0+r,t)\, a(t)\int_{t_0}^{t_0+r} b(s)\,\Delta s \\ &+ a(t)\int_{t_0+r}^{t} p(s)\, e_{-p}(s,t)\,\Delta s \\ &+ \frac{m-n}{n}\int_{t_0+r}^{t} p(s)\, e_{-p}(s,t)\,\Delta s \\ \le{}& c(t) + e_{-p}(t_0+r,t)\, c(t)\int_{t_0}^{t_0+r} b(s)\,\Delta s \\ &+ c(t)\int_{t_0+r}^{t} p(s)\, e_{-p}(s,t)\,\Delta s \\ ={}& c(t)\, e_{-p}(t_0+r,t)\left(1+\int_{t_0}^{t_0+r} b(s)\,\Delta s\right) \\ \le{}& c(t)\, e_{-b}(t_0,t). \end{aligned} \tag{22}$$

The proof is complete. □

Theorem 4. *Assume that $u(t)$ satisfies condition (8), $a(t) \ge 0$, $K := 2^{m-1}\Gamma^{m-1}(p\beta - p + 1)(m/pn)^{\beta m-1}e^{-nr}$, $b_1(t) := (n/m)Kb^m(t)$, $-Kb^m \in \mathscr{R}^+$; then*

$$\begin{aligned} u(t) &\le e^t\left[w_1(t) + y_1(t)\right]^{1/m}, \quad t \in [t_0+r, T)_{\mathbb{T}}, \\ u(t) &\le a(t) + \int_{t_0}^{t} (t-s)^{\beta-1} b(s)\,\phi^{n/m}(s-r)\,\Delta s, \\ & \qquad t \in [t_0, t_0+r)_{\mathbb{T}}, \end{aligned} \tag{23}$$

where $w_1(t) := 2^{m-1}a^m(t)e^{-mt_0}$, $\phi_1(t) := e^{-t_0}e^r\phi(t)$, and $y_1(t) := \int_{t_0+r}^{t} b_1(s)w_1(s-r)e_{-b_1}(s,t)\Delta s + e_{-b_1}(t_0+r,t)\int_{t_0}^{t_0+r} Kb^m(s)\phi_1^n(s-r)\Delta s + ((m-n)/n)(e_{-b_1}(t_0+r,t)-1)$.

If, in addition, $a(t)$ and $\phi(t)$ are nondecreasing, and $a^m(t_0) = 2^{1-m}e^{(m-n)t_0}e^{nr}\phi^n(t_0)$, then

$$u(t) \le e^t\left[\alpha(t)\, e_{-Kb^m}(t_0,t)\right]^{1/m}, \quad t \in [t_0, T)_{\mathbb{T}}, \tag{24}$$

where $\alpha(t) := w_1(t) + (m-n)/n$

Proof. The second inequality in (23) is obvious. Next, we will prove the first inequality in (23). For $t \in [t_0, T)_{\mathbb{T}}$, using Hölder's inequality with indices p and m, we obtain from (8)

$$\begin{aligned} u(t) &\le a(t) + \int_{t_0}^{t} (t-s)^{\beta-1} e^{ns/m} b(s)\, e^{-ns/m} u^{n/m}(s-r)\,\Delta s \\ &\le a(t) + \left(\int_{t_0}^{t} (t-s)^{p\beta-p} e^{pns/m}\Delta s\right)^{1/p} \\ &\quad \times \left(\int_{t_0}^{t} b^m(s)\, e^{-ns} u^n(s-r)\,\Delta s\right)^{1/m}. \end{aligned} \tag{25}$$

By Jensen's inequality $(\sum_{i=1}^{n} x_i)^{\sigma} \le n^{\sigma-1}(\sum_{i=1}^{n} x_i^{\sigma})$, we get

$$\begin{aligned} u^m(t) \le{}& 2^{m-1}a^m(t) \\ &+ 2^{m-1}\left(\int_{t_0}^{t} (t-s)^{p\beta-p} e^{pns/m}\Delta s\right)^{m/p} \\ &\times \int_{t_0}^{t} b^m(s)\, e^{-ns} u^n(s-r)\,\Delta s. \end{aligned} \tag{26}$$

For the first integral in (26), we have the estimate

$$\begin{aligned} &\int_{t_0}^{t} (t-s)^{p\beta-p} e^{pns/m}\Delta s \\ &\quad = \int_{0}^{t-t_0} \tau^{p\beta-p} e^{pn(t-\tau)/m}\Delta\tau \\ &\quad \le e^{pnt/m}\int_{0}^{t} \tau^{p\beta-p} e^{-pn\tau/m}\Delta\tau \\ &\quad = e^{pnt/m}\left(\frac{m}{pn}\right)^{p\beta-p+1}\int_{0}^{pnt/m} \sigma^{p\beta-p} e^{-\sigma}\Delta\sigma \\ &\quad < e^{pnt/m}\left(\frac{m}{pn}\right)^{p\beta-p+1}\Gamma(p\beta-p+1). \end{aligned} \tag{27}$$

Hence,

$$u^m(t) \le 2^{m-1}a^m(t) + 2^{m-1}e^{nt}\Gamma^{m-1}(p\beta - p + 1) \times \left(\frac{m}{pn}\right)^{\beta m-1} \int_{t_0}^{t} b^m(s)e^{-ns}u^n(s-r)\Delta s \tag{28}$$

and so

$$\left(u(t)e^{-t}\right)^m \le 2^{m-1}a^m(t)e^{-mt_0} + 2^{m-1}\Gamma^{m-1}(p\beta - p + 1)\left(\frac{m}{pn}\right)^{\beta m-1} \times \int_{t_0}^{t} b^m(s)e^{-ns}u^n(s-r)\Delta s. \tag{29}$$

Let $v(t) := e^{-t}u(t)$; then we have

$$v^m(t) \le w_1(t) + K\int_{t_0}^{t} b^m(s)v^n(s-r)\Delta s, \quad t \in [t_0, T)_{\mathbb{T}}. \tag{30}$$

For $t \in [t_0 - r, t_0)_{\mathbb{T}}$, we have $e^{-t}u(t) \le e^{-t}\phi(t) \le e^r e^{-t_0}\phi(t)$; that is, $v(t) \le \phi_1(t)$. By Lemma 3, we get

$$\begin{aligned} v^m(t) \le{}& w_1(t) + \int_{t_0+r}^{t} b_1(s)w_1(s-r)e_{-b_1}(s,t)\Delta s \\ &+ e_{-b_1}(t_0+r,t)\int_{t_0}^{t_0+r} Kb^m(s)\phi_1^n(s-r)\Delta s \\ &+ \frac{m-n}{n}\left(e_{-b_1}(t_0+r,t) - 1\right). \end{aligned} \tag{31}$$

Hence, the first inequality in (23) follows.

Finally, if $a(t)$ and $\phi(t)$ are nondecreasing, and $a^m(t_0) = 2^{1-m}e^{(m-n)t_0}\phi^n(t_0)e^{nr}$, by Lemma 3, we have

$$u(t) \le e^t\left[\alpha(t)e_{-Kb^m}(t_0,t)\right]^{1/m}, \quad t \in [t_0, T)_{\mathbb{T}}. \tag{32}$$

The proof is complete. □

Lemma 5. *Let $a(t) \ge 0$, $b(t) > 0$, $c(t) > 0$, $p(t) := (nb(t)/m)$, $q(t) := (nc(t)/m)$, $\gamma(t) := a(t) + (m-n)/n$ and $-p, -(p+c) \in \mathcal{R}^+$ and let $\phi(t) \ge 0$ be rd-continuous on $[t_0 - r, t_0]_{\mathbb{T}}$, where $r \ge 0$ and $m \ge n > 0$ are real constants. If $u(t) \ge 0$ is rd-continuous and*

$$\begin{gathered} u^m(t) \le a(t) + \int_{t_0}^{t}[b(s)u^n(s) + c(s)u^n(s-r)]\Delta s, \quad t \in [t_0, T)_{\mathbb{T}}, \\ u(t) \le \phi(t), \quad t \in [t_0 - r, t_0)_{\mathbb{T}}, \end{gathered} \tag{33}$$

then

$$\begin{aligned} u^m(t) \le{}& a(t) \\ &+ \int_{t_0+r}^{t}[p(s)\gamma(s) + q(s)\gamma(s-r)]e_{-(p+q)}(s,t)\Delta s \\ &+ e_{-(p+q)}(t_0+r,t) \\ &\times \int_{t_0}^{t_0+r}[p(s)\gamma(s) + c(s)\phi^n(s-r)]e_{-p}(s,t_0+r)\Delta s \end{aligned} \tag{34}$$

for $t \in [t_0 + r, T)_{\mathbb{T}}$ and

$$u^m(t) \le a(t) + \int_{t_0}^{t}[p(s)\gamma(s) + c(s)\phi^n(s-r)]e_{-p}(s,t)\Delta s \tag{35}$$

for $t \in [t_0, t_0 + r)_{\mathbb{T}}$.

Furthermore, if $a(t)$ and $\phi(t)$ are nondecreasing with $a(t_0) = \phi^n(t_0)$, then

$$u^m(t) \le \gamma(t)e_{-(p+c)}(t_0,t), \quad t \in [t_0, T)_{\mathbb{T}}. \tag{36}$$

Proof. Let $z(t) = \int_{t_0}^{t}[b(s)u^n(s) + c(s)u^n(s-r)]\Delta s$. Then, $z(t_0) = 0$, $u^m(t) \le a(t) + z(t)$, $z(t)$ is positive and nondecreasing for $t \in [t_0, T)_{\mathbb{T}}$. Further, we have

$$z^{\Delta}(t) = b(t)u^n(t) + c(t)u^n(t-r). \tag{37}$$

For $t \in [t_0, t_0 + r)_{\mathbb{T}}$, using Lemma 2, we have

$$\begin{aligned} z^{\Delta}(t) &\le b(t)(a(t) + z(t))^{n/m} + c(t)\phi^n(t-r) \\ &\le b(t)\left[\frac{n}{m}(a(t) + z(t)) + \frac{m-n}{m}\right] + c(t)\phi^n(t-r) \\ &\le p(t)\gamma(t) + p(t)z(\sigma(t)) + c(t)\phi^n(t-r), \\ \left(e_{-p}(t,t_0)z(t)\right)^{\Delta} &\le \left(p(t)\gamma(t) + c(t)\phi^n(t-r)\right)e_{-p}(t,t_0). \end{aligned} \tag{38}$$

Integrating both sides from t_0 to t, we obtain

$$z(t) \le \int_{t_0}^{t}[p(s)\gamma(s) + c(s)\phi^n(s-r)]e_{-p}(s,t)\Delta s. \tag{39}$$

For $t \in [t_0 + r, T)_{\mathbb{T}}$,

$$\begin{aligned} z^{\Delta}(t) &\le b(t)[a(t) + z(t)]^{n/m} \\ &\quad + c(t)[a(t-r) + z(t-r)]^{n/m} \\ &\le b(t)\left(\frac{n}{m}(a(t) + z(t)) + \frac{m-n}{m}\right) \\ &\quad + c(t)\left(\frac{n}{m}(a(t-r) + z(t-r)) + \frac{m-n}{m}\right) \\ &\le \left(\frac{n}{m}b(t) + \frac{n}{m}c(t)\right)z(\sigma(t)) + \frac{n}{m}b(t)a(t) \\ &\quad + \frac{n}{m}c(t)a(t-r) + \frac{m-n}{m}b(t) + \frac{m-n}{m}c(t) \\ &\le (p(t) + q(t))z(\sigma(t)) + p(t)\gamma(t) + q(t)\gamma(t-r). \end{aligned} \tag{40}$$

Hence, we get

$$\begin{aligned} &\left(e_{-(p+q)}(t, t_0 + r)z(t)\right)^{\Delta} \\ &\quad \le (p(t)\gamma(t) + q(t)\gamma(t-r))e_{-(p+q)}(t, t_0 + r). \end{aligned} \tag{41}$$

Integrating both sides from $t_0 + r$ to t, we obtain

$$\begin{aligned} z(t) &\le e_{-(p+q)}(t_0 + r, t)z(t_0 + r) \\ &\quad + e_{-(p+q)}(t_0 + r, t) \\ &\quad \times \int_{t_0+r}^{t} [p(s)\gamma(s) + q(s)\gamma(s-r)]e_{-(p+q)}(s, t_0 + r)\Delta s \\ &\le e_{-(p+q)}(t_0 + r, t) \\ &\quad \times \int_{t_0}^{t_0+r} [p(s)\gamma(s) + c(s)\phi^n(s-r)]e_{-p}(s, t_0 + r)\Delta s \\ &\quad + \int_{t_0+r}^{t} [p(s)\gamma(s) + q(s)\gamma(s-r)]e_{-(p+q)}(s, t)\Delta s. \end{aligned} \tag{42}$$

Using $u^m(t) \le a(t) + z(t)$, we get inequalities (34) and (35).

Finally, if $a(t)$ and $\phi(t)$ are nondecreasing, then, by (35),

$$\begin{aligned} u^m(t) &\le \gamma(t)\left(1 + \int_{t_0}^{t}(p(s) + c(s))e_{-p}(s, t)\Delta s\right) \\ &\le \gamma(t)\left(1 + \int_{t_0}^{t}(p(s) + c(s))e_{-(p+c)}(s, t)\Delta s\right) \\ &\le \gamma(t)e_{-(p+c)}(t_0, t) \end{aligned} \tag{43}$$

for $t \in [t_0, t_0 + r)_{\mathbb{T}}$. Furthermore, by (34),

$$\begin{aligned} u^m(t) &\le \gamma(t) + \gamma(t)e_{-(p+q)}(t_0 + r, t) \\ &\quad \times \int_{t_0}^{t_0+r}(p(s) + c(s))e_{-p}(s, t_0 + r)\Delta s \\ &\quad + \gamma(t)\int_{t_0+r}^{t}(p(s) + q(s))e_{-(p+q)}(s, t)\Delta s \\ &\le \gamma(t)e_{-(p+q)}(t_0 + r, t) \\ &\quad \times \left(1 + \int_{t_0}^{t_0+r}(p(s) + c(s))e_{-(p+c)}(s, t_0 + r)\Delta s\right) \\ &= \gamma(t)e_{-(p+c)}(t_0, t) \end{aligned} \tag{44}$$

for $t \in [t_0 + r, T)_{\mathbb{T}}$. The proof is complete. □

Theorem 6. *Assume that $u(t)$ satisfies condition* (9), $a(t) \ge 0$, $K := 3^{m-1}\Gamma^{m-1}(p\beta - p + 1)(m/pn)^{\beta m-1}$, $p(t) := nKb^m(t)/m$, $c_1(t) := Ke^{-nr}c^m(t)$, $q(t) := (n/m)c_1(t)$, $-p, -(p + c_1) \in \mathscr{R}^+$.

If, in addition, $a(t)$ and $\phi(t)$ are nondecreasing, and $a^m(t_0) = 3^{1-m}e^{(m-n)t_0}e^{nr}\phi^n(t_0)$, then

$$u(t) \le e^t\left[\gamma(t)e_{-(p+c_1)}(t_0, t)\right]^{1/m}, \qquad t \in [t_0, T)_{\mathbb{T}}, \tag{45}$$

where $\gamma(t) = 3^{m-1}a^m(t)e^{-mt_0} + (m-n)/n$.

Proof. For $t \in [t_0, T)_{\mathbb{T}}$, using Hölder's inequality with indices p and m, we obtain from (9) that

$$\begin{aligned} u(t) &\le a(t) + \int_{t_0}^{t}(t-s)^{\beta-1}e^{ns/m}b(s)e^{-ns/m}u^{n/m}(s)\Delta s \\ &\quad + \int_{t_0}^{t}(t-s)^{\beta-1}e^{ns/m}c(s)e^{-ns/m}u^{n/m}(s-r)\Delta s \\ &\le a(t) + \left(\int_{t_0}^{t}(t-s)^{p\beta-p}e^{pns/m}\Delta s\right)^{1/p} \\ &\quad \times \left(\int_{t_0}^{t}b^m(s)e^{-ns}u^n(s)\Delta s\right)^{1/m} \\ &\quad + \left(\int_{t_0}^{t}(t-s)^{p\beta-p}e^{pns/m}\Delta s\right)^{1/p} \\ &\quad \times \left(\int_{t_0}^{t}c^m(s)e^{-ns}u^n(s-r)\Delta s\right)^{1/m} \end{aligned}$$

$$
\begin{aligned}
&\leq a(t) + e^{nt/m}\left(\frac{m}{pn}\right)^{(\beta-1+1/p)}\Gamma^{1/p}(p\beta-p+1)\\
&\quad\times\left[\left(\int_{t_0}^{t} b^m(s)\,e^{-ns}u^n(s)\,\Delta s\right)^{1/m}\right.\\
&\quad\left.+\left(\int_{t_0}^{t} c^m(s)\,e^{-ns}u^n(s-r)\,\Delta s\right)^{1/m}\right].
\end{aligned}
\tag{46}
$$

By Jensen's inequality $(\sum_{i=1}^{n} x_i)^{\sigma} \leq n^{\sigma-1}(\sum_{i=1}^{n} x_i^{\sigma})$, we get

$$
\begin{aligned}
&u^m(t)\\
&\leq 3^{m-1}a^m(t) + 3^{m-1}e^{nt}\left(\frac{m}{pn}\right)^{(m\beta-1)}\Gamma^{m-1}(p\beta-p+1)\\
&\quad\times\left(\int_{t_0}^{t} b^m(s)\,e^{-ns}u^n(s)\,\Delta s + \int_{t_0}^{t} c^m(s)\,e^{-ns}u^n(s-r)\,\Delta s\right).
\end{aligned}
\tag{47}
$$

So,

$$
\begin{aligned}
&\left(u(t)\,e^{-t}\right)^m\\
&\leq 3^{m-1}a^m(t)\,e^{-mt_0}\\
&\quad+3^{m-1}\left(\frac{m}{pn}\right)^{(m\beta-1)}\Gamma^{m-1}(p\beta-p+1)\\
&\quad\times\left(\int_{t_0}^{t} b^m(s)\,e^{-ns}u^n(s)\,\Delta s + \int_{t_0}^{t} c^m(s)\,e^{-ns}u^n(s-r)\,\Delta s\right).
\end{aligned}
\tag{48}
$$

Let $v(t) := e^{-t}u(t)$, $w_2(t) := 3^{m-1}a^m(t)e^{-mt_0}$; we have

$$
\begin{aligned}
v^m(t) &\leq w_2(t) + \int_{t_0}^{t} Kb^m(s)\,v^n(s)\,\Delta s\\
&\quad+\int_{t_0}^{t} Ke^{-nr}c^m(s)\,v^n(s-r)\,\Delta s
\end{aligned}
\tag{49}
$$

for $t \in [t_0, T)_{\mathbb{T}}$. For $t \in [t_0 - r, t_0)_{\mathbb{T}}$, we have $e^{-t}u(t) \leq e^{-t}\phi(t) \leq e^{-t_0}e^{r}\phi(t)$; that is, $v(t) \leq \phi_1(t)$. By Lemma 5, we get

$$
u(t) \leq e^{t}\left[\gamma(t)\,e_{-(p+c_1)}(t_0,t)\right]^{1/m}, \quad t \in [t_0, T)_{\mathbb{T}}. \tag{50}
$$

The proof is complete. □

The following is a simple consequence of Theorem 4.

Corollary 7. *Suppose that $m = n = 2$,*

$$
\begin{aligned}
u(t) &\leq a(t) + \int_{t_0}^{t}(t-s)^{\beta-1}b(s)\,u(s-r)\,\Delta s,\\
&\qquad t \in [t_0, T)_{\mathbb{T}},\\
u(t) &\leq \phi(t), \quad t \in [t_0 - r, t_0)_{\mathbb{T}};
\end{aligned}
\tag{51}
$$

then

$$
\begin{aligned}
u(t) &\leq e^{t}\left[w_1(t) + \int_{t_0+r}^{t} Kb^2(s)\,w_1(s-r)\,e_{-Kb^2}(s,t)\,\Delta s\right.\\
&\quad+e_{-Kb^2}(t_0+r,t)\\
&\quad\left.\times\int_{t_0}^{t_0+r} Kb^2(s)\,\phi_1^2(s-r)\,\Delta s\right]^{1/2},\\
&\qquad t \in [t_0+r, T)_{\mathbb{T}},\\
u(t) &\leq a(t) + \int_{t_0}^{t}(t-s)^{\beta-1}b(s)\,\phi(s-r)\,\Delta s,\\
&\qquad t \in [t_0, t_0+r)_{\mathbb{T}},
\end{aligned}
\tag{52}
$$

where $K := \Gamma(2\beta-1)e^{-2r}\cdot(1/4^{\beta-1})$, $w_1(t) := 2a^2(t)e^{-2t_0}$, $\phi_1(t) := e^{-t_0}e^{r}\phi(t)$.

If $\mathbb{T} = \mathbb{R}$, then the conclusion reduces to that of Theorem A for $\beta > 1/2$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] R. P. Agarwal, S. Deng, and W. Zhang, "Generalization of a retarded Gronwall-like inequality and its applications," *Applied Mathematics and Computation*, vol. 165, no. 3, pp. 599–612, 2005.

[2] B. G. Pachpatte, "Explicit bounds on certain integral inequalities," *Journal of Mathematical Analysis and Applications*, vol. 267, no. 1, pp. 48–61, 2002.

[3] W.-S. Cheung, "Some new nonlinear inequalities and applications to boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 9, pp. 2112–2128, 2006.

[4] C.-J. Chen, W.-S. Cheung, and D. Zhao, "Gronwall-Bellman-type integral inequalities and applications to BVPs," *Journal of Inequalities and Applications*, vol. 2009, Article ID 258569, 15 pages, 2009.

[5] Y. G. Sun, "On retarded integral inequalities and their applications," *Journal of Mathematical Analysis and Applications*, vol. 301, no. 2, pp. 265–275, 2005.

[6] H. Zhang and F. Meng, "On certain integral inequalities in two independent variables for retarded equations," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 608–616, 2008.

[7] H. Ye and J. Gao, "Henry-Gronwall type retarded integral inequalities and their applications to fractional differential

equations with delay," *Applied Mathematics and Computation*, vol. 218, no. 8, pp. 4152–4160, 2011.

[8] O. Lipovan, "A retarded Gronwall-like inequality and its applications," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 1, pp. 389–401, 2000.

[9] O. Lipovan, "A retarded integral inequality and its applications," *Journal of Mathematical Analysis and Applications*, vol. 285, no. 2, pp. 436–443, 2003.

[10] F. Jiang and F. Meng, "Explicit bounds on some new nonlinear integral inequalities with delay," *Journal of Computational and Applied Mathematics*, vol. 205, no. 1, pp. 479–486, 2007.