Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 474160, 14 pages
http://dx.doi.org/10.1155/2014/474160

Research Article

Two-Level Brezzi-Pitkäranta Discretization Method Based on Newton Iteration for Navier-Stokes Equations with Friction Boundary Conditions

Rong An and Xian Wang

College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, China

Correspondence should be addressed to Rong An; anrong702@gmail.com

Received 9 December 2013; Revised 6 February 2014; Accepted 6 February 2014; Published 19 March 2014

Academic Editor: Grzegorz Lukaszewicz

We present a new stabilized finite element method for Navier-Stokes equations with friction slip boundary conditions based on Brezzi-Pitkäranta stabilized method. The stability and error estimates of numerical solutions in some norms are derived for standard one-level method. Combining the techniques of two-level discretization method, we propose two-level Newton iteration method and show the stability and error estimate. Finally, the numerical experiments are given to support the theoretical results and to check the efficiency of this two-level iteration method.

1. Introduction

Consider the steady incompressible flows governed by the following steady incompressible Navier-Stokes equations

$$\begin{gathered} -\mu\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = \mathbf{f}, \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, \quad \text{in } \Omega. \end{gathered} \tag{1}$$

Here $\Omega \subset \mathbb{R}^2$ ia a bounded domain and is assumed to have Lipschitz continuous boundary $\partial\Omega$. $\mathbf{u} = (u_1, u_2)$ denotes the velocity vector of the flows, p denotes the pressure, and $\mathbf{f} = (f_1, f_2)$ denote the body force vector. The constant $\mu > 0$ is the viscous coefficient. The solenoidal condition $\operatorname{div}\mathbf{u} = 0$ indicates that the flows are incompressible.

In this paper, we consider the following friction slip boundary conditions:

$$\begin{gathered} \mathbf{u} = 0, \quad \text{on } \Gamma, \\ \mathbf{u}_n = 0, \quad -\sigma_\tau(\mathbf{u}) \in g\partial\left|\mathbf{u}_\tau\right|, \quad \text{on } S, \end{gathered} \tag{2}$$

where $\Gamma \cap S = \emptyset$, $\overline{\Gamma \cup S} = \partial\Omega$. g is a scalar function. $\mathbf{u}_n$ and $\mathbf{u}_\tau$ are the normal and tangential components of the velocity. $\sigma_\tau(\mathbf{u}) = \sigma - \sigma_n\mathbf{n}$, independent of p, is the tangential components of the stress vector σ which is defined by $\sigma_i = \sigma_i(\mathbf{u}, p) = (\mu e_{ij}(\mathbf{u}) - p\delta_{ij})n_j$ with $e_{ij}(\mathbf{u}) = \partial u_i/\partial x^j + \partial u_j/\partial x^i$, $i, j = 1, 2$. The set $\partial\psi(a)$ defined in next section denotes a subdifferential set of the function ψ at $a \in L^2(S)$.

Navier-Stokes equations (1) with friction boundary conditions (2) are introduced by Fujita [1] to describe some problems in hydrodynamics. Subsequently, some well-posedness problems about the solutions to the steady and nonstationary problems are studied by many scholars, such as Fujita [2–4], Y. Li and K. Li [5, 6], Saito and Fujita [7], Saito [8], and references cited therein.

Compared with Navier-Stokes equations with homogeneous Dirichlet boundary conditions, the variational formulation of the problem (1) and (2) is the variational inequality problem of the second kind. There exist some references about the finite element methods for solving the numerical solution of the problem (1) and (2). For example, using $P_1b - P_1$ element, Ayadi et al. studied the finite element approximation for Stokes problem and the error estimate derived is suboptimal [9]. Kashiwabara obtained the optimal error estimate by defining the different numerical integration of the nondifferential term on the boundary S corresponding to the different finite element pairs [10, 11]. Djoko and Mbehou studied the direct finite element approximation for steady Stokes problem [12] and the fully discretization scheme for nonstationary Stokes problem [13]. Li and An discussed the penalty and stabilized finite element approximation and corresponding two-level methods for the steady Navier-Stokes equations [14–16]. From the computational cost point of view, the $P_1 - P_1$ element is of practical importance in scientific computation with the lower computational cost. Therefore, much attention has been attracted for simulating the incompressible flows. Y. Li and K. Li [17] applied the pressure projection stabilized method to solving the numerical solution of the problem (1) and (2). Subsequently, An and his collaborates studied the corresponding two-level Stokes/Oseen/Newton iteration methods [18, 19], from which we observe that if the coarse mesh H and the fine mesh h are selected appropriately, then two-level iteration methods provide the same convergence rate as the standard one-level method. Moreover, CPU time can be largely saved.

On the other hand, in computational fluid dynamics, it is very important in searching the appropriate mixed finite element approximation to solve the numerical solutions of the problem (1) quickly and efficiently. Generally, the selected finite element spaces are required to satisfy the discrete inf-sup condition, such as the finite element space constructed by Taylor-Hood element ($P_2 - P_1$ pair). However, from the computational cost point of view, the $P_1 - P_1$ pair is of practical importance in scientific computation with the lower computational cost than the $P_2 - P_1$ pair. Therefore, much attention has been attracted by the $P_1 - P_1$ pair for simulating the incompressible flow. But the discrete inf-sup condition does not hold for $P_1 - P_1$ pair. A usual technique is to introduce the stabilized term in the finite element variational equation. There exist many stabilized methods, such as Brezzi-Pitkäranta stabilized method [20], locally stabilized method [21, 22], pressure stabilized method [23], stream upwind Petrov-Galerkin method [24], Douglas-Wang absolutely stabilized method [25], pressure projection stabilized method [26, 27], and references cited therein. Most of these stabilized methods necessarily introduce the stabilized parameters and are conditionally stable.

In this paper, we combine the Brezzi-Pitkäranta stabilized method [20], which is unconditionally stable [28], with two-level discretization technique to approximate the problem (1) and (2) under a uniqueness condition. Two-level discretization method has become a powerful tool in solving nonlinear partial differential equations. The basic idea is to capture "large eddies" by computing the initial approximation on the coarse mesh and then to obtain the fine approximation by solving a linearized problem corresponding to nonlinear partial differential equations on the fine mesh. More details can be referred to the works of Xu [29, 30]. Since the variational formulation of the problem (1) and (2) is of the form of variational inequality, in this paper, we solve nonlinear Navier-Stokes type variational inequality problem on the coarse mesh with mesh size H and solve a linearized Navier-Stokes type variational inequality problem corresponding to Newton iteration method on the fine mesh with mesh size h. Denote $(\mathbf{u}^h, p^h)$ the finite element approximation solution on the fine mesh. If we suppose that the solution $(\mathbf{u}, p)$ to the problem (1) and (2) belongs to $(H^2(\Omega)^2, H^1(\Omega))$, then the error estimate derived is

$$\left\|\mathbf{u} - \mathbf{u}^h\right\|_V + \left\|p - p^h\right\| \le c\left(h^{3/4} + H^{3/2}\right), \tag{3}$$

where $c > 0$ is independent of h and H and the norms $\|\cdot\|_V$ and $\|\cdot\|$ are defined in next section. Thus, we show that if $H = O(h^{1/2})$, then two-level method proposed in this paper provides the same convergence order as the usual one-level method.

This paper is organized as follows. In Section 2, we introduce some function spaces and some theoretical results about the problem (1) and (2). In Section 3, the Brezzi-Pitkäranta stabilized finite element approximation will be applied and the error estimates about the velocity in H^1-norm and the pressure in L^2-norm are derived. In Section 4, the two-level Newton iteration method is proposed and the error estimate (3) is shown. In Section 5, we give the program implementation to solve the subproblems in two-level method based on Uzawa iteration. In final section, the numerical experiments are displayed to support the theoretical results.

2. Navier-Stokes Equations with Friction Boundary Conditions

In what follows, we employ the standard notation $H^l(\Omega)$ and $\|\cdot\|_l$, $l \ge 0$, for the Sobolev spaces of all functions having square integrable derivatives up to order l in Ω and the standard Sobolev norm. In particular for $l = 0$, we write $L^2(\Omega)$ and $\|\cdot\|$ instead of $H^0(\Omega)$ and $\|\cdot\|_0$, respectively. We use the boldface Sobolev spaces $\mathbf{H}^l(\Omega)$ and $\mathbf{L}^2(\Omega)$ to denote the vector Sobolev spaces $H^l(\Omega)^2$ and $L^2(\Omega)^2$, respectively. Throughout this paper, the symbol c always denotes some positive constant which is independent of the mesh parameter h, H and can be a different constant even in the same formulation.

First, we recall the definition of the subdifferential set. Let ψ be a given function which is of convexity and weak semicontinuity from below. The set $\partial\psi(a)$ is a subdifferential of the function ψ at $a \in L^2(S)$ if and only if

$$\partial\psi(a) = \Big\{b \in L^2(S) : \psi(h) - \psi(a) \ge b(h-a), \forall h \in L^2(S)\Big\}. \tag{4}$$

For the mathematical setting, we introduce the following function spaces usually used in this paper:

$$\mathbf{V} = \left\{\mathbf{u} \in \mathbf{H}^1(\Omega), \mathbf{u}|_\Gamma = 0, \mathbf{u}_n|_S = 0\right\}, \qquad \mathbf{V}_0 = \mathbf{H}_0^1(\Omega),$$
$$\mathbf{V}_\sigma = \{\mathbf{u} \in \mathbf{V}, \operatorname{div}\mathbf{u} = 0\},$$
$$M = L_0^2(\Omega) = \left\{q \in L^2(\Omega), \int_\Omega q\,dx = 0\right\}. \tag{5}$$

We define the norm in $\mathbf{V}$ by

$$\|\mathbf{v}\|_V = \left(\int_\Omega |\nabla \mathbf{v}|^2 dx\right)^{1/2}, \quad \forall \mathbf{v} \in \mathbf{V}. \tag{6}$$

Then $\|\cdot\|_V$ is equivalent to the standard Sobolev norm $\|\cdot\|_1$ due to Poincaré inequality.

We also introduce the following continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $\mathbf{V}\times\mathbf{V}$ and $\mathbf{V}\times M$, respectively, by

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \mu\sum_{i=1}^{2}\sum_{j=1}^{2}\int_\Omega e_{ij}(\mathbf{u}) : e_{ij}(\mathbf{v})\,dx, \quad \forall \mathbf{u},\mathbf{v}\in\mathbf{V}, \\ d(\mathbf{v},q) &= \int_\Omega q \operatorname{div}\mathbf{v}\,dx, \quad \forall \mathbf{v}\in\mathbf{V},\ q\in M \end{aligned} \tag{7}$$

and a trilinear form on $\mathbf{V}\times\mathbf{V}\times\mathbf{V}$ by

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &= \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx - \frac{1}{2}\int_\Omega \operatorname{div}\mathbf{u}\mathbf{v}\cdot\mathbf{w}\,dx \\ &= \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\,dx, \\ &\qquad \forall \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}. \end{aligned} \tag{8}$$

It is well known that from Korn's inequality that

$$\kappa_0\mu\|\mathbf{u}\|_V^2 \le a(\mathbf{u},\mathbf{u}) \le \kappa_1\mu\|\mathbf{u}\|_V^2, \quad \forall \mathbf{u}\in\mathbf{V}, \tag{9}$$

where κ_0 and κ_1 both are some positive constants. Moreover, it is easy to check that this trilinear form satisfies the following important properties [31, 32]:

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}), \tag{10}$$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \le N\|\mathbf{u}\|_V\|\mathbf{v}\|_V\|\mathbf{w}\|_V, \tag{11}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}$ and

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| + |b(\mathbf{v},\mathbf{u},\mathbf{w})| + |b(\mathbf{w},\mathbf{u},\mathbf{v})| \le N\|\mathbf{u}\|_V\|\mathbf{v}\|_2\,\|\mathbf{w}\|, \tag{12}$$

for all $\mathbf{u}\in\mathbf{V}, \mathbf{v}\in\mathbf{H}^2(\Omega), \mathbf{w}\in\mathbf{L}^2(\Omega)$, where $N>0$ depends only on Ω.

Given $\mathbf{f}\in\mathbf{L}^2(\Omega)$ and $g\in L^2(S)$ with $g>0$ on S, based on the above notations, the variational formulation of the problem (1) and (2) reads as follows: find $(\mathbf{u},p)\in\mathbf{V}\times M$ such that for all $(\mathbf{v},q)\in\mathbf{V}\times M$

$$\begin{aligned} a(\mathbf{u},\mathbf{v}-\mathbf{u}) + b(\mathbf{u},\mathbf{u},\mathbf{v}-\mathbf{u}) + j(\mathbf{v}_\tau) - j(\mathbf{u}_\tau) - d(\mathbf{v}-\mathbf{u},p) \\ \ge (\mathbf{f},\mathbf{v}-\mathbf{u}), \\ d(\mathbf{u},q) = 0 \end{aligned} \tag{13}$$

with $j(\mathbf{v}_\tau) = \int_S g|\mathbf{v}_\tau|ds$, which is the variational inequality problem of the second kind with Navier-Stokes operator and is called Navier-Stokes type variational inequality problem. Moreover, the variational inequality problem (13) is equivalent to the following: find $\mathbf{u}\in\mathbf{V}_\sigma$ such that for all $\mathbf{v}\in\mathbf{V}_\sigma$,

$$a(\mathbf{u},\mathbf{v}-\mathbf{u}) + b(\mathbf{u},\mathbf{u},\mathbf{v}-\mathbf{u}) + j(\mathbf{v}_\tau) - j(\mathbf{u}_\tau) \ge (\mathbf{f},\mathbf{v}-\mathbf{u}), \tag{14}$$

which is from the inf-sup condition derived by Saito [8].

Now we recall the existence and uniqueness result about the solution to the problem (14) under the uniqueness condition (15), which has been shown by Y. Li and K. Li [17].

Theorem 1. *If the following uniqueness condition holds:*

$$\frac{4\kappa_2 N\left(\|\mathbf{f}\| + \|g\|_{L^2(S)}\right)}{\kappa_0^2\mu^2} < 1, \tag{15}$$

then the variational inequality problem (14) *admits a unique solution* $\mathbf{u}\in\mathbf{V}_\sigma$ *with*

$$\|\mathbf{u}\|_V \le \frac{2\kappa_2}{\kappa_0\mu}\left(\|\mathbf{f}\| + \|g\|_{L^2(S)}\right) < \frac{\kappa_0\mu}{2N}, \tag{16}$$

where $\kappa_2>0$ *satisfies*

$$\left|(\mathbf{f},\mathbf{v}) - j(\mathbf{v}_\tau)\right| \le \kappa_2\left(\|\mathbf{f}\| + \|g\|_{L^2(S)}\right)\|\mathbf{v}\|_V, \quad \forall \mathbf{v}\in\mathbf{V}_\sigma. \tag{17}$$

3. Stabilized Finite Element Approximation

In this section, we assume that Ω is a convex polygon. Let $\mathcal{T}_h$ be a quasiuniform family of triangular partition of Ω. The corresponding ordered triangles are denoted by $K_1, K_2, \ldots, K_n$. Let $h_i = \operatorname{diam}(K_i)$, $i=1,\ldots,n$, and $h=\max\{h_1,h_2,\ldots,h_n\}$. For every $K\in\mathcal{T}_h$, let $P_r(K)$ denote the space of the polynomials on K of degree at most r. The finite element spaces $\mathbf{V}_h$ and M_h are constructed by

$$\begin{aligned} \mathbf{W}_h &= \left\{\mathbf{v}_h\in\mathbf{C}\left(\overline{\Omega}\right), \mathbf{v}_h|_K\in\mathbf{P}_1(K), \forall K\in\mathcal{T}_h\right\}, \\ \mathbf{V}_h &= \mathbf{W}_h\cap\mathbf{V}, \mathbf{V}_{0h} = \mathbf{W}_h\cap\mathbf{V}_0\subset\mathbf{V}_h, \\ M_h &= \left\{q_h\in C\left(\overline{\Omega}\right),\ q_h|_K\in P_1(K),\ \forall K\in\mathcal{T}_h\right\}\cap M. \end{aligned} \tag{18}$$

Then the Brezzi-Pitkäranta stabilized finite element approximation formulation of the problem (13) reads as follows: find $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times M_h$ such that for all $(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times M_h$

$$\begin{aligned} a(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h) + b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h) + j(\mathbf{v}_{h\tau}) \\ - j(\mathbf{u}_{h\tau}) - d(\mathbf{v}_h-\mathbf{u}_h,p_h) \ge (\mathbf{f},\mathbf{v}_h-\mathbf{u}_h), \\ d(\mathbf{u}_h,q_h) + \alpha C_h(p_h,q_h) = 0, \end{aligned} \tag{19}$$

with the stable parameter $\alpha>0$, where the stabilized term $C_h(\cdot,\cdot)$ on $M_h\times M_h$ is defined by

$$C_h(p_h,q_h) = \sum_{i=1}^{n} h_i^2\int_{K_i}\nabla p_h\cdot\nabla q_h dx, \quad \forall p_h,q_h\in M_h. \tag{20}$$

Define a mesh-dependent norm $[\cdot]_h$ on M_h by

$$[q_h]_h = \left[C_h(q_h,q_h)\right]^{1/2}, \quad \forall q_h\in M_h. \tag{21}$$

Then, it holds that $C_h(p_h,q_h)\le[p_h]_h[q_h]_h$ for all $p_h,q_h\in M_h$ and

$$d(\mathbf{v},q_h) \le ch^{-1}\|\mathbf{v}\|\,[q_h]_h, \quad \forall v\in\mathbf{V},\ q_h\in M_h, \tag{22}$$

which has been shown by Latché and Vola [33]. Moreover, $C_h(p,q)$ also is defined for any couple of functions $p, q \in H^1(\Omega)$ and satisfies

$$[q]_h \le ch\|q\|_1, \quad \forall q \in H^1(\Omega). \tag{23}$$

Now, we introduce a generalized bilinear form $\mathscr{B}_h(\cdot,\cdot;\cdot,\cdot)$ on $(\mathbf{V}_h, M_h)\times(\mathbf{V}_h, M_h)$ by

$$\begin{aligned}\mathscr{B}_h(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = a(\mathbf{u}_h, \mathbf{v}_h) - d(\mathbf{v}_h, p_h) + d(\mathbf{u}_h, q_h) \\ + \alpha C_h(p_h, q_h).\end{aligned} \tag{24}$$

Then, in this case, the discrete problem (19) can be rewritten as follows:

$$\begin{aligned}\mathscr{B}_h(\mathbf{u}_h, p_h; \mathbf{v}_h - \mathbf{u}_h, q_h - p_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \\ + j(\mathbf{v}_{h\tau}) - j(\mathbf{u}_{h\tau}) \ge (\mathbf{f}, \mathbf{v}_h - \mathbf{u}_h).\end{aligned} \tag{25}$$

From the classical result for variational inequality problem of the second kind in finite dimension [34], it is easy to show, under the uniqueness condition (15), the problem (25) admits a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ with

$$\|\mathbf{u}_h\|_V \le \frac{2\kappa_2}{\kappa_0\mu}\left(\|\mathbf{f}\| + \|g\|_{L^2(S)}\right) < \frac{\kappa_0\mu}{2N}. \tag{26}$$

To obtain the existence and uniqueness of the solution to the problem (25), we recall the stable result shown in [28]; that is, there exists some positive $\beta > 0$ such that

$$\beta\left(\|\mathbf{w}_h\|_V + \|r_h\|\right) \le \sup_{(\mathbf{v}_h, q_h)\in(\mathbf{V}_h, M_h)} \frac{\mathscr{B}_h(\mathbf{w}_h, r_h; \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_V + \|q_h\|}. \tag{27}$$

Define the following Galerkin projection operators $R_h : \mathbf{V} \to \mathbf{V}_h$ and $Q_h : M \to M_h$ defined by

$$\mathscr{B}_h(R_h\mathbf{w}, Q_h r; \mathbf{w}_h, r_h) = \mathscr{B}(\mathbf{w}, r; \mathbf{w}_h, r_h) \tag{28}$$

for each $(\mathbf{w}, r) \in \mathbf{V}\times M$ and all $(\mathbf{w}_h, r_h) \in \mathbf{V}_h \times M_h$. It is obvious that

$$\mathscr{B}_h(R_h\mathbf{w}, Q_h r; \mathbf{w}_h, r_h) = \mathscr{B}_h(\mathbf{w}, r; \mathbf{w}_h, r_h) - C_h(r, r_h). \tag{29}$$

Moreover, the following approximation properties about the Galerkin projection operators R_h and Q_h have been derived in [28]:

$$\begin{aligned}\|\mathbf{w} - R_h\mathbf{w}\| + h\|\mathbf{w} - R_h\mathbf{w}\|_V + h\|r - Q_h r\| + h[r - Q_h r]_h \\ \le ch^2\left(\|\mathbf{w}\|_2 + \|r\|_1\right)\end{aligned} \tag{30}$$

for any $\mathbf{w} \in \mathbf{H}^2(\Omega)\cap\mathbf{V}$ and $r \in H^1(\Omega)\cap M$. In terms of the trace inequality $\|\mathbf{v}\|_{L^2(S)} \le c\|\mathbf{v}\|^{1/2}\|\mathbf{v}\|_V^{1/2}$, there holds

$$\begin{aligned}\|\mathbf{w} - R_h\mathbf{w}\|_{L^2(S)} &\le c\|\mathbf{w} - R_h\mathbf{w}\|^{1/2}\|\mathbf{w} - R_h\mathbf{w}\|_V^{1/2} \\ &\le ch^{3/2}\|\mathbf{w}\|_2.\end{aligned} \tag{31}$$

Based on the above assumptions and notations, the H^1 and L^2 error estimates for the velocity and pressure in one-level finite element approximation (25) are derived.

Theorem 2. *Under the uniqueness condition* (15)*, suppose that* $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega)\cap\mathbf{V}\times H^1(\Omega)\cap M$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ *are the solutions to the problems* (13) *and* (25)*, respectively; then, one has the following error estimate:*

$$\|\mathbf{u} - \mathbf{u}_h\|_V + \|p - p_h\| + [p - p_h]_h \le ch^{3/4}. \tag{32}$$

Proof. It follows from (24) that

$$\begin{aligned}\kappa_0\mu\|\mathbf{u}_h - R_h\mathbf{u}\|_V^2 &+ \alpha[p_h - Q_h p]_h^2 \\ &= \mathscr{B}_h(\mathbf{u}_h - R_h\mathbf{u}, p_h - Q_h p; \mathbf{u}_h - R_h\mathbf{u}, p_h - Q_h p) \\ &= \mathscr{B}_h(\mathbf{u}_h, p_h; \mathbf{u}_h - R_h\mathbf{u}, p_h - Q_h p) \\ &\quad - \mathscr{B}_h(R_h\mathbf{u}, Q_h p; \mathbf{u}_h - R_h\mathbf{u}, p_h - Q_h p) \\ &= a(\mathbf{u}_h, \mathbf{u}_h - R_h\mathbf{u}) - d(\mathbf{u}_h - R_h\mathbf{u}, p_h) \\ &\quad + d(\mathbf{u}_h, p_h - Q_h p) + \alpha C_h(p_h, p_h - Q_h p) \\ &\quad - \mathscr{B}_h(R_h\mathbf{u}, Q_h p; \mathbf{u}_h - R_h\mathbf{u}, p_h - Q_h p) \\ &\le (f, \mathbf{u}_h - R_h\mathbf{u}) - b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_h - R_h\mathbf{u}) \\ &\quad + j(R_h\mathbf{u}_\tau) - j(\mathbf{u}_{h\tau}) \\ &\quad - \mathscr{B}_h(R_h\mathbf{u}, Q_h p; \mathbf{u}_h - R_h\mathbf{u}, p_h - Q_h p).\end{aligned} \tag{33}$$

Taking $\mathbf{v} = \mathbf{u}_h$ and $\mathbf{v} = 2\mathbf{u} - R_h\mathbf{u}$ in the first inequality of (13) and adding them yielded

$$\begin{aligned}a(\mathbf{u}, \mathbf{u}_h - R_h\mathbf{u}) + b(\mathbf{u}, \mathbf{u}, \mathbf{u}_h - R_h\mathbf{u}) - d(\mathbf{u}_h - R_h\mathbf{u}, p) \\ + j((2\mathbf{u} - R_h\mathbf{u})_\tau) + j(\mathbf{u}_{h\tau}) - 2j(\mathbf{u}_\tau) \ge (f, \mathbf{u}_h - R_h\mathbf{u}).\end{aligned} \tag{34}$$

Substituting the above inequality into (33), we obtain

$$\begin{aligned}\kappa_0\mu\|\mathbf{u}_h - R_h\mathbf{u}\|_V^2 &+ \alpha[p_h - Q_h p]_h^2 \\ &\le \underbrace{a(\mathbf{u} - R_h\mathbf{u}, \mathbf{u}_h - R_h\mathbf{u})}_{I_1} \\ &\quad \underbrace{+b(\mathbf{u}, \mathbf{u}, \mathbf{u}_h - R_h\mathbf{u}) - b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_h - R_h\mathbf{u})}_{I_2} \\ &\quad \underbrace{-d(\mathbf{u}_h - R_h\mathbf{u}, p - Q_h p)}_{I_3} \\ &\quad \underbrace{+j((2\mathbf{u} - R_h\mathbf{u})_\tau) - 2j(\mathbf{u}_\tau) + j(R_h\mathbf{u}_\tau)}_{I_4} \\ &\quad \underbrace{+d(\mathbf{u} - R_h\mathbf{u}, p_h - Q_h p)}_{I_5}\underbrace{-\alpha C_h(Q_h p, p_h - Q_h p)}_{I_6}.\end{aligned} \tag{35}$$

From Hölder inequality and Young inequality, I_1 can be estimated by

$$\begin{aligned} a(\mathbf{u}-R_h\mathbf{u},\mathbf{u}_h-R_h\mathbf{u}) &\le \kappa_1\mu\|\mathbf{u}-R_h\mathbf{u}\|_V\|\mathbf{u}_h-R_h\mathbf{u}\|_V \\ &\le \frac{\kappa_0\mu}{8}\|\mathbf{u}_h-R_h\mathbf{u}\|_V^2 \\ &\quad + \frac{2\kappa_1^2\mu}{\kappa_0}\|\mathbf{u}-R_h\mathbf{u}\|_V^2. \end{aligned} \tag{36}$$

Similarly, I_3 and I_5 satisfy

$$\begin{aligned} d(\mathbf{u}_h-R_h\mathbf{u},p-Q_hp) &\le \|\mathbf{u}_h-R_h\mathbf{u}\|_V\|p-Q_hp\| \\ &\le \frac{\kappa_0\mu}{8}\|\mathbf{u}_h-R_h\mathbf{u}\|_V^2+\frac{2}{\kappa_0\mu}\|p-Q_hp\|^2, \\ d(\mathbf{u}-R_h\mathbf{u},p_h-Q_hp) &\le ch^{-1}\|\mathbf{u}-R_h\mathbf{u}\|[p_h-Q_hp]_h \\ &\le \frac{\alpha}{4}[p_h-Q_hp]_h^2+\frac{c}{\alpha h^2}\|\mathbf{u}-R_h\mathbf{u}\|^2, \end{aligned} \tag{37}$$

where we use the inequality (22). We rewrite I_2 as

$$\begin{aligned} &b(\mathbf{u},\mathbf{u},\mathbf{u}_h-R_h\mathbf{u})-b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{u}_h-R_h\mathbf{u}) \\ &\quad = b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},\mathbf{u}_h-R_h\mathbf{u})+b(\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h,\mathbf{u}_h-R_h\mathbf{u}) \\ &\quad = b(\mathbf{u}-R_h\mathbf{u},\mathbf{u},\mathbf{u}_h-R_h\mathbf{u})+b(R_h\mathbf{u}-\mathbf{u}_h,\mathbf{u},\mathbf{u}_h-R_h\mathbf{u}) \\ &\qquad + b(\mathbf{u}_h,\mathbf{u}-R_h\mathbf{u},\mathbf{u}_h-R_h\mathbf{u}). \end{aligned} \tag{38}$$

Then from (11), (16), and (26), it is estimated by

$$\begin{aligned} &b(\mathbf{u},\mathbf{u},\mathbf{u}_h-R_h\mathbf{u})-b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{u}_h-R_h\mathbf{u}) \\ &\quad \le N(\|\mathbf{u}\|_V+\|\mathbf{u}_h\|_V)\|\mathbf{u}-R_h\mathbf{u}\|_V\|\mathbf{u}_h-R_h\mathbf{u}\|_V \\ &\qquad + N\|\mathbf{u}\|_V\|\mathbf{u}_h-R_h\mathbf{u}\|_V^2 \\ &\quad \le \kappa_0\mu\|\mathbf{u}-R_h\mathbf{u}\|_V\|\mathbf{u}_h-R_h\mathbf{u}\|_V+\frac{\kappa_0\mu}{2}\|\mathbf{u}_h-R_h\mathbf{u}\|_V^2 \\ &\quad \le \left(\frac{\kappa_0\mu}{8}+\frac{\kappa_0\mu}{2}\right)\|\mathbf{u}_h-R_h\mathbf{u}\|_V^2+2\kappa_0\mu\|\mathbf{u}-R_h\mathbf{u}\|_V^2. \end{aligned} \tag{39}$$

It follows from triangular inequality that I_4 satisfies

$$j((2\mathbf{u}-R_h\mathbf{u})_\tau)-2j(\mathbf{u}_\tau)+j(R_h\mathbf{u}_\tau)\le c\|\mathbf{u}-R_h\mathbf{u}\|_{L^2(S)}. \tag{40}$$

Finally, we estimate I_6 by

$$\begin{aligned} &\alpha C_h(Q_hp,p_h-Q_hp) \\ &\quad = \alpha C_h(Q_hp-p,p_h-Q_hp)+\alpha C_h(p,p_h-Q_hp) \\ &\quad \le \alpha([Q_hp-p]_h+[p]_h)[p_h-Q_hp]_h \\ &\quad \le \frac{\alpha}{4}[p_h-Q_hp]_h^2+2\alpha([Q_hp-p]_h^2+[p]_h^2). \end{aligned} \tag{41}$$

Substituting (36)–(41) into (35), we get

$$\begin{aligned} &\|\mathbf{u}_h-R_h\mathbf{u}\|_V+[p_h-Q_hp]_h \\ &\quad \le c\Big(\|\mathbf{u}-R_h\mathbf{u}\|_V+\|p-Q_hp\|+\frac{1}{h}\|\mathbf{u}-R_h\mathbf{u}\| \\ &\qquad + \|\mathbf{u}-R_h\mathbf{u}\|_{L^2(S)}^{1/2}+[p-Q_hp]_h+[p]_h\Big), \end{aligned} \tag{42}$$

which together with (23), (30), (31), and triangular inequality shows

$$\|\mathbf{u}-\mathbf{u}_h\|_V+[p-p_h]_h\le ch^{3/4}. \tag{43}$$

Next, we give the error estimate for the pressure. We rewrite (13) as

$$\begin{aligned} &\mathscr{B}_h(\mathbf{u},p;\mathbf{v}-\mathbf{u},q-p)+b(\mathbf{u},\mathbf{u},\mathbf{v}-\mathbf{u})+j(\mathbf{v}_\tau)-j(\mathbf{u}_\tau) \\ &\quad \ge (\mathbf{f},\mathbf{v}-\mathbf{u})+\alpha C_h(p,q-p). \end{aligned} \tag{44}$$

For all $\mathbf{w}_h\in\mathbf{V}_{0h}$ and $r_h\in M_h$, taking $(\mathbf{v},q)=(\mathbf{u}\pm\mathbf{w}_h,p\pm r_h)$ and $(\mathbf{v}_h,q_h)=(\mathbf{u}_h\pm\mathbf{w}_h,p_h\pm r_h)$ in (44) and (19), respectively, and subtracting them yielded

$$\begin{aligned} &\mathscr{B}_h(\mathbf{u}_h-\mathbf{u},p_h-p;\mathbf{w}_h,r_h) \\ &\quad = b(\mathbf{u},\mathbf{u},\mathbf{w}_h)-b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{w}_h)-\alpha C_h(p,r_h). \end{aligned} \tag{45}$$

Then in terms of the stable result (27), there holds

$$\begin{aligned} &\beta(\|\mathbf{u}_h-R_h\mathbf{u}\|_V+\|p_h-Q_hp\|) \\ &\le \sup_{(\mathbf{w}_h,r_h)\in(\mathbf{V}_h,M_h)}\frac{\mathscr{B}_h(\mathbf{u}_h-R_h\mathbf{u},p_h-Q_hp;\mathbf{w}_h,r_h)}{\|\mathbf{w}_h\|_V+\|r_h\|} \\ &= \sup_{(\mathbf{w}_h,r_h)\in(\mathbf{V}_h,M_h)}\Big((\mathscr{B}_h(\mathbf{u}_h-\mathbf{u},p_h-p;\mathbf{w}_h,r_h) \\ &\qquad +\mathscr{B}_h(\mathbf{u}-R_h\mathbf{u},p-Q_hp;\mathbf{w}_h,r_h)) \\ &\qquad \times(\|\mathbf{w}_h\|_V+\|r_h\|)^{-1}\Big) \\ &= \sup_{(\mathbf{w}_h,r_h)\in(\mathbf{V}_h,M_h)}\Big((b(\mathbf{u},\mathbf{u},\mathbf{w}_h)-b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{w}_h)-\alpha C_h(p,r_h) \\ &\qquad +\mathscr{B}_h(\mathbf{u}-R_h\mathbf{u},p-Q_hp;\mathbf{w}_h,r_h)) \\ &\qquad \times(\|\mathbf{w}_h\|_V+\|r_h\|)^{-1}\Big). \end{aligned} \tag{46}$$

It follows from (11), (16), (23), (26), (30), (43), and inverse inequality that

$$\begin{aligned}
& b(\mathbf{u},\mathbf{u},\mathbf{w}_h) - b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{w}_h) - \alpha C_h(p,r_h) \\
&\quad \leq N\left(\|\mathbf{u}\|_V + \|\mathbf{u}_h\|_V\right)\|\mathbf{u}-\mathbf{u}_h\|_V\|\mathbf{w}_h\|_V + \alpha[p]_h[r_h]_h \\
&\quad \leq \kappa_0\mu\|\mathbf{u}-\mathbf{u}_h\|_V\|\mathbf{w}_h\|_V + ch\|p\|_1\|r_h\| \\
&\quad \leq ch^{3/4}\left(\|\mathbf{w}_h\|_V + \|r_h\|\right), \\
& \mathscr{B}_h\left(\mathbf{u}-R_h\mathbf{u}, p-Q_hp; \mathbf{w}_h, r_h\right) \\
&\quad = a\left(\mathbf{u}-R_h\mathbf{u},\mathbf{w}_h\right) - d\left(w_h, p-Q_hp\right) \\
&\qquad + d\left(\mathbf{u}-R_h\mathbf{u}, r_h\right) + \alpha C_h\left(p-Q_hp, r_h\right) \\
&\quad \leq \kappa_1\mu\|\mathbf{u}-R_h\mathbf{u}\|_V\|\mathbf{w}_h\|_V + \|\mathbf{w}_h\|_V\|p-Q_hp\| \\
&\qquad + \|\mathbf{u}-R_h\mathbf{u}\|_V\|r_h\| + \alpha[r_h][p-Q_hp] \\
&\quad \leq ch\left(\|\mathbf{w}_h\|_V + \|r_h\|\right).
\end{aligned} \tag{47}$$

Thus, we get from (46) and triangular inequality that

$$\|p-p_h\| \leq ch^{3/4}. \tag{48}$$

□

4. Two-Level Newton Iteration Method

In this section, based on Brezzi-Pitkäranta stabilized finite element approximation, the two-level Newton iteration methods for (13) are proposed. From now on, H and h with $h < H < 1$ are two real positive parameter. The coarse mesh triangulation $\mathscr{T}_H$ is made as like in Section 3. And a fine mesh triangulation $\mathscr{T}_h$ is generated by a mesh refinement process to $\mathscr{T}_H$. The finite element space pairs $(\mathbf{V}_h, M_h)$ and $(\mathbf{V}_H, M_H) \subset (\mathbf{V}_h, M_h)$ corresponding to the triangulations $\mathscr{T}_h$ and $\mathscr{T}_H$, respectively, are constructed as in Section 3. With the above notations, we propose the following two-level Newton iteration scheme.

Step 1. We solve the problem (25) on the coarse mesh; that is, find $(\mathbf{u}_H, p_H) \in \mathbf{V}_H \times M_H$ such that for all $(\mathbf{v}_H, q_H) \in \mathbf{V}_H \times M_H$

$$\begin{aligned}
& \mathscr{B}_h\left(\mathbf{u}_H, p_H; \mathbf{v}_H - \mathbf{u}_H, q_H - p_H\right) + b\left(\mathbf{u}_H,\mathbf{u}_H,\mathbf{v}_H-\mathbf{u}_H\right) \\
&\qquad + j\left(\mathbf{v}_{H\tau}\right) - j\left(\mathbf{u}_{H\tau}\right) \geq \left(\mathbf{f}, \mathbf{v}_H - \mathbf{u}_H\right).
\end{aligned} \tag{49}$$

Step 2. We solve a linearized Navier-Stokes type variational inequality problem according to Newton iteration on the fine mesh; that is, find $(\mathbf{u}^h, p^h) \in \mathbf{V}_h \times M_h$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$

$$\begin{aligned}
& \mathscr{B}_h\left(\mathbf{u}^h, p^h; \mathbf{v}_h - \mathbf{u}^h, q_h - p^h\right) + b\left(\mathbf{u}^h, \mathbf{u}_H, \mathbf{v}_h - \mathbf{u}^h\right) \\
&\qquad + b\left(\mathbf{u}_H, \mathbf{u}^h, \mathbf{v}_h - \mathbf{u}^h\right) + j\left(\mathbf{v}_{h\tau}\right) - j\left(\mathbf{u}^h_\tau\right) \\
&\quad \geq \left(\mathbf{f}, \mathbf{v}_h - \mathbf{u}^h\right) + b\left(\mathbf{u}_H, \mathbf{u}_H, \mathbf{v}_h - \mathbf{u}^h\right).
\end{aligned} \tag{50}$$

In this section, we will assume that the following stable condition holds:

$$\frac{8\kappa_2 N\left(\|\mathbf{f}\| + \|g\|_{L^2(S)}\right)}{\kappa_0^2\mu^2} < 1. \tag{51}$$

In this case, the problem (49) exists a unique solution $(\mathbf{u}_H, p_H) \in \mathbf{V}_H \times M_H$ with

$$\|\mathbf{u}_H\|_V \leq \frac{2\kappa_2}{\kappa_0\mu}\left(\|\mathbf{f}\| + \|g\|_{L^2(S)}\right) < \frac{\kappa_0\mu}{4N}. \tag{52}$$

Taking $(\mathbf{v}_h, q_h) = (2\mathbf{u}^h, 2p^h)$ and $(\mathbf{v}_h, q_h) = (0, 0)$ in (50), respectively, it yields

$$\begin{aligned}
& \mathscr{B}_h\left(\mathbf{u}^h, p^h; \mathbf{u}^h, p^h\right) + b\left(\mathbf{u}^h, \mathbf{u}_H, \mathbf{u}^h\right) \\
&\quad = \left(\mathbf{f}, \mathbf{u}^h\right) - j\left(\mathbf{u}^h_\tau\right) + b\left(\mathbf{u}_H, \mathbf{u}_H, \mathbf{u}^h\right).
\end{aligned} \tag{53}$$

Then from (11) and (52), the left-hand side of (53) satisfies

$$\mathscr{B}_h\left(\mathbf{u}^h, p^h; \mathbf{u}^h, p^h\right) + b\left(\mathbf{u}^h, \mathbf{u}_H, \mathbf{u}^h\right) \geq \frac{3\kappa_0\mu}{4}\left\|\mathbf{u}^h\right\|_V^2 + \alpha\left[p^h\right]_h^2, \tag{54}$$

which implies that the problem (50) admits a unique solution $(\mathbf{u}^h, p^h) \in \mathbf{V}_h \times M_h$. Moreover, it is easy to check that $\mathbf{u}^h$ satisfies

$$\begin{aligned}
\left\|\mathbf{u}^h\right\|_V &\leq \frac{4}{3\kappa_0\mu}\left[\kappa_2\left(\|\mathbf{f}\| + \|g\|_{L^2(S)}\right) + N\|\mathbf{u}_H\|_V^2\right] \\
&\leq \frac{4\kappa_2}{3\kappa_0\mu}\left(\|\mathbf{f}\| + \|g\|_{L^2(S)}\right) + \frac{1}{3}\|\mathbf{u}_H\|_V \leq \frac{\kappa_0\mu}{4N}.
\end{aligned} \tag{55}$$

On the other hand, according to Theorem 2, the finite element approximation solution $(\mathbf{u}_H, p_H) \in \mathbf{V}_H \times M_H$ on the coarse mesh satisfies the following error estimate:

$$\|\mathbf{u}-\mathbf{u}_H\|_V + \|p-p_H\| + [p-p_H]_H \leq cH^{3/4}. \tag{56}$$

The error estimate for the two-level Newton iteration scheme is derived in the following theorem.

Theorem 3. *Under the stable condition* (51)*, suppose that* $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \cap \mathbf{V} \times H^1(\Omega) \cap M$ *and* $(\mathbf{u}^h, p^h) \in \mathbf{V}_h \times M_h$ *are the solutions to the problems* (13) *and* (50)*, respectively; then, for sufficiently small H, one has the following error estimate:*

$$\left\|\mathbf{u}-\mathbf{u}^h\right\|_V + \left\|p-p^h\right\| \leq c\left(h^{3/4} + H^{3/2}\right). \tag{57}$$

Proof. Proceeding as in the proof of Theorem 2, we have

$$\begin{aligned}
&\kappa_0\mu\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V^2 + \alpha\left[p^h - Q_h p\right]_h^2 \\
&\quad\le \underbrace{a\left(\mathbf{u} - R_h\mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right)}_{K_1} \\
&\qquad \underbrace{+ j\left((2\mathbf{u} - R_h\mathbf{u})_\tau\right) - 2j(\mathbf{u}_\tau) + j(R_h\mathbf{u}_\tau)}_{K_2} \\
&\qquad \underbrace{+ d\left(\mathbf{u} - R_h\mathbf{u}, p^h - Q_h p\right) - d\left(\mathbf{u}^h - R_h\mathbf{u}, p - Q_h p\right)}_{K_3} \\
&\qquad \underbrace{- \alpha C_h\left(Q_h p, p^h - Q_h p\right)}_{K_4} \\
&\qquad \underbrace{+ b(\mathbf{u}, \mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}) - b(\mathbf{u}^h, \mathbf{u}_H, \mathbf{u}^h - R_h\mathbf{u}) - b(\mathbf{u}_H, \mathbf{u}^h, \mathbf{u}^h - R_h\mathbf{u}) + b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{u}^h - R_h\mathbf{u})}_{K_5}.
\end{aligned} \tag{58}$$

Moreover, the terms $K_1 - K_4$ can be estimated, respectively, by

$$\begin{aligned}
&a\left(\mathbf{u} - R_h\mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\qquad \le \frac{\kappa_0\mu}{8}\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V^2 + \frac{2\kappa_1^2\mu}{\kappa_0}\left\|\mathbf{u} - R_h\mathbf{u}\right\|_V^2, \\
&j\left((2\mathbf{u} - R_h\mathbf{u})_\tau\right) - 2j\left(\mathbf{u}_\tau\right) + j\left(R_h\mathbf{u}_\tau\right) \\
&\qquad \le c\left\|\mathbf{u} - R_h\mathbf{u}\right\|_{L^2(S)}, \\
&d\left(\mathbf{u} - R_h\mathbf{u}, p^h - Q_h p\right) - d\left(\mathbf{u}^h - R_h\mathbf{u}, p - Q_h p\right) \\
&\qquad \le \frac{\kappa_0\mu}{8}\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V^2 + \frac{2}{\kappa_0\mu}\left\|p - Q_h p\right\|^2 \\
&\qquad\quad + \frac{\alpha}{4}\left[p^h - Q_h p\right]_h^2 + \frac{c}{\alpha h^2}\left\|\mathbf{u} - R_h\mathbf{u}\right\|^2, \\
&\alpha C_h\left(Q_h p, p^h - Q_h p\right) \\
&\qquad = \alpha C_h\left(Q_h p - p, p^h - Q_h p\right) + \alpha C_h\left(p, p^h - Q_h p\right) \\
&\qquad \le \frac{\alpha}{4}\left[p^h - Q_h p\right]_h^2 + 2\alpha\left(\left[Q_h p - p\right]_h^2 + \left[p\right]_h^2\right).
\end{aligned} \tag{59}$$

Since K_5 can be rewritten as

$$\begin{aligned}
&b\left(\mathbf{u}, \mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right) - b\left(\mathbf{u}_H, \mathbf{u}^h, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\qquad - b\left(\mathbf{u}^h, \mathbf{u}_H, \mathbf{u}^h - R_h\mathbf{u}\right) + b\left(\mathbf{u}_H, \mathbf{u}_H, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\quad = b\left(\mathbf{u} - \mathbf{u}^h, \mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right) + b\left(\mathbf{u}^h, \mathbf{u} - \mathbf{u}^h, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\qquad + b\left(\mathbf{u}^h - \mathbf{u}_H, \mathbf{u}^h - \mathbf{u}_H, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\quad = b\left(\mathbf{u} - R_h\mathbf{u}, \mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right) + b\left(R_h\mathbf{u} - \mathbf{u}^h, \mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\qquad + b\left(\mathbf{u}^h, \mathbf{u} - R_h\mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\qquad + b\left(\mathbf{u}^h - \mathbf{u}_H, R_h\mathbf{u} - \mathbf{u}_H, \mathbf{u}^h - R_h\mathbf{u}\right),
\end{aligned} \tag{60}$$

then from (11), (16), (55), and (56), all terms in the right-hand side of (60) are estimated, respectively, by

$$\begin{aligned}
&b\left(\mathbf{u} - R_h\mathbf{u}, \mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\qquad \le N\|\mathbf{u}\|_V\left\|\mathbf{u} - R_h\mathbf{u}\right\|_V\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V \\
&\qquad \le \frac{\kappa_0\mu}{4}\left\|\mathbf{u} - R_h\mathbf{u}\right\|_V\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V \\
&\qquad \le \frac{\kappa_0\mu}{8}\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V^2 + \frac{\kappa_0\mu}{8}\left\|\mathbf{u} - R_h\mathbf{u}\right\|_V^2,
\end{aligned} \tag{61}$$

where we use $\|\mathbf{u}\|_V \le \kappa_0\mu/4N$ under the condition (51) and

$$\begin{aligned}
&b\left(R_h\mathbf{u} - \mathbf{u}^h, \mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\qquad = N\|\mathbf{u}\|_V\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V^2 \le \frac{\kappa_0\mu}{4}\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V^2, \\
&b\left(\mathbf{u}^h, \mathbf{u} - R_h\mathbf{u}, \mathbf{u}^h - R_h\mathbf{u}\right) \\
&\qquad \le N\left\|\mathbf{u}^h\right\|_V\left\|\mathbf{u} - R_h\mathbf{u}\right\|_V\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V \\
&\qquad \le \frac{\kappa_0\mu}{4}\left\|\mathbf{u} - R_h\mathbf{u}\right\|_V\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V \\
&\qquad \le \frac{\kappa_0\mu}{8}\left\|\mathbf{u}^h - R_h\mathbf{u}\right\|_V^2 + \frac{\kappa_0\mu}{8}\left\|\mathbf{u} - R_h\mathbf{u}\right\|_V^2,
\end{aligned}$$

$$\begin{aligned} &b\left(\mathbf{u}^h-\mathbf{u}_H, R_h\mathbf{u}-\mathbf{u}_H, \mathbf{u}^h-R_h\mathbf{u}\right)\\ &\quad= b\left(\mathbf{u}^h-R_h\mathbf{u}, R_h\mathbf{u}-\mathbf{u}_H, \mathbf{u}^h-R_h\mathbf{u}\right)\\ &\qquad+ b\left(R_h\mathbf{u}-\mathbf{u}_H, R_h\mathbf{u}-\mathbf{u}_H, \mathbf{u}^h-R_h\mathbf{u}\right)\\ &\quad\le N\left\|R_h\mathbf{u}-\mathbf{u}_H\right\|_V\left\|\mathbf{u}^h-R_h\mathbf{u}\right\|_V^2\\ &\qquad+ N\left\|R_h\mathbf{u}-\mathbf{u}_H\right\|_V^2\left\|\mathbf{u}^h-R_h\mathbf{u}\right\|_V\\ &\quad\le cH^{3/4}\left\|\mathbf{u}^h-R_h\mathbf{u}\right\|_V^2+\frac{\kappa_0\mu}{8}\left\|\mathbf{u}^h-R_h\mathbf{u}\right\|_V^2\\ &\qquad+ c\left\|R_h\mathbf{u}-\mathbf{u}_H\right\|_V^4. \end{aligned} \tag{62}$$

Combining (59)–(62) with (58), for sufficiently small H such that $cH^{3/4} < \kappa_0\mu/8$, we get

$$\begin{aligned} &\left\|\mathbf{u}^h-R_h\mathbf{u}\right\|_V\\ &\quad\le c\bigg(\left\|\mathbf{u}-R_h\mathbf{u}\right\|_V+\left\|p-Q_hp\right\|+\frac{1}{h}\left\|\mathbf{u}-R_h\mathbf{u}\right\|\\ &\qquad+\left\|\mathbf{u}-R_h\mathbf{u}\right\|_{L^2(S)}^{1/2}+\left[p-Q_hp\right]_h\\ &\qquad+\left[p\right]_h+\left\|R_h\mathbf{u}-\mathbf{u}_H\right\|_V^2\bigg). \end{aligned} \tag{63}$$

Thus, we obtain from triangular inequality and (30), (31), and (56) that

$$\left\|\mathbf{u}-\mathbf{u}^h\right\|_V \le c\left(h^{3/4}+H^{3/2}\right). \tag{64}$$

Next, we show the error estimate for the pressure. For all $\mathbf{w}_h \in \mathbf{V}_{0h}$ and $r_h \in M_h$, taking $(\mathbf{v}, q) = (\mathbf{u}\pm\mathbf{w}_h, p\pm r_h)$ and $(\mathbf{v}_h, q_h) = (\mathbf{u}^h\pm\mathbf{w}_h, p^h\pm r_h)$ in (44) and (50), respectively, and subtracting them yielded

$$\begin{aligned} &\mathscr{B}_h\left(\mathbf{u}^h-\mathbf{u}, p^h-p; \mathbf{w}_h, r_h\right)\\ &\quad= b\left(\mathbf{u},\mathbf{u},\mathbf{w}_h\right)-b\left(\mathbf{u}^h,\mathbf{u}_H,\mathbf{w}_h\right)-b\left(\mathbf{u}_H,\mathbf{u}^h,\mathbf{w}_h\right)\\ &\qquad+b\left(\mathbf{u}_H,\mathbf{u}_H,\mathbf{w}_h\right)+\alpha C_h\left(p,r_h\right). \end{aligned} \tag{65}$$

Since there holds

$$\begin{aligned} &b\left(\mathbf{u},\mathbf{u},\mathbf{w}_h\right)-b\left(\mathbf{u}^h,\mathbf{u}_H,\mathbf{w}_h\right)-b\left(\mathbf{u}_H,\mathbf{u}^h,\mathbf{w}_h\right)\\ &\qquad+b\left(\mathbf{u}_H,\mathbf{u}_H,\mathbf{w}_h\right)+\alpha C_h\left(p,r_h\right)\\ &\quad= b\left(\mathbf{u}-\mathbf{u}^h,\mathbf{u},\mathbf{w}_h\right)+b\left(\mathbf{u},\mathbf{u}-R_h\mathbf{u},\mathbf{w}_h\right)\\ &\qquad-b\left(\mathbf{u}-\mathbf{u}^h,\mathbf{u}-R_h\mathbf{u},\mathbf{w}_h\right)\\ &\qquad+b\left(\mathbf{u}^h-\mathbf{u}_H,R_h\mathbf{u}-\mathbf{u}_H,\mathbf{w}_h\right)\\ &\qquad-b\left(\mathbf{u}_H,\mathbf{u}^h-R_h\mathbf{u},\mathbf{w}_h\right)+\alpha C_h\left(p,r_h\right)\\ &\quad\le\left(N\|\mathbf{u}\|_V\left\|\mathbf{u}-\mathbf{u}^h\right\|_V+N\|\mathbf{u}\|_V\left\|\mathbf{u}-R_h\mathbf{u}\right\|_V\right.\\ &\qquad\left.+\left\|\mathbf{u}-R_h\mathbf{u}\right\|_V\left\|\mathbf{u}-\mathbf{u}^h\right\|_V\right)\left\|\mathbf{w}_h\right\|_V\\ &\qquad+\left(\left\|\mathbf{u}-\mathbf{u}^h\right\|_V+\left\|\mathbf{u}-\mathbf{u}_H\right\|_V\right)\\ &\qquad\times\left(\left\|\mathbf{u}-R_h\mathbf{u}\right\|_V+\left\|\mathbf{u}-\mathbf{u}_H\right\|_V\right)\left\|\mathbf{v}_h\right\|_V\\ &\qquad+N\left\|\mathbf{u}_H\right\|_V\left(\left\|\mathbf{u}-\mathbf{u}^h\right\|_V+\left\|\mathbf{u}-R_h\mathbf{u}\right\|_V\right)\left\|\mathbf{w}_h\right\|_V\\ &\qquad+ch\|p\|_1\left\|r_h\right\|, \end{aligned} \tag{66}$$

then it is easy to show that

$$\begin{aligned} \left\|p^h-Q_hp\right\| &\le c\left(\left\|\mathbf{u}-\mathbf{u}^h\right\|_V+\left\|\mathbf{u}-R_h\mathbf{u}\right\|_V+\left\|p-Q_hp\right\|\right)\\ &\quad+c\left\|\mathbf{u}-R_h\mathbf{u}\right\|_V\left\|\mathbf{u}-\mathbf{u}^h\right\|_V\\ &\quad+c\left(\left\|\mathbf{u}-\mathbf{u}^h\right\|_V+\left\|\mathbf{u}-\mathbf{u}_H\right\|_V\right)\\ &\quad\times\left(\left\|\mathbf{u}-R_h\mathbf{u}\right\|_V+\left\|\mathbf{u}-\mathbf{u}_H\right\|_V\right)\\ &\quad+c\left(\left[p-Q_hp\right]_h+h\|p\|_1\right), \end{aligned} \tag{67}$$

where we use (27) and (65). Therefore, the estimates (30), (56), and (64) imply that

$$\left\|p-p^h\right\| \le c\left(h^{3/4}+H^{3/2}\right). \tag{68}$$

□

5. Program Implementation

Since the subproblems (49) and (50) in two-level Newton iteration method both are nonlinear variational inequality problems, then the appropriate numerical iteration schemes are required. Here, we use Uzawa iteration method discussed by Y. Li and K. Li in [35], which is based on the following equivalence relationship. It is easy to show that Navier-Stokes type variational inequality problem (13) is equivalent to the following variational equation:

$$\begin{gathered} a(\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{u},\mathbf{v})-d(\mathbf{v},p)+\int_S g\lambda\mathbf{v}_\tau ds=(\mathbf{f},\mathbf{v}),\\ \forall\mathbf{v}\in\mathbf{V},\\ d(\mathbf{u},q)=0,\quad \forall q\in M,\\ \lambda\mathbf{u}_\tau=\left|\mathbf{u}_\tau\right|,\quad \text{on } S, \end{gathered} \tag{69}$$

where

$$\lambda\in\Lambda=\left\{\gamma\in L^2(S):|\gamma(x)|\le 1 \text{ on } S\right\}. \tag{70}$$

Then we use the following Uzawa iteration scheme to solve two-level Newton iteration scheme (49) and (50).

Step 1. Denote $\Lambda_H = \{v_H|_S : v_H \in W_H\} \cap \Lambda$.

$$\lambda_H^0 \in \Lambda_H \text{ is given,} \tag{71}$$

and then we solve $(\mathbf{u}_H^m, p_H^m)$ and λ_H^m with $m \in \mathbb{N}^+$ on the coarse mesh by

$$\begin{aligned} &\mathscr{B}_H\left(\mathbf{u}_H^m, p_H^m; \mathbf{v}_H, q_H\right) + b\left(\mathbf{u}_H^m, \mathbf{u}_H^{m-1}, \mathbf{v}_H\right) + b\left(\mathbf{u}_H^{m-1}, \mathbf{u}_H^m, \mathbf{v}_H\right) \\ &\quad = (\mathbf{f}, \mathbf{v}_H) + b\left(\mathbf{u}_H^{m-1}, \mathbf{u}_H^{m-1}, \mathbf{v}_H\right) \\ &\qquad - \int_S g\lambda_H^{m-1}\mathbf{v}_{H\tau} ds, \quad \forall (v_H, q_H) \in \mathbf{V}_H \times M_H, \\ &\qquad \lambda_H^m = P_\Lambda\left(\lambda_H^{m-1} + \rho g \mathbf{u}_{H\tau}^m\right), \quad \rho > 0, \end{aligned} \tag{72}$$

where

$$P_{\Lambda_H}(\gamma) = \sup(-1, \inf(1, \gamma)), \quad \forall \gamma \in L^2(S). \tag{73}$$

The condition of iteration stop is $\|\mathbf{u}_H^m - \mathbf{u}_H^{m-1}\| < 10^{-6}$.

Step 2. Denote $\Lambda_h = \{v_h|_S : v_h \in W_h\} \cap \Lambda$.

$$\lambda_0^h \in \Lambda_h \text{ is given,} \tag{74}$$

and we solve $(\mathbf{u}_n^h, p_n^h)$ and λ_n^h with $n \in \mathbb{N}^+$ on the fine mesh by

$$\begin{aligned} &\mathscr{B}_h\left(\mathbf{u}_n^h, p_n^h; \mathbf{v}_h, q_h\right) + b\left(\mathbf{u}_H^m, \mathbf{u}_n^h, \mathbf{v}_h\right) + b\left(\mathbf{u}_n^h, \mathbf{u}_H^m, \mathbf{v}_h\right) \\ &\quad = (\mathbf{f}, \mathbf{v}_h) + b\left(\mathbf{u}_H^m, \mathbf{u}_H^m, \mathbf{v}_h\right) \\ &\qquad - \int_S g\lambda_{n-1}^h \mathbf{v}_{h\tau} ds, \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h, \\ &\qquad \lambda_n^h = P_{\Lambda_h}\left(\lambda_{n-1}^h + \rho g \mathbf{u}_{n,\tau}^h\right). \end{aligned} \tag{75}$$

The condition of iteration stop is $\|\mathbf{u}_n^h - \mathbf{u}_{n-1}^h\| < 10^{-6}$.

6. Numerical Results

In this section, we give the numerical experiments to support the theoretical results derived in Sections 3 and 4. The testing example is quoted from [19]; namely, the exact solution is chosen as

$$\begin{aligned} \mathbf{u}(x, y) &= (u_1(x, y), u_2(x, y)), \\ p(x, y) &= (2x - 1)(2y - 1), \\ u_1(x, y) &= -x^2 y(x - 1)(3y - 2), \\ u_2(x, y) &= xy^2(y - 1)(3x - 2) \end{aligned} \tag{76}$$

in the unit square $\Omega = (0, 1) \times (0, 1)$ (see Figure 1). The body force $\mathbf{f}$ is determined by the first equation in (1).

It is easy to verify that the exact solution $\mathbf{u}$ satisfies $\mathbf{u} = 0$ on Γ and $\mathbf{u}_n = 0$ on $S = S_1 \cup S_2$. The tangential vector τ on S_1 and S_2 had been $(0, 1)$ and $(-1, 0)$. Thus, we select

$$\begin{aligned} \sigma_\tau &= 4\mu y^2(y - 1), \quad \text{on } S_1, \\ \sigma_\tau &= 4\mu x^2(x - 1), \quad \text{on } S_2. \end{aligned} \tag{77}$$

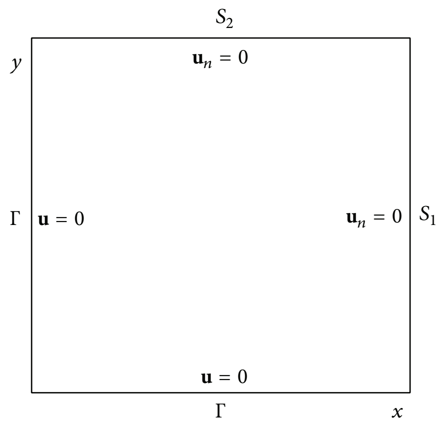

Figure 1: Unit square.

On the other hand, from the friction slip boundary conditions (2), there holds

$$|\sigma_\tau| \le g, \tag{78}$$

and then the function g can be chosen as $g = -\sigma_\tau \ge 0$ on S_1 and S_2.

In all numerical experiments, the viscous coefficient and the stable parameter are chosen as $\mu = 0.01$ and $\alpha = 0.01$. The parameter ρ in Uzawa iteration scheme is chosen as $\rho = 0.5\,\mu$. According to Theorem 3, we choose $H = h^{1/2}$; then, two-level finite element approximation solution is of the following error estimate:

$$\left\|\mathbf{u} - \mathbf{u}^h\right\|_V + \left\|p - p^h\right\| \le ch^{3/4}. \tag{79}$$

Here we select eight fine mesh values $h = 1/4^2, 1/6^2, \ldots, 1/18^2$. Then the corresponding coarse mesh values are obtained. These fine mesh values also are used in the numerical experiment for one-level finite element approximation.

The numerical results are displayed in Tables 1 and 2, from which we observe the following conclusions. Based on Table 1, the numerical convergence orders reach the theoretical convergence orders derived in Theorem 2, namely, $O(h^{3/4})$ for the velocity in H^1-norm and the pressure in L^2-norm. We also observe that if $h = 1/18^2$, in this case, the standard one-level method cannot work and does not obtain the predicted numerical results. From Table 2, we can see that if $H = h^{1/2}$, two-level Newton iteration scheme can reach the theoretical convergence orders of $O(h^{3/4})$ for both velocity and pressure, in H^1-norm and L^2-norm, respectively. Besides, we find that the current method also achieves the predicted convergence order of $O(h^{7/4})$ for velocity in the sense of L^2-norm. From the view of computational cost, we can obviously observe by comparing Tables 1 and 2 that two-level Newton iteration method significantly saves CPU time than one-level method and, meanwhile, obtains nearly the same approximation results.

Finally, we show the contour plots of the exact solution and the numerical solution to exhibit the approximation

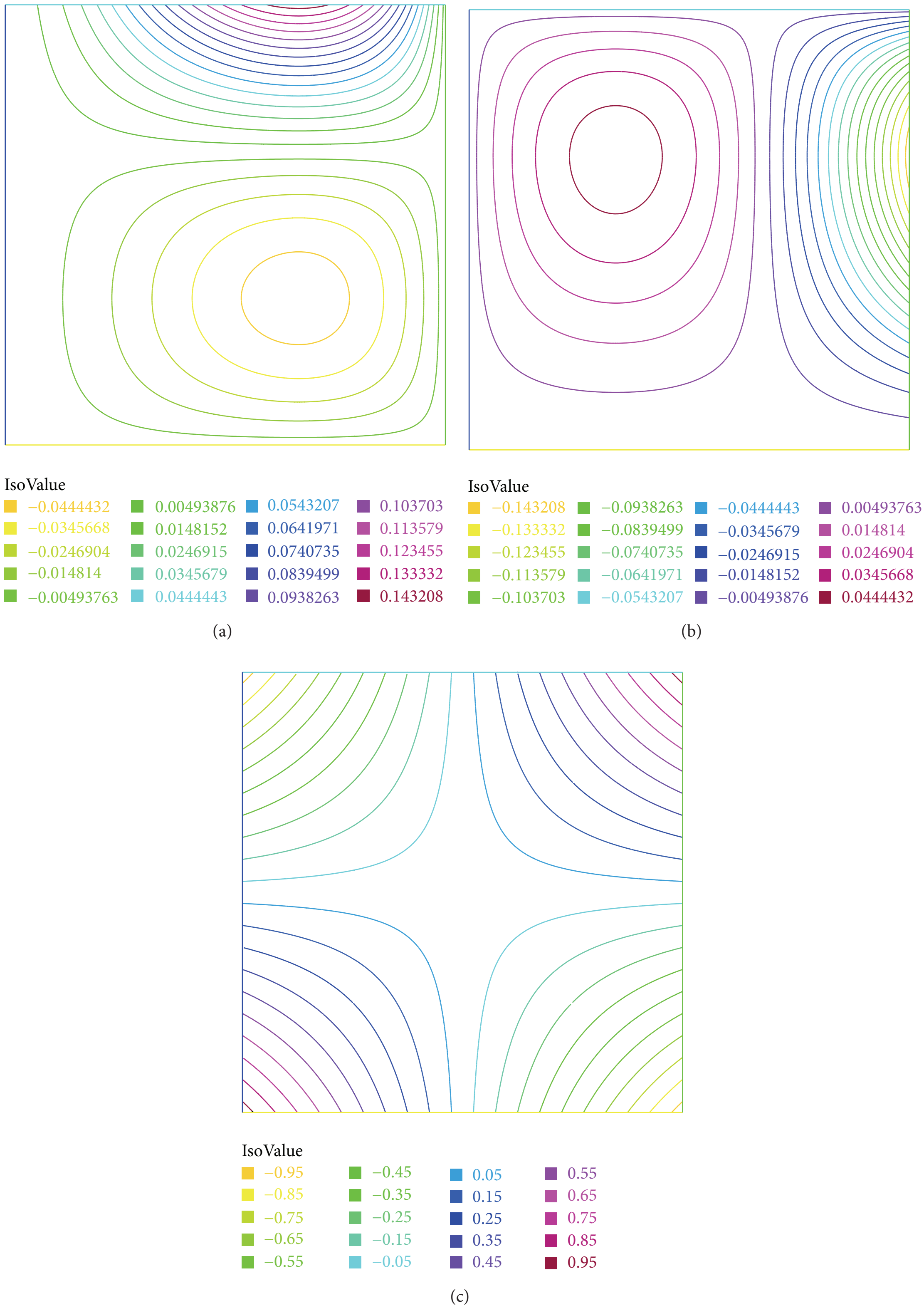

FIGURE 2: Contour plots of exact solution. From (a) to (c): two components of velocity and pressure.

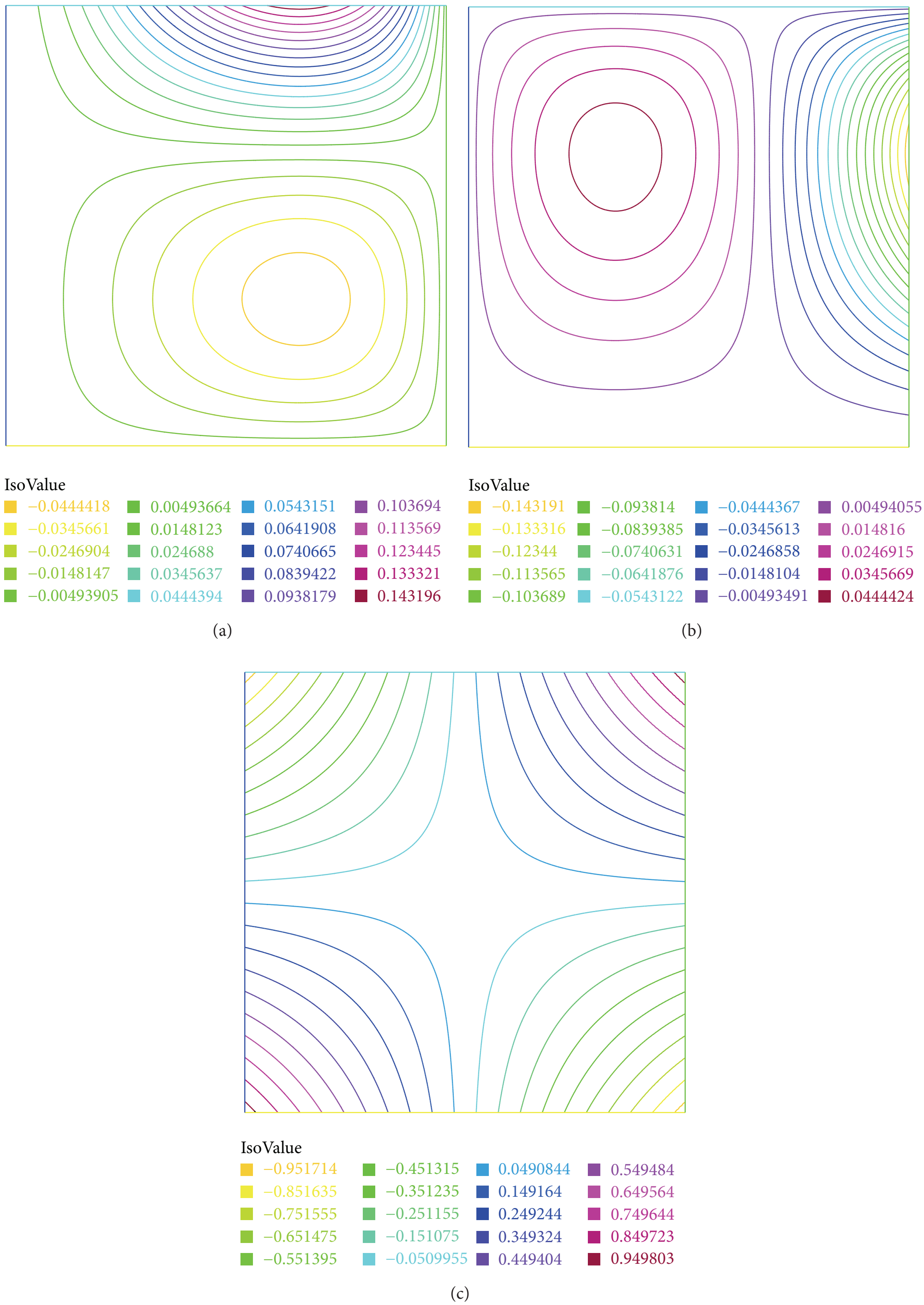

FIGURE 3: Contour plots of numerical solution by one-level method. From (a) to (c): two components of velocity and pressure.

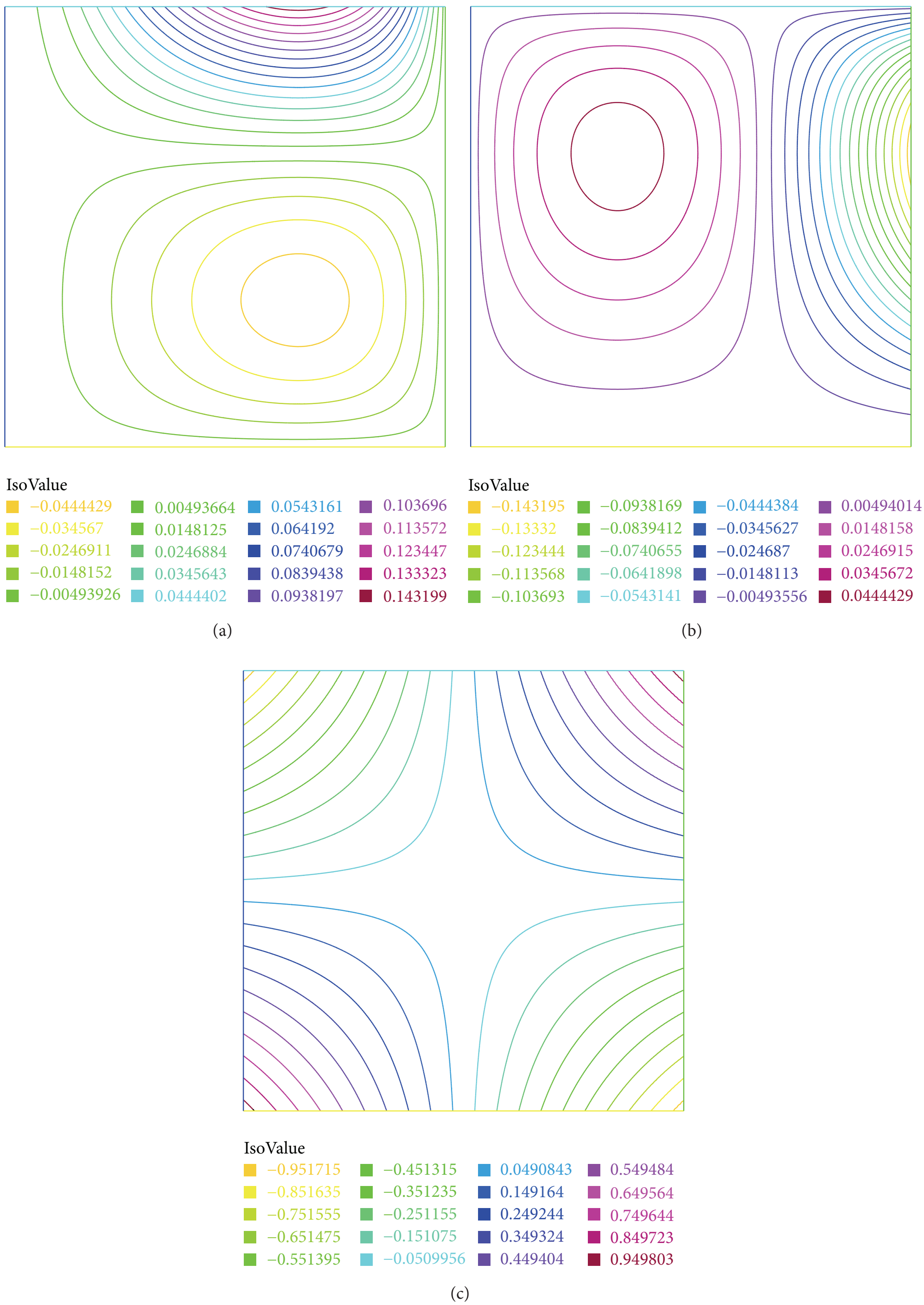

Figure 4: Contour plots of numerical solution by two-level Newton method. From (a) to (c): two components of velocity and pressure.

Table 1: Convergence of one-level method.

$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}^h\|}{\|\mathbf{u}\|}$	Rate	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_V}{\|\mathbf{u}\|_V}$	Rate	$\frac{\|p-p^h\|}{\|p\|}$	Rate	CPU (s)
4^2	$2.52501e-02$	/	$1.77812e-01$	/	$1.11990e-02$	/	0.717
6^2	$4.31365e-03$	2.1790	$7.17050e-02$	1.1199	$5.03212e-03$	0.9865	4.633
8^2	$1.24103e-03$	2.1653	$3.83421e-02$	1.0880	$2.44663e-03$	1.2533	14.602
10^2	$4.84344e-04$	2.1083	$2.39285e-02$	1.0564	$1.32260e-03$	1.3783	36.925
12^2	$2.30749e-04$	2.0334	$1.63920e-02$	1.0374	$7.82664e-04$	1.4388	79.451
14^2	$1.26874e-04$	1.9401	$1.19446e-02$	1.0266	$4.98557e-04$	1.4628	148.934
16^2	$7.80965e-05$	1.8170	$9.09623e-03$	1.0201	$3.36414e-04$	1.4730	265.996
18^2	OUT		OF		MEMORY		

Table 2: Convergence of two-level Newton iteration scheme.

$1/H$	$1/h$	$\frac{\|\mathbf{u}-\mathbf{u}^h\|}{\|\mathbf{u}\|}$	Rate	$\frac{\|\mathbf{u}-\mathbf{u}^h\|_V}{\|\mathbf{u}\|_V}$	Rate	$\frac{\|p-p^h\|}{\|p\|}$	Rate	CPU (s)
4	4^2	$2.52556e-02$	/	$1.77813e-01$	/	$1.11990e-02$	/	0.414
6	6^2	$4.30902e-03$	2.1806	$7.17049e-02$	1.1199	$5.03215e-03$	0.9865	1.522
8	8^2	$1.23630e-03$	2.1701	$3.83421e-02$	1.0880	$2.44665e-03$	1.2533	4.553
10	10^2	$4.79543e-04$	2.1221	$2.39285e-02$	1.0564	$1.32261e-03$	1.3783	11.493
12	12^2	$2.25836e-04$	2.0651	$1.63919e-02$	1.0374	$7.82669e-04$	1.4388	25.997
14	14^2	$1.21790e-04$	2.0029	$1.19445e-02$	1.0266	$4.98560e-04$	1.4628	45.606
16	16^2	$7.27796e-05$	1.9279	$9.09620e-03$	1.0200	$3.36415e-04$	1.4730	77.029
18	18^2	$4.72790e-04$	1.8312	$7.16069e-03$	1.0156	$2.37461e-04$	1.4787	129.046

profiles. Figures 2, 3, and 4 display the exact solution and the numerical solution by one-level method and two-level Newton method, respectively. From these three groups of contour plots, we can observe the good coincidence with each other to illustrate the stability of the present stabilized methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is based upon work funded by the National Natural Science Foundation of China under Grant Nos. 10901122 and 11001205 and by Zhejiang Provincial Natural Science Foundation of China under Grant No. LY12A01015.

References

[1] H. Fujita, *Flow Problems with Unilateral Boundary Conditions*, Collège de France, Lecons, France, 1993.

[2] H. Fujita, "A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions," *RIMS Kokyuroku*, no. 888, pp. 199–216, 1994.

[3] H. Fujita, "Non-stationary Stokes flows under leak boundary conditions of friction type," *Journal of Computational Mathematics*, vol. 19, no. 1, pp. 1–8, 2001.

[4] H. Fujita, "A coherent analysis of Stokes flows under boundary conditions of friction type," *Journal of Computational and Applied Mathematics*, vol. 149, no. 1, pp. 57–69, 2002.

[5] Y. Li and K. Li, "Existence of the solution to stationary Navier-Stokes equations with nonlinear slip boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 381, no. 1, pp. 1–9, 2011.

[6] Y. Li and K. Li, "Global strong solutions of two-dimensional Navier-Stokes equations with nonlinear slip boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 393, no. 1, pp. 1–13, 2012.

[7] N. Saito and H. Fujita, "Regularity of solutions to the Stokes equations under a certain nonlinear boundary condition," in *The Navier-Stokes Equations: Theory and Numerical Methods*, vol. 223 of *Lecture Notes in Pure and Applied Mathematics*, pp. 73–86, Dekker, New York, NY, USA, 2002.

[8] N. Saito, "On the Stokes equation with the leak and slip boundary conditions of friction type: regularity of solutions," *Publications of RIMS, Kyoto University*, vol. 40, no. 2, pp. 345–383, 2004.

[9] M. Ayadi, M. K. Gdoura, and T. Sassi, "Mixed formulation for Stokes problem with Tresca friction," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 348, no. 19-20, pp. 1069–1072, 2010.

[10] T. Kashiwabara, "On a finite element approximation of the Stokes equations under a slip boundary condition of the friction type," *Japan Journal of Industrial and Applied Mathematics*, vol. 30, no. 1, pp. 227–261, 2013.

[11] T. Kashiwabara, "Finite element method for Stokes equations under leak boundary condition of friction type," *SIAM Journal on Numerical Analysis*, vol. 51, no. 4, pp. 2448–2469, 2013.

[12] J. Djoko and M. Mbehou, "Finite element analysis for Stokes and Navier-Stokes equations driven by threshold slip boundary conditions," *International Journal of Numerical Analysis and Modeling B*, vol. 4, no. 3, pp. 235–255, 2013.

[13] J. K. Djoko, "On the time approximation of the Stokes equations with nonlinear slip boundary conditions," *International Journal of Numerical Analysis and Modeling*, vol. 11, no. 1, pp. 34–53, 2014.

[14] R. An and Y. Li, "Two-level penalty finite element methods for Navier-Stokes equations with nonlinear slip boundary conditions," *International Journal of Numerical Analysis and Modeling*. In press.

[15] Y. Li and R. An, "Penalty finite element method for Navier-Stokes equations with nonlinear slip boundary conditions," *International Journal for Numerical Methods in Fluids*, vol. 69, no. 3, pp. 550–566, 2012.

[16] Y. Li and R. An, "Two-level iteration penalty methods for the Navier-Stokes equations with friction boundary conditions," *Abstract and Applied Analysis*, vol. 2013, Article ID 125139, 17 pages, 2013.

[17] Y. Li and K. Li, "Pressure projection stabilized finite element method for Navier-Stokes equations with nonlinear slip boundary conditions," *Computing*, vol. 87, no. 3-4, pp. 113–133, 2010.

[18] R. An and H. Qiu, "Two-level Newton iteration methods for Navier-Stokes type variational inequality problem," *Advances in Applied Mathematics and Mechanics*, vol. 5, no. 1, pp. 36–54, 2013.

[19] Y. Li and R. An, "Two-level pressure projection finite element methods for Navier-Stokes equations with nonlinear slip boundary conditions," *Applied Numerical Mathematics*, vol. 61, no. 3, pp. 285–297, 2011.

[20] F. Brezzi and J. Pitkäranta, "On the stabilization of finite element approximations of the Stokes equations," in *Efficient Solutions of Elliptic Systems*, W. Hackbusch, Ed., vol. 10, pp. 11–19, Vieweg, Braunschweig, Germany, 1984.

[21] N. Kechkar and D. Silvester, "Analysis of locally stabilized mixed finite element methods for the Stokes problem," *Mathematics of Computation*, vol. 58, no. 197, pp. 1–10, 1992.

[22] D. J. Silvester and N. Kechkar, "Stabilised bilinear-constant velocity-pressure finite elements for the conjugate gradient solution of the Stokes problem," *Computer Methods in Applied Mechanics and Engineering*, vol. 79, no. 1, pp. 71–87, 1990.

[23] R. Becker and P. Hansbo, "A simple pressure stabilization method for the Stokes equations," *Communications in Numerical Methods in Engineering with Biomedical Applications*, vol. 24, no. 11, pp. 1421–1431, 2008.

[24] A. N. Brooks and T. J. R. Hughes, "Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations," *Computer Methods in Applied Mechanics and Engineering*, vol. 32, no. 1–3, pp. 199–259, 1982.

[25] J. Douglas and J. P. Wang, "An absolutely stabilized finite element method for the Stokes problem," *Mathematics of Computation*, vol. 52, no. 186, pp. 495–508, 1989.

[26] P. B. Bochev, C. R. Dohrmann, and M. D. Gunzburger, "Stabilization of low-order mixed finite element," *SIAM Journal on Numerical Analysis*, vol. 44, no. 1, pp. 82–101, 2006.

[27] J. Li and Y. He, "A stabilized finite element method based on two local Gauss integrations for the Stokes equations," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 58–65, 2008.

[28] R. An, "Error estimates of fully discrete stabilized finite element methods for two-dimensional bingham fluid flow," Preprint.

[29] J. Xu, "A novel two-grid method for semilinear elliptic equations," *SIAM Journal on Scientific Computing*, vol. 15, no. 1, pp. 231–237, 1994.

[30] J. Xu, "Two-grid discretization techniques for linear and non-linear PDEs," *SIAM Journal on Numerical Analysis*, vol. 33, no. 5, pp. 1759–1777, 1996.

[31] Y. He and K. Li, "Two-level stabilized finite element methods for the steady Navier-Stokes problem," *Computing*, vol. 74, no. 4, pp. 337–351, 2005.

[32] J. Shen, "On error estimates of the penalty method for unsteady Navier-Stokes equations," *SIAM Journal on Numerical Analysis*, vol. 32, no. 2, pp. 386–403, 1995.

[33] J.-C. Latché and D. Vola, "Analysis of the Brezzi-Pitkäranta stabilized Galerkin scheme for creeping flows of Bingham fluids," *SIAM Journal on Numerical Analysis*, vol. 42, no. 3, pp. 1208–1225, 2004.

[34] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer, New York, NY, USA, 1984.

[35] Y. Li and K. Li, "Uzawa iteration method for Stokes type variational inequality of the second kind," *Acta Mathematicae Applicatae Sinica*, vol. 27, no. 2, pp. 303–316, 2011.